J. Math. Computer Sci., 28 (2023), 60–71

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Oscillation of fourth-order neutral differential equations with distributed deviating arguments

A. A. El-Gaber*, M. M. A. El-Sheikh

Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Shebin El-Koom, Egypt.

Abstract

A general class of fourth-order neutral differential equations with distributed deviating arguments is considered. New oscillation criteria are deduced in both canonical and noncanonical cases. Two illustrative examples are given.

Keywords: Oscillation, fourth-order, neutral differential equations.

2020 MSC: 34C10, 34K11.

1. Introduction

In this paper, we are concerned with the oscillation of fourth-order half-linear neutral differential equations of the form

$$\left(r(t)\left(z'''(t)\right)^{\alpha}\right)' + \int_c^d q(t,\zeta) f(x(\sigma(t,\zeta)))\, d\zeta = 0, \qquad t \geqslant t_0, \tag{1.1}$$

where $z(t) = x(t) + \int_a^b p(t,\mu) x(\tau(t,\mu))\, d\mu$ and $\alpha \geqslant 1$ is a quotient of odd positive integers under the conditions

$$R(t_0) = \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty, \tag{1.2}$$

and

$$R(t_0) = \int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}(t)} dt < \infty. \tag{1.3}$$

Throughout the paper, we assume the following assumptions:

(A_1) $r(t) \in C^1([t_0,\infty),(0,\infty))$, $r'(t) \geqslant 0$;

(A_2) $p(t,\mu) \in C([t_0,\infty)\times[a,b],[0,\infty))$, $0 \leqslant \int_a^b p(t,\mu)\, d\mu \leqslant P < 1$;

(A_3) $\tau(t,\mu) \in C([t_0,\infty)\times[a,b],\mathbb{R})$ is a nondecreasing function for μ satisfying $\tau(t,\mu) \leqslant t$ and $\liminf_{t\to\infty} \tau(t,\mu) = \infty$;

*Corresponding author

Email address: amina.aboalnour@science.menofia.edu.eg (A. A. El-Gaber)

doi: 10.22436/jmcs.028.01.06

Received: 2021-09-11 *Revised:* 2022-02-19 *Accepted:* 2022-03-10

(A_4) $q(t,\zeta) \in C([t_0,\infty)\times[c,d],(0,\infty))$, $f \in C(R,R)$, $f(x)/x^{\alpha} \geqslant K$ for all $x \neq 0$, and for some $K > 0$;

(A_5) $\sigma(t,\zeta) \in C([t_0,\infty)\times[c,d],R)$ is a nondecreasing function for ξ satisfying $\sigma(t,\xi) \leqslant t, \sigma_1'(t) > 0$, where $\sigma_1(t) = \sigma(t,c)$, and $\liminf_{t\to\infty}\sigma(t,\xi) = \infty$.

By a solution of (1.1), we mean a nontrivial real function $x(t)$ such that

$$r(t)\left(\left[x(t)+\int_a^b p(t,\mu)\,x(\tau(t,\mu))\,d\mu\right]'''\right)^{\alpha}$$

is continuously differentiable satisfying (1.1) for any $t_1 \geqslant t_0$. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

In recent years, there has been a great deal of interest in studying the oscillatory behavior of solutions of various types of differential equations; see [1–16, 18, 19, 21–25, 27–36, 38–40]. The half-linear equations have numerous applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous medium, etc; see, e.g., [6–8, 20] for more details. Moreover, in the frame of continuous PDEs, and in particular in dynamical models, delay and oscillatory-type effects are often modeled by external sources perturbing the natural evolution of the related systems, some of these contributions on parabolic chemotaxis model with nonlinear diffusions can be found in [17, 26, 37], which are connected to mathematical biology. In particular, the papers [6–8, 12, 21, 23–25] were concerned with the oscillation of various classes of half-linear differential equations, whereas the papers [22, 27, 39] were concerned with the oscillatory behavior of different classes of fourth-order differential equations. In the following, we show some previous results in the literature which related to this paper. The authors in [3, 4, 28] discussed the oscillatory behavior of solutions of the fourth-order neutral differential equation

$$\left(r(t)\left([x(t)+p(t)x(\tau(t))]'''\right)^{\alpha}\right)' + q(t)\,x^{\beta}(\delta(t)) = 0, \tag{1.4}$$

under the condition (1.2).

In [11] Dassios and Bazighifan discussed the oscillation of Eq. (1.4) under the condition (1.3).

In [19] Li et al. studied the oscillation of the fourth-order neutral differential equations with p-laplacian like operators of the type

$$\left(r(t)\left|z'''(t)\right|^{p-2} z'''(t)\right)' + \sum_{i=1}^{l} q_i(t)\,|x(\tau_i(t))|^{p-2}\,x(\tau_i(t)) = 0,$$

where $z(t) = x(t) + a(t)x(\sigma(t))$, under the condition $\int_{t_0}^{\infty}\frac{1}{r^{\frac{1}{p-2}}(t)}dt < \infty$.

In [5] Bazighifan et al. discussed the asymptotic behavior of solutions of the fourth-order neutral differential equations

$$\left(r(t)\left([x(t)+p(t)x(\phi(t))]'''\right)^{\alpha}\right)' + \int_a^b q(t,\theta)\,x^{\beta}(\delta(t,\theta))\,d\theta = 0,$$

where α, β are quotients of odd positive integers and $\beta \geqslant \alpha$ under the condition (1.2).

The aim of this paper is to employ generalized Riccati transformation to establish some new conditions for the oscillation of all solutions of equation (1.1), under the conditions (1.2) and (1.3).

2. Preliminaries

We first outline some lemmas which will be needed for the proofs of the main results.

