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Intuitionistic fuzzy Hilbert algebras

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Abstract

The notions of intuitionistic fuzzy subalgebras and intuitionistic fuzzy ideals of Hilbert algebras are introduced and studied in this work, as well as some of their properties. Under intuitionistic fuzzy ideals, we also investigate inverse images of homomorphisms. Finally, several equivalence relations on the class of all intuitionistic fuzzy ideals are examined.

Keywords: Hilbert algebra, intuitionistic fuzzy subalgebra, intuitionistic fuzzy ideal.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [21]. Many scholars have researched the theory of fuzzy sets and their several applications in real-life situations. After introducing the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 5, 8]. The idea of intuitionistic fuzzy sets, suggested by Atanassov [4] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making [13–15]. The concept of Hilbert algebra was introduced in the early 50-ties by Henkin and Skolem for some investigations of implications in intuitionistic and other non-classical logics. In the 60-ties, these algebras were studied, especially by Horn and Diego, from an algebraic point of view. Diego [10] proved that Hilbert algebras form a locally finite variety. Hilbert algebras were treated by Busneag [6, 7] and Jun [16] and some of their filters forming deductive systems were recognized. Dudek [11] considered the fuzzification of subalgebras and deductive systems in Hilbert algebras.

In this paper, we introduce and study the concepts of intuitionistic fuzzy subalgebras and intuitionistic fuzzy ideals of Hilbert algebras and investigate some of their properties. We also study inverse images of homomorphisms under intuitionistic fuzzy ideals. Finally, we study some equivalence relations on the class of all intuitionistic fuzzy ideals.

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2. Preliminaries

Definition 2.1 ([10]). A Hilbert algebra is a triplet $H = (H, \cdot, 1)$, where H is a nonempty set, "$\cdot$" is a binary operation on H, and "1" is the fixed element of H such that the following axioms hold:

1. $(\forall x, y \in H)(x \cdot (y \cdot x) = 1)$;
2. $(\forall x, y, z \in H)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$;
3. $(\forall x, y \in H)((x \cdot y = 1, y \cdot x = 1) \Rightarrow x = y)$.

The following result was proved in [11].

Lemma 2.2. *Let $H = (H, \cdot, 1)$ be a Hilbert algebra. Then,*

1. $(\forall x \in H)(x \cdot x = 1)$;
2. $(\forall x \in H)(1 \cdot x = x)$;
3. $(\forall x \in H)(x \cdot 1 = 1)$;
4. $(\forall x, y, z \in H)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

It is easily checked that in a Hilbert algebra H, the relation $\leqslant$ is defined by $x \leqslant y \Leftrightarrow x \cdot y = 1$ as a partial order on H with 1 is the largest element.

Definition 2.3 ([9]). A nonempty subset I of a Hilbert algebra $H = (H, \cdot, 1)$ is called an ideal of H if,

1. $1 \in I$;
2. $(\forall x \in H, \forall y \in I)(x \cdot y \in I)$;
3. $(\forall x \in H, \forall y_1, y_2 \in I)((y_2 \cdot (y_1 \cdot x)) \cdot x \in I)$.

A fuzzy set [21] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.4 ([12]). A fuzzy set μ in a Hilbert algebra H is said to be a fuzzy ideal of H if the following conditions hold:

1. $(\forall x \in H)(\mu(1) \geqslant \mu(x))$;
2. $(\forall x, y \in H)(\mu(x \cdot y) \geqslant \mu(y))$;
3. $(\forall x, y_1, y_2 \in H)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.5 ([4]). An intuitionistic fuzzy set on a nonempty set H is defined to be a structure

$$A := \{\langle x, \mu_A(x), \gamma_A(x) \mid x \in H\}, \tag{2.1}$$

where $\mu_A : H \to [0, 1]$ is the degree of membership of x and $\gamma_A : H \to [0, 1]$ is the degree of non-membership of x such that $0 \leqslant \mu_A(x) + \gamma_A(x) \leqslant 1$, and the intuitionistic fuzzy set in (2.1) is simply denoted by $A = (\mu_A, \gamma_A)$.

3. Intuitionistic fuzzy Hilbert algebras

Definition 3.1. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is called an intuitionistic fuzzy subalgebra of H if the following conditions hold:

$$(\forall x, y \in H)\left(\begin{array}{l} \mu_A(x \cdot y) \geqslant \min\{\mu_A(x), \mu_A(y)\} \\ \gamma_A(x \cdot y) \leqslant \max\{\gamma_A(x), \gamma_A(y)\} \end{array}\right).$$

Example 3.2. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then H is a Hilbert algebra. We define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as follows:

H	1	x	y	z	0
μ_A	1	0.8	0.8	0.7	0.4
γ_A	0.3	0.5	0.7	0.3	0.6

Then A is an intuitionistic fuzzy subalgebra of H.