Lemma 2.1 ([29]). *Let $z(t)$ be a positive and n-times differentiable function on an interval $[T,\infty)$, with non - positive nth derivative $z^{(n)}(t)$ on $[T,\infty)$ which is not identically zero on any interval of the form $[T',\infty)$, $T'\geqslant T$, and such that $z^{(n-1)}(t)z^{(n)}(t)\leqslant 0$. Then there exist constants $0<\theta<1$ and $N>0$ such that $z'(\theta t)\geqslant Nt^{n-2}z^{(n-1)}(t)$ for all sufficient large t.*

Lemma 2.2 ([28]). *Let $z^{(n)}(t)$ be of fixed sign and $z^{(n-1)}(t)z^{(n)}(t)\leqslant 0$, for all $t\geqslant t_1$. If $\lim_{t\to\infty} z(t)\neq 0$, then for every $\lambda\in(0,1)$ there exists $t_\lambda\geqslant t_1$ such that $z(t)\geqslant \frac{\lambda}{(n-1)!}t^{n-1}\left|z^{(n-1)}(t)\right|$ for $t\geqslant t_\lambda$.*

Lemma 2.3 ([2]). *If α is a ratio of two odd numbers with $V>0$ and U are constants, then $UY-VY^{\frac{(\alpha+1)}{\alpha}} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{U^{\alpha+1}}{V^\alpha}$.*

Lemma 2.4. *Assume that $x(t)$ is an eventually positive solution of (1.1), and $z'(t)>0$, then*

$$\left(r(t)\left(z'''(t)\right)^\alpha\right)'\leqslant -q_1(t)z^\alpha(\sigma_1(t)), \tag{2.1}$$

where $q_1(t)=K(1-P)^\alpha\int_c^d q(t,\zeta)\,d\zeta$, $\sigma_1(t)=\sigma(t,c)$.

Proof. Since $x(t)$ is an eventually positive solution of (1.1), then there exists a $t_1\geqslant t_0$ such that $x(t)>0, x(\sigma(t,\xi))>0$ and $x(\tau(t,\mu))>0$ for $t\geqslant t_1$. Now from the definition of z we have

$$\begin{aligned}
x(t) &= z(t)-\int_a^b p(t,\mu)\,x(\tau(t,\mu))\,d\mu\\
&\geqslant z(t)-\int_a^b p(t,\mu)\,z(\tau(t,\mu))\,d\mu\\
&\geqslant z(t)-z(\tau(t,b))\int_a^b p(t,\mu)\,d\mu \geqslant \left(1-\int_a^b p(t,\mu)\,d\mu\right)z(t)\geqslant(1-P)\,z(t).
\end{aligned}$$

Using Eq. (1.1), we get

$$\begin{aligned}
\left(r(t)\left(z'''(t)\right)^\alpha\right)' &\leqslant -K\int_c^d q(t,\xi)\,x^\alpha(\sigma(t,\xi))\,d\xi\\
&\leqslant -K(1-P)^\alpha\int_c^d q(t,\xi)\,z^\alpha(\sigma(t,\xi))\,d\xi\\
&\leqslant -K(1-P)^\alpha z^\alpha(\sigma(t,c))\int_c^d q(t,\xi)\,d\xi=-q_1(t)\,z^\alpha(\sigma_1(t)).
\end{aligned}$$

Thus the proof is completed. □

The following two auxiliary results are very similar to those in [3, 11].

Lemma 2.5. *Let $x(t)$ be a positive solution of (1.1). If (1.2) is satisfied, then there exists $t\geqslant t_1$ such that*

$$z(t)>0, z'(t)>0, z'''(t)>0, z^{(4)}(t)<0, \left(r(t)\left(z'''(t)\right)^\alpha\right)'\leqslant 0.$$

Lemma 2.6. *Let $x(t)$ be a positive solution of (1.1). If (1.3) is satisfied, then there exist three possible cases for sufficiently large $t\geqslant t_1$:*

(S_1) $z(t)>0, z'(t)>0, z'''(t)>0$, *and* $z^{(4)}(t)\leqslant 0$;
(S_2) $z(t)>0, z'(t)>0, z''(t)>0$, *and* $z'''(t)<0$;
(S_3) $z(t)>0, z'(t)<0, z''(t)>0$, *and* $z'''(t)<0$.

3. Main results

In this section, we start with the case $R(t_0) = \infty$.

Lemma 3.1. *Let x be an eventually positive solution of (1.1). If there exist $\rho(t) \in C^1([t_0,\infty),(0,\infty))$, $a(t) \in C^1([t_0.\infty),[0,\infty))$, $\delta \in (0,1)$, and $\epsilon > 0$, such that*

$$\vartheta(t) = \rho(t)\left[\frac{r(t)\left(z'''(t)\right)^{\alpha}}{z^{\alpha}(\delta\sigma_1(t))} + r(t)a(t)\right], \ \mathbf{t} \geqslant \mathbf{t}_1, \tag{3.1}$$

then

$$\vartheta'(t) \leqslant -Q(t) + A(t)\vartheta(t) - B(t)\vartheta^{1+\frac{1}{\alpha}}(t), \tag{3.2}$$

where

$$A(t) = \frac{\rho'(t)}{\rho(t)} + (\alpha+1)\delta\epsilon\sigma_1^2(t)\sigma_1'(t)a^{\frac{1}{\alpha}}(t), \qquad B(t) = \frac{\alpha\delta\epsilon\sigma_1^2(t)\sigma_1'(t)}{[r(t)\rho(t)]^{\frac{1}{\alpha}}},$$

and

$$Q(t) = q_1(t)\rho(t) - \rho(t)[r(t)a(t)]' + \delta\epsilon\sigma_1^2(t)\sigma_1'(t)r(t)\rho(t)a^{1+\frac{1}{\alpha}}(t).$$

Proof. Assume that x is an eventually positive solution of (1.1). Using Lemma 2.4, we obtain (2.1). It is clear by (3.1) that $\vartheta(t) > 0$ for $t \geqslant t_1$, and

$$\vartheta'(t) = \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)a(t)]' + \rho(t)\frac{\left(r(t)\left(z'''(t)\right)^{\alpha}\right)'}{z^{\alpha}(\delta\sigma_1(t))} - \rho(t)\frac{\alpha\delta r(t)\sigma_1'(t)\left(z'''(t)\right)^{\alpha}z'(\delta\sigma_1(t))}{z^{\alpha+1}(\delta\sigma_1(t))},$$

i.e.,

$$\vartheta'(t) \leqslant \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)a(t)]' - \rho(t)\frac{q_1(t)z^{\alpha}(\sigma_1(t))}{z^{\alpha}(\delta\sigma_1(t))} - \rho(t)\frac{\alpha\delta r(t)\sigma_1'(t)\left(z'''(t)\right)^{\alpha}z'(\delta\sigma_1(t))}{z^{\alpha+1}(\delta\sigma_1(t))}.$$

By Lemma 2.1, we have

$$z'(\delta\sigma_1(t)) \geqslant \epsilon\sigma_1^2(t)z'''(\sigma_1(t)).$$

Since $z(t)$ is increasing, then we have

$$z^{\alpha}(\sigma_1(t)) \geqslant z^{\alpha}(\delta\sigma_1(t)),$$

then

$$\vartheta'(t) \leqslant \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)a(t)]' - \rho(t)q_1(t) - \rho(t)\frac{\alpha\delta\epsilon r(t)\sigma_1'(t)\left(z'''(t)\right)^{\alpha}\sigma_1^2(t)z'''(\sigma_1(t))}{z^{\alpha+1}(\delta\sigma_1(t))}.$$

But since

$$z'''(\sigma_1(t)) \geqslant z'''(t),$$

then

$$\vartheta'(t) \leqslant \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)a(t)]' - \rho(t)q_1(t) - \rho(t)\alpha\epsilon\delta r(t)\sigma_1'(t)\sigma_1^2(t)\left(\frac{z'''(t)}{z(\delta\sigma_1(t))}\right)^{\alpha+1}.$$

Moreover since from (3.1), we have

$$\frac{z'''(t)}{z(\delta\sigma_1(t))} = \frac{1}{r^{\frac{1}{\alpha}}(t)}\left[\frac{\vartheta(t)}{\rho(t)} - [r(t)\,a(t)]\right]^{\frac{1}{\alpha}},$$

then

$$\vartheta'(t) \leqslant \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)\,a(t)]' - \rho(t)\,q_1(t) - \alpha\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\frac{\rho(t)}{r^{\frac{1}{\alpha}}(t)}\left(\frac{\vartheta(t)}{\rho(t)} - [r(t)\,a(t)]\right)^{\frac{\alpha+1}{\alpha}}. \quad (3.3)$$

Following [35], we define

$$M = \frac{\vartheta(t)}{\rho(t)} \quad \text{and} \quad N = r(t)\,a(t),$$

using the inequality

$$M^{1+\frac{1}{\alpha}} - (M-N)^{1+\frac{1}{\alpha}} \leqslant N^{\frac{1}{\alpha}}\left[\left(1+\frac{1}{\alpha}\right)M - \frac{1}{\alpha}N\right], \quad MN \geqslant 0, \quad \alpha \geqslant 1,$$

we have

$$\left(\frac{\vartheta(t)}{\rho(t)} - [r(t)\,a(t)]\right)^{\frac{\alpha+1}{\alpha}} \geqslant \left[\frac{\vartheta(t)}{\rho(t)}\right]^{1+\frac{1}{\alpha}} + \frac{1}{\alpha}[r(t)\,a(t)]^{1+\frac{1}{\alpha}} - \left(1+\frac{1}{\alpha}\right)\frac{[r(t)\,a(t)]^{\frac{1}{\alpha}}}{\rho(t)}\vartheta(t). \quad (3.4)$$

Using the inequalities (3.3) and (3.4), for $t \geqslant T$, we have

$$\begin{aligned}\vartheta'(t) \leqslant\ & \frac{\rho'(t)}{\rho(t)}\vartheta(t) + \rho(t)[r(t)\,a(t)]' - \rho(t)\,q_1(t) \\ & + \alpha\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\frac{\rho(t)}{r^{\frac{1}{\alpha}}(t)}\left[\left(1+\frac{1}{\alpha}\right)\frac{[r(t)\,a(t)]^{\frac{1}{\alpha}}}{\rho(t)}\vartheta(t) - \frac{1}{\alpha}[r(t)\,a(t)]^{1+\frac{1}{\alpha}} - \frac{\vartheta^{1+\frac{1}{\alpha}}(t)}{\rho^{1+\frac{1}{\alpha}}(t)}\right].\end{aligned}$$

Then

$$\begin{aligned}\vartheta'(t) \leqslant\ & \rho(t)\left([r(t)\,a(t)]' - q_1(t)\right) + \left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)\,\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,a^{\frac{1}{\alpha}}(t)\right]\vartheta(t) \\ & - \frac{\alpha\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)}{r^{\frac{1}{\alpha}}(t)\,\rho^{\frac{1}{\alpha}}(t)}\vartheta^{1+\frac{1}{\alpha}}(t) - \delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,r(t)\,\rho(t)\,a^{1+\frac{1}{\alpha}}(t).\end{aligned}$$

Thus we obtain

$$\vartheta'(t) \leqslant -Q(t) + A(t)\,\vartheta(t) - B(t)\,\vartheta^{1+\frac{1}{\alpha}}(t).$$

This completes the proof. □

In the following theorem we establish a Kamenev-type oscillation criterion for (1.1) under the condition (1.2).