Proposition 3.3. *Every intuitionistic fuzzy subalgebra $A = (\mu_A, \gamma_A)$ of a Hilbert algebra H satisfies*

$$\mu_A(1) \geqslant \mu_A(x), \quad \text{and} \quad \gamma_A(1) \leqslant \gamma_A(x),$$

for all $x \in H$.

Proof. For any $x \in H$, we have

$$\mu_A(1) = \mu_A(x \cdot x) \geqslant \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x),$$

and

$$\gamma_A(1) = \gamma_A(x \cdot x) \leqslant \max\{\gamma_A(x), \gamma_A(x)\} = \gamma_A(x).$$

□

Proposition 3.4. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra H into a Hilbert algebra Y and $A = (\mu_A, \gamma_A)$ an intuitionistic fuzzy subalgebra of Y. Then the inverse image $f^{-1}(A)$ of A is an intuitionistic fuzzy subalgebra of H.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} \mu_{f^{-1}(A)}(x \cdot y) &= \mu_A(f(x \cdot y)) \\ &= \mu_A(f(x) \cdot f(y)) \\ &\geqslant \min\{\mu_A(f(x)), \mu_A(f(y))\} \\ &= \min\{\mu_{f^{-1}(A)}(x), \mu_{f^{-1}(A)}(y)\}, \end{aligned}$$

and

$$\begin{aligned} \gamma_{f^{-1}(A)}(x \cdot y) &= \gamma_A(f(x \cdot y)) \\ &= \gamma_A(f(x) \cdot f(y)) \\ &\leqslant \max\{\gamma_A(f(x)), \gamma_A(f(y))\} \\ &= \max\{\gamma_{f^{-1}(A)}(x), \gamma_{f^{-1}(A)}(y)\}. \end{aligned}$$

Hence, $f^{-1}(A)$ is an intuitionistic fuzzy subalgebra of H. □

Definition 3.5. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to have the sup-inf property if for any subset $T \subset H$, there exists $t_0 \in T$ such that $\mu_A(t_0) = \sup_{t \in T} \mu_A(t)$ and $\gamma_A(t_0) = \inf_{t \in T} \mu_A(t)$.

From Example 3.2, we have A is an intuitionistic fuzzy subalgebra of H that has the sup-inf property.

Proposition 3.6. *Let* $f : X \to Y$ *be a homomorphism of a Hilbert algebra* H *onto a Hilbert algebra* Y *and let* $A = (\mu_A, \gamma_A)$ *be an intuitionistic fuzzy subalgebra of* H *that has the sup-inf property. Then the image* $f(A)$ *of* A *is an intuitionistic fuzzy subalgebra of* Y.

Proof. For $u, v \in Y$, let $x_0 \in f^{-1}(u)$, $y_0 \in f^{-1}(v)$ such that $\mu_A(x_0) = \sup_{t \in f^{-1}(u)} \mu_A(t)$, $\mu_A(y_0) = \sup_{t \in f^{-1}(v)} \mu_A(t)$, $\gamma_A(x_0) = \inf_{t \in f^{-1}(u)} \gamma_A(t)$ and $\gamma_A(y_0) = \inf_{t \in f^{-1}(v)} \gamma_A(t)$. Then, by the definition of $\mu_{f(A)}$, we have

$$\begin{aligned}
\mu_{f(A)}(u \cdot v) &= \sup_{t \in f^{-1}(u \cdot v)} \mu_A(t) \\
&\geqslant \mu_A(x_0 \cdot y_0) \\
&\geqslant \min\{\mu_A(x_0), \mu_A(y_0)\} \\
&= \min\{\sup_{t \in f^{-1}(u)} \mu_A(t), \sup_{t \in f^{-1}(v)} \mu_A(t)\} \\
&= \min\{\mu_{f(A)}(u), \mu_{f(A)}(v)\},
\end{aligned}$$

and

$$\begin{aligned}
\gamma_{f(A)}(u \cdot v) &= \inf_{t \in f^{-1}(u \cdot v)} \gamma_A(t) \\
&\leqslant \gamma_A(x_0 \cdot y_0) \\
&\leqslant \max\{\gamma_A(x_0), \gamma_A(y_0)\} \\
&= \max\{\inf_{t \in f^{-1}(u)} \gamma_A(t), \inf_{t \in f^{-1}(v)} \gamma_A(t)\} \\
&= \max\{\gamma_{f(A)}(u), \gamma_{f(A)}(v)\}.
\end{aligned}$$