Theorem 3.2. *If*

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s) - \frac{r(s)\,\rho(s)}{(\alpha+1)^{\alpha+1}}\frac{\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,a^{\frac{1}{\alpha}}(s)\right]^{\alpha+1}}{\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\right]^{\alpha}}\right]ds = \infty, \quad (3.5)$$

then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1) on $[t_0, \infty)$. Without loss of generality, we may assume that x is an eventually positive. Using Lemma 3.1, we get (3.2). Now let

$$U = \frac{\rho'(t)}{\rho(t)} + (\alpha+1)\,\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,a^{\frac{1}{\alpha}}(t), \qquad V = \alpha\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,\frac{1}{[r(t)\,\rho(t)]^{\frac{1}{\alpha}}}, \quad \text{and } \; Y = \vartheta(t).$$

Thus by Lemma 2.3, we obtain

$$\begin{aligned}&\left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)\,\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,a^{\frac{1}{\alpha}}(t)\right]\vartheta(t) - \left[\alpha\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,\frac{1}{[r(t)\,\rho(t)]^{\frac{1}{\alpha}}}\right]\vartheta^{\frac{\alpha+1}{\alpha}}(t)\\ &\leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\,\frac{r(t)\,\rho(t)\left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)\,\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,a^{\frac{1}{\alpha}}(t)\right]^{\alpha+1}}{\alpha^\alpha\left[\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\right]^\alpha}.\end{aligned}$$

Thus we have

$$\vartheta'(t) \leqslant -Q(t) + \frac{r(t)\,\rho(t)\left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)\,\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\,a^{\frac{1}{\alpha}}(t)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta\epsilon\sigma_1'(t)\,\sigma_1^2(t)\right]^\alpha},$$

and

$$-\int_{t_0}^{t}(t-s)^n\,\vartheta'(s)\,ds \geqslant \int_{t_0}^{t}(t-s)^n\left[Q(s) - \frac{r(s)\,\rho(s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,a^{\frac{1}{\alpha}}(s)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\right]^\alpha}\right]ds. \tag{3.6}$$

Since

$$\int_{t_0}^{t}(t-s)^n\,\vartheta'(s)\,ds = n\int_{t_0}^{t}(t-s)^{n-1}\,\vartheta(s)\,ds - (t-t_0)^n\,\vartheta(t_0),$$

then from (3.6), we get

$$\begin{aligned}&(t-t_0)^n\,\vartheta(t_0) - n\int_{t_0}^{t}(t-s)^{n-1}\,\vartheta(s)\,ds\\ &\geqslant \int_{t_0}^{t}(t-s)^n\left[Q(s) - \frac{r(s)\,\rho(s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,a^{\frac{1}{\alpha}}(s)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\right]^\alpha}\right]ds.\end{aligned}$$

Hence

$$\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s) - \frac{r(s)\,\rho(s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,a^{\frac{1}{\alpha}}(s)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\right]^\alpha}\right]ds \leqslant \left(\frac{t-t_0}{t}\right)^n\vartheta(t_0),$$

and so

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}(t-s)^n\left[Q(s) - \frac{r(s)\,\rho(s)\left[\frac{\rho'(s)}{\rho(s)} + (\alpha+1)\,\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,a^{\frac{1}{\alpha}}(s)\right]^{\alpha+1}}{(\alpha+1)^{\alpha+1}\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\right]^\alpha}\right]ds \to \vartheta(t_0),$$

which contradicts (3.5) and this completes the proof. □

In the following, we establish Philos-type oscillation criteria for Eq. (1.1) under the condition (1.2). We first need the following definition.

Definition 3.3. Let $D = \{(t,s) \in R^2 : t \geqslant s \geqslant t_0\}$ and $D_0 = \{(t,s) \in R^2 : t > s \geqslant t_0\}$. The functions $H_i(t,s) \in C(D,R)$, $i = 1,2$ are said to belong to the class X written as $H_i \in X$ if they satisfy:

I) $H_i(t,t) = 0$ for $t \geqslant t_0$, $H_i(t,s) > 0$, $(t,s) \in D_0$;

II) $\frac{\partial H_i(t,s)}{\partial s} \leqslant 0$, and there exist $\eta(t) \in C^1([t_0,\infty),(0,\infty))$ and $h_i(t,s) \in C(D,R)$ satisfying

$$-\frac{\partial H_1(t,s)}{\partial s} = H_1(t,s)\left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)\delta\epsilon\sigma_1'(t)\sigma_1^2(t)a^{\frac{1}{\alpha}}(t)\right] + h_1(t,s),$$

and

$$\frac{\partial H_2(t,s)}{\partial s} + \frac{\eta'(t)}{\eta(t)}H_2(t,s) = \frac{h_2(t,s)}{\eta(t)}[H_2(t,s)]^{\frac{\alpha}{\alpha+1}}.$$

Theorem 3.4. *If there exists a function* $H_1 \in X$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H_1(t,t_0)}\int_{t_0}^{t}\left[H_1(t,s)Q(s) - \frac{r(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\frac{[|h_1(t,s)|]^{\alpha+1}}{\left[\delta\epsilon\sigma_1'(s)\sigma_1^2(s)H_1(t,s)\right]^{\alpha}}\right]ds = \infty, \tag{3.7}$$

then every solution of (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that x is an eventually positive solution of (1.1). Now from Lemma 3.1 we get (3.2). Multiplying the inequality (3.2) by $H_1(t,s)$ and integrating the resulting inequality from T to t, we have

$$\begin{aligned}\int_T^t H_1(t,s)Q(s)\,ds &\leqslant \int_T^t H_1(t,s)[-\vartheta'(s) + A(s)\vartheta(s) - B(s)\vartheta^{1+\frac{1}{\alpha}}(s)]ds\\ &= H_1(t,T)\vartheta(T) + \int_T^t\left[\frac{\partial H_1(t,s)}{\partial s} + H_1(t,s)A(s)\right]\vartheta(s)\,ds - \int_T^t H_1(t,s)B(s)\vartheta^{1+\frac{1}{\alpha}}(s)\,ds\\ &= H_1(t,T)\vartheta(T) - \int_T^t h_1(t,s)\vartheta(s)\,ds - \int_T^t H_1(t,s)B(s)\vartheta^{1+\frac{1}{\alpha}}(s)]ds,\\ &\leqslant H_1(t,T)\vartheta(T) + \int_T^t\left[|h_1(t,s)|\vartheta(s) - H_1(t,s)B(s)\vartheta^{1+\frac{1}{\alpha}}(s)\right]ds.\end{aligned}$$

Letting $U = |h_1(t,s)|$, $V = H_1(t,s)B(s)$, and using Lemma 2.3, we obtain

$$|h_1(t,s)|\vartheta(s) - H_1(t,s)B(s)\vartheta^{1+\frac{1}{\alpha}}(s) \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{|h_1(t,s)|^{\alpha+1}}{[H_1(t,s)B(s)]^\alpha}.$$

Then

$$\int_T^t H_1(t,s)Q(s)\,ds \leqslant H_1(t,T)\vartheta(T) + \int_T^t\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{r(s)\rho(s)|h_1(t,s)|^{\alpha+1}}{\alpha^\alpha[H_1(t,s)]^\alpha\left[\delta\epsilon\sigma_1'(s)\sigma_1^2(s)\right]^\alpha}ds.$$

Hence

$$\frac{1}{H_1(t,T)}\int_T^t\left[H_1(t,s)Q(s) - \frac{r(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\frac{|h_1(t,s)|^{\alpha+1}}{\left[\delta\epsilon\sigma_1'(s)\sigma_1^2(s)H_1(t,s)\right]^\alpha}\right]ds \leqslant \vartheta(T),$$

for all sufficiently large t, which contradicts (3.7). □

4. The case $R(t_0) < \infty$

In this section, we discuss the oscillation of Eq. (1.1) under the condition (1.3) . We first need the following lemma.

Lemma 4.1. *Assume that* x *is an eventually positive solution of Eq.* (1.1) *and* (S_2) *holds. If*

$$\Phi(t) = \eta(t) \frac{r(t)\left[z'''(t)\right]^{\alpha}}{\left[z''(t)\right]^{\alpha}}, \tag{4.1}$$

then

$$\Phi'(t) \leqslant \frac{\eta'(t)}{\eta(t)}\Phi(t) - \eta(t) q_1(t)\left[\frac{\lambda}{2}\sigma_1^2(t)\right]^{\alpha} - \frac{\alpha\Phi^{\alpha+1}(t)}{r^{\frac{1}{\alpha}}(t)\,\eta^{\frac{1}{\alpha}}(t)}, \quad \lambda \in (0,1). \tag{4.2}$$

Proof. Assume that x is an eventually positive solution of Eq. (1.1) and (S_2) holds. Since $z' > 0$, then by using Lemma 2.4, we obtain (2.1). Now from (4.1) we see that $\Phi(t) < 0$, for $t \geqslant t_1$, and

$$\Phi'(t) = \frac{\eta'(t)}{\eta(t)}\Phi(t) + \eta(t)\frac{\left[r(t)\left[z'''(t)\right]^{\alpha}\right]'}{\left[z''(t)\right]^{\alpha}} - \frac{\alpha\eta(t) r(t)\left[z'''(t)\right]^{\alpha+1}}{\left[z''(t)\right]^{\alpha+1}}.$$

This with (2.1) and (4.1) leads to

$$\Phi'(t) \leqslant \frac{\eta'(t)}{\eta(t)}\Phi(t) - \eta(t)\frac{q_1(t) z^{\alpha}(\sigma_1(t))}{\left[z''(t)\right]^{\alpha}} - \frac{\alpha\Phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)\,\eta^{\frac{1}{\alpha}}(t)},$$

i.e.,

$$\Phi'(t) \leqslant \frac{\eta'(t)}{\eta(t)}\Phi(t) - \eta(t)\frac{q_1(t) z^{\alpha}(\sigma_1(t))\left[z''(\sigma_1(t))\right]^{\alpha}}{\left[z''(\sigma_1(t))\right]^{\alpha}\left[z''(t)\right]^{\alpha}} - \frac{\alpha\Phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)\,\eta^{\frac{1}{\alpha}}(t)}.$$

Since $z''(t)$ is decreasing, then $-\frac{z''(\sigma_1(t))}{z''(t)} \leqslant -1$, and from Lemma 2.2, we obtain $z(\sigma_1(t)) \geqslant \frac{\lambda}{2}\sigma_1^2(t) z''(\sigma_1(t))$. Then

$$\Phi'(t) \leqslant \frac{\eta'(t)}{\eta(t)}\Phi(t) - \eta(t) q_1(t)\left[\frac{\lambda}{2}\sigma_1^2(t)\right]^{\alpha} - \frac{\alpha\Phi^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)\,\eta^{\frac{1}{\alpha}}(t)}.$$

Thus the proof is completed. □

Theorem 4.2. *Assume that* (3.7) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[H_2(t,s)\eta(s) q_1(s)\left[\frac{\lambda}{2}\sigma_1^2(s)\right]^{\alpha} - \frac{r(s)}{(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)}\left[h_2(t,s)\right]^{\alpha+1}\right] ds > 0, \tag{4.3}$$

and one of the following conditions holds

$$\int_{t_0}^{\infty} R(s)\, ds = \infty, \tag{4.4}$$

or

$$\int_{t_0}^{\infty}\int_{u}^{\infty} R(s)\, ds du = \infty, \tag{4.5}$$

then Eq. (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality we may assume that x is eventually positive. From Lemma 2.6, we have that three possible cases hold. Letting (S_1) holds, then by Theorem 3.4 we see that every solution of (1.1) is oscillatory when condition (3.7) holds. Now if (S_2) holds, then from Lemma 4.1, we have (4.2). Multiplying (4.2) by $H_2(t,s)$ and integrating from t_1 to t, we obtain

$$\begin{aligned}
&\int_{t_1}^{t} H_2(t,s)\,\eta(s)\,q_1(s)\left[\frac{\lambda}{2}\sigma_1^2(s)\right]^{\alpha} ds\\
&\leqslant H_2(t,t_1)\,\Phi(t_1)+\int_{t_1}^{t}\left[\frac{\partial H_2(t,s)}{\partial s}+\frac{\eta'(s)}{\eta(s)}H_2(t,s)\right]\Phi(s)\,ds-\alpha\int_{t_1}^{t} H_2(t,s)\,\frac{\Phi^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)\,\eta^{\frac{1}{\alpha}}(s)}ds,\\
&= H_2(t,t_1)\,\Phi(t_1)+\int_{t_1}^{t}\frac{h_2(t,s)}{\eta(s)}\left[H_2(t,s)\right]^{\frac{\alpha}{\alpha+1}}\Phi(s)\,ds-\alpha\int_{t_1}^{t} H_2(t,s)\,\frac{\Phi^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)\,\eta^{\frac{1}{\alpha}}(s)}ds.
\end{aligned}$$

Set

$$V=\frac{\alpha H_2(t,s)}{r^{\frac{1}{\alpha}}(s)\,\eta^{\frac{1}{\alpha}}(s)},\quad U=\frac{h_2(t,s)}{\eta(s)}\left[H_2(t,s)\right]^{\frac{\alpha}{\alpha+1}},\ \text{and}\ Y=\Phi(s).$$

Then by Lemma 2.3, we have

$$\frac{h_2(t,s)}{\eta(s)}\left[H_2(t,s)\right]^{\frac{\alpha}{\alpha+1}}\Phi(s)-\frac{\alpha H_2(t,s)\,\Phi^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)\,\eta^{\frac{1}{\alpha}}(s)}\leqslant\frac{1}{(\alpha+1)^{\alpha+1}}\left[h_2(t,s)\right]^{(\alpha+1)}\frac{r(s)}{\eta^{\alpha}(s)}.$$

Hence

$$\int_{t_1}^{t}\left[H_2(t,s)\,\eta(s)\,q_1(s)\left[\frac{\lambda}{2}\sigma_1^2(s)\right]^{\alpha}-\frac{1}{(\alpha+1)^{\alpha+1}}\left[h_2(t,s)\right]^{(\alpha+1)}\frac{r(s)}{\eta^{\alpha}(s)}\right]ds\leqslant H_2(t,t_1)\,\Phi(t_1)<0,$$

which contradicts (4.3). Now consider the case (S_3). Assume that $z(t)$ satisfies (S_3). Noting that $r(t)\left(z'''(t)\right)^{\alpha}$ is nonincreasing, we have

$$r^{\frac{1}{\alpha}}(s)\left(z'''(s)\right)\leqslant r^{\frac{1}{\alpha}}(t)\left(z'''(t)\right),\ s\geqslant t\geqslant t_1.$$

Going through as in the proof of Theorem 2.3 case 1 in [19], we get a contradiction with (4.4) and (4.5) and so the proof is completed. □

5. Examples

Example 5.1. For $t\geqslant 1$ and $q_0>0$, consider the fourth-order differential equation

$$\left(t\left[x(t)+\int_1^2\frac{\mu}{t+1}x\left(\frac{t+\mu}{3}\right)d\mu\right]'''\right)'+\int_0^1\frac{2q_0\xi}{t^3}x\left(\frac{t+\xi}{2}\right)d\xi=0.\tag{5.1}$$

Here $\alpha=1, a=1, b=2, c=0, d=1, K=1, r(t)=t, p(t,\mu)=\frac{\mu}{t+1}, \tau(t,\mu)=\frac{t+\mu}{3}, q(t,\xi)=\frac{2q_0\xi}{t^3}$, and $\sigma(t,\xi)=\frac{t+\xi}{2}$. Then

$$\int_a^b p(t,\mu)\,d\mu=\int_1^2\frac{\mu}{t+1}d\mu\leqslant\frac{3}{4},\quad \sigma_1(t)=\sigma(t,c)=\frac{t}{2},\quad \sigma_1'(t)=\frac{1}{2}>0,\ \text{and}\ \int_1^{\infty}\frac{1}{r(s)}ds=\infty.$$

Therefore the conditions (A_1)-(A_5) and (1.2) are satisfied. Choosing $P=\frac{3}{4}, \rho(t)=t^2, a(t)=\frac{1}{t^3}$, and $H_1(t,s)=(t-s)^2$, then $h_1(t,s)=(t-s)\left[\left(4+\frac{\delta\epsilon}{4}\right)-\frac{8+\delta\epsilon}{4}ts^{-1}\right], Q(t)=\left[\frac{q_0}{4}+2+\frac{\delta\epsilon}{8}\right]\frac{1}{t}$, and

$$\limsup_{t\to\infty}\frac{1}{H_1(t,t_0)}\int_{t_0}^{t}\left[H_1(t,s)\,Q(s)-\frac{r(s)\,\rho(s)}{(\alpha+1)^{\alpha+1}}\frac{\left[h_1(t,s)\right]^{\alpha+1}}{\left[\delta\epsilon\sigma_1'(s)\,\sigma_1^2(s)\,H_1(t,s)\right]^{\alpha}}\right]ds$$

$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \left[(t-s)^2 \left[\frac{q_0}{4} + 2 + \frac{\delta\epsilon}{8} \right] \frac{1}{s} - \frac{2s}{\delta\epsilon} \left[\left(4 + \frac{\delta\epsilon}{4} \right) - \frac{8+\delta\epsilon}{4} ts^{-1} \right]^2 \right] ds = \infty.$$

Therefore, by Theorem 3.4, every solution of (5.1) is oscillatory, if $q_0 > \frac{32}{\delta\epsilon}$ for some $\epsilon > 0$ and $\delta \in (0,1)$.

Example 5.2. For $t \geqslant 1$ and $q_0 > 0$, consider the fourth-order differential equation

$$\left(t^2 \left[x(t) + \int_{\frac{1}{2}}^1 \frac{4\mu}{3t^2} x\left(\frac{t+\mu}{3} \right) d\mu \right]''' \right)' + \int_0^1 \frac{32 q_0 \xi}{t^2} x\left(\frac{t+\xi}{2} \right) d\xi = 0. \tag{5.2}$$

Here $\alpha = 1, a = \frac{1}{2}, b = 1, c = 0, d = 1, K = 1, r(t) = t^2, p(t,\mu) = \frac{4\mu}{3t^2}, \tau(t,\mu) = \frac{t+\mu}{3}, q(t,\xi) = \frac{32 q_0 \xi}{t^2}$, and $\sigma(t,\xi) = \frac{t+\xi}{2}$. Then

$$\int_a^b p(t,\mu)\, d\mu = \int_{\frac{1}{2}}^1 \frac{4\mu}{3t^2} d\mu \leqslant \frac{1}{2}, \qquad \sigma_1(t) = \sigma(t,c) = \frac{t}{2}, \qquad \sigma_1'(t) = \frac{1}{2} > 0,$$
$$\int_{t_0}^\infty \frac{1}{r(s)} ds = \int_1^\infty \frac{1}{s^2} ds < \infty, \qquad \int_{t_0}^\infty R(s)\, ds = \infty, \qquad \int_{t_0}^\infty \int_u^\infty R(s)\, ds du = \infty.$$

Therefore the conditions (A_1)-(A_5), (1.3), (4.4), and (4.5) are satisfied. Choose $P = \frac{1}{2}, \rho(t) = t, \eta(t) = 1, a(t) = \frac{1}{t^3}$ and $H_1(t,s) = H_2(t,s) = (t-s)^2$. Then

$$h_1(t,s) = (t-s) \left[\left(3 + \frac{\delta\epsilon}{4} \right) - \frac{4+\delta\epsilon}{4} ts^{-1} \right],$$
$$h_2(t,s) = -2,$$
$$q_1 = 8\frac{q_0}{t^2},$$
$$Q(t) = \left[8q_0 + 1 + \frac{\delta\epsilon}{8} \right] \frac{1}{t},$$
$$\limsup_{t\to\infty} \frac{1}{H_1(t,t_0)} \int_{t_0}^t \left[H_1(t,s) Q(s) - \frac{r(s)\rho(s)}{(\alpha+1)^{\alpha+1}} \frac{[h_1(t,s)]^{\alpha+1}}{[\delta\epsilon\sigma_1'(s)\sigma_1^2(s) H_1(t,s)]^\alpha} \right] ds$$
$$= \limsup_{t\to\infty} \frac{1}{(t-1)^2} \int_1^t \left[(t-s)^2 \left[8q_0 + 1 + \frac{\delta\epsilon}{8} \right] \frac{1}{s} - \frac{2s}{\delta\epsilon} \left[\left(3 + \frac{\delta\epsilon}{4} \right) - \frac{4+\delta\epsilon}{4} ts^{-1} \right]^2 \right] ds = \infty,$$

for any $\epsilon > 0, \delta \in (0,1)$, and $q_0 > \frac{1}{4\delta\epsilon}$. Moreover

$$\limsup_{t\to\infty} \int_{t_0}^t \left[H_2(t,s)\eta(s) q_1(s) \left[\frac{\lambda}{2} \sigma_1^2(s) \right]^\alpha - \frac{r(s)}{(\alpha+1)^{\alpha+1} \eta^\alpha(s)} [h_2(t,s)]^{\alpha+1} \right] ds$$
$$= \limsup_{t\to\infty} \int_1^t \left[\lambda q_0 (t-s)^2 - s^2 ds \right] > 0,$$

for any $\lambda \in (0,1)$, and $q_0 > \frac{1}{\lambda}$. Therefore, by Theorem 4.2, every solution of (5.2) is oscillatory where $q_0 > \frac{1}{4\delta\epsilon}$ and $q_0 > \frac{1}{\lambda}$.

6. Conclusions

In this work, we discuss the oscillation of fourth-order neutral differential equations with distributed deviating arguments of the type (1.1) in both cases $\int_{t_0}^\infty \frac{1}{r^{\frac{1}{\alpha}}(t)} dt = \infty$ and $\int_{t_0}^\infty \frac{1}{r^{\frac{1}{\alpha}}(t)} dt < \infty$. We establish new oscillation criteria using Riccati and generalized Riccati transformation. For interested researchers, there is a good deal of finding new results for (1.1) with different neutral coefficients see, e.g., the papers [7, 12, 21–23, 25, 27].

Acknowledgments

The authors of the paper are grateful to the editorial board and reviewers for the careful reading and helpful suggestions which led to an improvement of our original manuscript.

References

[1] R. P. Agarwal, M. Bohner, T. Li, Ch. Zhang, *Oscillation of third-order nonlinear delay differential equations*, Taiwanese J. Math., **17** (2013), 545–558. 1
[2] R. P. Agarwal, C. Zhang, T. Li, *Some remarks on oscillation of second order neutral differential equations*, Appl. Math. Comput., **274** (2016), 178–181. 2.3
[3] O. Bazighifan, *Kamenev and Philos-types oscillation criteria for fourth-order neutral differential equations*, Adv. Differ. Equ., **2020** (2020), 1–12. 1, 2
[4] O. Bazighifan, C. Cesarano, *A Philos-type oscillation criteria for fourth-order neutral differential equations*, Symmetry, **379** (2020), 10 pages. 1
[5] O. Bazighifan, F. Minhos, O. Moaaz, *Sufficient conditions for oscillation of fourth-order neutral differential equations with distributed deviating arguments*, Axioms, **9** (2020), 11 pages. 1
[6] M. Bohner, T. S. Hassan, T. Li, *Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments*, Indag. Math., **29** (2018), 548–560. 1
[7] M. Bohner, T. Li, *Oscillation of second-order* p*-Laplace dynamic equations with a nonpositive neutral coefficient*, Appl. Math. Lett., **37** (2014), 72–76. 6
[8] M. Bohner, T. Li, *Kamenev-type criteria for nonlinear damped dynamic equations*, Sci. China Math., **58** (2015), 1445–1452. 1
[9] G. E. Chatzarakis, E. M. Elabbasy, O. Bazighifan, *An oscillation criterion in 4th- order neutral differential equations with a continuously distributed delay*, Adv. Differ. Equ., **336** (2019), 9 pages.
[10] K.-S. Chiu, T. Li, *Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments*, Math. Nachr., **292** (2019), 2153–2164.
[11] I. Dassios, O. Bazighifan, *Oscillation conditions for certain fourth-order non-Linear neutral differential equation*, Symmetry, **12** (2020), 9 pages. 1, 2
[12] J. Džurina, S. R. Grace, I. Jadlovsk, T. Li, *Oscillation criteria for second-order Emden-Fowler delay differential equations with a sublinear neutral term*, Math. Nachr., **293** (2020), 910–922. 1, 6
[13] M. M. A. El-Sheikh, *Oscillation and nonoscillation criteria for second order nonlinear differential equations*, J. Math. Anal. Appl., **179** (1993), 14–27.
[14] M. M. A. El-Sheikh, R. A. Sallam, D. Elimy, *Oscillation criteria for second order nonlinear equations with damping*, Adv. Differ. Equ. Control Process, **8** (2011), 127–142.
[15] M. M. A. El-Sheikh, R. A.Sallam, E. El-Saedy, *On the oscillatory behavior of solutions of second order nonlinear neutral delay differential equations*, Wseas Trans. Math., **17** (2018), 51–57.
[16] M. M. A. El-Sheikh, R.A. Sallam, S. Salem, *Oscillation of nonlinear third-order differential equations with several sublinear neutral terms*, Math. Solovaca, **71** (2021), 1411–1426. 1
[17] S. Frassu, C. van der Mee, G. Viglialoro, *Boundedness in a nonlinear attraction-repulsion Keller-Segel system with production and consumption*, J. Math. Anal. Appl., **504** (2021), 13 pages. 1
[18] C. Jiang, Y. Jiang, T. Li, *Asymptotic behavior of third-order differential equations with nonpositive neutral coefficients and distributed deviating arguments*, Adv. Differ. Equ., **2016** (2016), 14 pages. 1
[19] T. Li, B. Baculikova, J. Dzurina, C. Zhang, *Oscillation of fourth-order neutral differential equations with p-laplacian like operators*, Bound. value prob., **2014** (2014), 9 pages. 1, 1, 4
[20] T. Li, N. Pintus, G. Viglialoro, *Properties of solutions to porous medium problems with different sources and boundary conditions*, Z. Angew. Math. Phys., **70** (2019), 1–18. 1
[21] T. Li, Y. V. Rogovchenko, *Oscillation of second-order neutral differential equations*, Math. Nachr., **288** (2015), 1150–1162. 1, 6
[22] T. Li, Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett., **61** (2016), 35–41. 1
[23] T. Li, Y. V. Rogovchenko, *Oscillation criteria for second-order superlinear Emden-Fowler neutral differential equations*, Monatsh. Math., **184** (2017), 489–500. 1, 6
[24] T. Li, Y. V. Rogovchenko, *On asymptotic behavior of solutions to higher-order sublinear Emden-Fowler delay differential equations*, Appl. Math. Lett., **67** (2017), 53–59.
[25] T. Li, Y. V. Rogovchenko, *On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations*, Appl. Math. Lett., **105** (2020), 7 pages. 1, 6
[26] T. Li, G. Viglialoro, *Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime*, Differ. Integral Equ., **34** (2021), 315–336. 1
[27] T. Li, C. Zhang, E. Thandapani, *Asymptotic behavior of fourth-order neutral dynamic equations with noncanonical operators*, Taiwanese J. Math., **18** (2014), 1003–1019. 1, 6

[28] O. Moaaz, C. Cesarano, A. Muhib, *Some new oscillation results for fourth-order neutral differential equations*, European J. Appl. Math., **13** 2020, 185–199. 1, 2.2

[29] C. Philos, *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations will positive delays*, Arch. Math., **36** (1981), 168–178. 2.1

[30] G. Qin, C. Huang, Y. Xie, F. Wen, *Asymptotic behavior for third-order quasi-linear differential equations*, Adv. Differ. Equ., **2013** (2013), 8 pages.

[31] Y.-C. Qiu, I. Jadlovska, K.-S. Chiu, T. Li, *Existence of nonoscillatory solutions tending to zero of third-order neutral dynamic equations on time scales*, Adv. Differ. Equ., **2020** (2020), 9 pages.

[32] R. A. Sallam, M. M. A. El-Sheikh, E. I. El-Saedy, *On the oscillation of second order nonlinear neutral delay differential equations*, Math. Solovaca, **71** (2021), 859–870.

[33] R. A. Sallam, S. Salem, M. M. A. El-Sheikh, *Oscillation of solutions of third order nonlinear neutral differential equations*, Adv. Differ. Equ., **2020** (2020), 25 pages.

[34] E. Thandapani, M. M. A. El-Sheikh, R. Sallam, S. Salem, *On the oscillatory behavior of third order differential equations with a sublinear neutral term*, Math. Solovaca, **70** (2020), 95–106.

[35] Y. Tian, Y. Cail, Y. Fu, T. Li, *Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, Adv. Differ. Equ., **2015** (2015), 14 pages. 3

[36] A. Tiryaki, M. F. Aktas, *Oscillation criteria of a certain class of third order nonlinear delay differential equations with damping*, J. Math. Anal. Appl., **325** (2007), 54–68. 1

[37] G. Viglialoro, *Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source*, J. Math. Anal. Appl., **439** (2016), 197–212. 1

[38] H. Wang. G. Chen, Y. Jiang, C. Jiang, T. Li, *Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments*, J. Math. Computer Sci., **17** (2017), 194–199. 1

[39] C. Zhang, R. P. Agarwal, M. Bohner, T. Li, *Oscillation of fourth-order delay dynamic equations*, Sci. China Math., **58** (2015), 143–160. 1

[40] Q. Zhang, L. Gao, Y. Yu, *Oscillation criteria for third-order neutral differential equations with continuously distributed delay*, Appl. Math. Lett., **25** (2012), 1514–1519. 1