Hence, $f(A)$ is an intuitionistic fuzzy subalgebra of Y. □

Definition 3.7. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a Hilbert algebra H is said to be an intuitionistic fuzzy ideal of H if the following conditions hold:

$$(\forall x \in H)\begin{pmatrix} \mu_A(1) \geqslant \mu_A(x) \\ \gamma_A(1) \leqslant \gamma_A(x) \end{pmatrix}, \tag{3.1}$$

$$(\forall x, y \in H)\begin{pmatrix} \mu_A(x \cdot y) \geqslant \mu_A(y) \\ \gamma_A(x \cdot y) \leqslant \gamma_A(y) \end{pmatrix}, \tag{3.2}$$

$$(\forall x, y_1, y_2 \in H)\begin{pmatrix} \mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\} \\ \gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \max\{\gamma_A(y_1), \gamma_A(y_2)\} \end{pmatrix}. \tag{3.3}$$

Example 3.8. Let $H = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then H is a Hilbert algebra. We define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as follows:

H	1	x	y	z	0
μ_A	1	0.8	0.8	0.7	0.4
γ_A	0.3	0.5	0.7	0.3	0.6

Then A is an intuitionistic fuzzy ideal of H that has the sup-inf property.

Proposition 3.9. *If $A = (\mu_A, \gamma_A)$ is intuitionistic fuzzy ideal of a Hilbert algebra H, then*

$$(\forall x, y \in H)\left(\begin{array}{c}\mu_A((y \cdot x) \cdot x) \geqslant \mu_A(y) \\ \gamma_A((y \cdot x) \cdot x) \leqslant \gamma_A(y)\end{array}\right). \tag{3.4}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (3.4), we have

$$\mu_A((y \cdot x) \cdot x) \geqslant \min\{\mu_A(y), \mu_A(1)\} = \mu_A(y),$$

and

$$\gamma_A((y \cdot x) \cdot x) \leqslant \max\{\gamma_A(y), \gamma_A(1)\} = \gamma_A(y).$$

□

Lemma 3.10. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H, then we have the following*

$$(\forall x, y \in H)\left(x \leqslant y \Rightarrow \left\{\begin{array}{c}\mu_A(x) \leqslant \mu_A(y) \\ \gamma_A(x) \geqslant \gamma_A(y)\end{array}\right.\right). \tag{3.5}$$

Proof. Let $x, y \in H$ be such that $x \leqslant y$. Then $x \cdot y = 1$ and so

$$\begin{aligned}\mu_A(y) &= \mu_A(1 \cdot y) \\ &= \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\geqslant \min\{\mu_A(x \cdot y), \mu_A(x)\} \\ &\geqslant \min\{\mu_A(1), \mu_A(x)\} \\ &= \mu_A(x),\end{aligned}$$

and

$$\begin{aligned}\gamma_A(y) &= \gamma_A(1 \cdot y) \\ &= \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\leqslant \max\{\gamma_A(x \cdot y), \gamma_A(x)\} \\ &\leqslant \max\{\gamma_A(1), \gamma_A(x)\} \\ &= \gamma_A(x).\end{aligned}$$

□

Theorem 3.11. *Every intuitionistic fuzzy ideal of H is an intuitionistic fuzzy subalgebra of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of H. Since $y \leqslant x \cdot y$ for all $x, y \in H$ and from Lemma 3.10, we have

$$\mu_A(y) \geqslant \mu_A(x \cdot y), \gamma_A(y) \leqslant \gamma_A(x \cdot y).$$

It follows from (3.2) that

$$\begin{aligned}\mu_A(x \cdot y) &\geqslant \mu_A(y) \\ &\geqslant \min\{\mu_A(x \cdot y), \mu_A(x)\} \\ &\geqslant \min\{\mu_A(x), \mu_A(y)\},\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x\cdot y) &\leqslant \gamma_A(y)\\ &\leqslant \max\{\gamma_A(x\cdot y),\gamma_A(x)\}\\ &\leqslant \max\{\gamma_A(x),\gamma_A(y)\}.\end{aligned}$$

Hence, A is an intuitionistic fuzzy subalgebra of H. □

Proposition 3.12. *If* $\{(\mu_{A_i},\gamma_{A_i}) : i\in\Delta\}$ *is a family of intuitionistic fuzzy ideals of a Hilbert algebra* H*, then* $\bigwedge_{i\in\Delta} A_i$ *is an intuitionistic fuzzy ideal of* H.

Proof. Let $\{(\mu_{A_i},\gamma_{A_i}) : i\in\Delta\}$ be a family of intuitionistic fuzzy ideals of a Hilbert algebra H. Let $x\in H$. Then

$$(\bigwedge_{i\in\Delta}\mu_{A_i})(1)=\inf_{i\in\Delta}\{\mu_{A_i}(1)\}\geqslant\inf_{i\in\Delta}\{\mu_{A_i}(x)\}=(\bigwedge_{i\in\Delta}\mu_{A_i})(x),$$

and

$$(\bigwedge_{i\in\Delta}\gamma_{A_i})(1)=\sup_{i\in\Delta}\{\gamma_{A_i}(1)\}\leqslant\sup_{i\in\Delta}\{\gamma_{A_i}(x)\}=(\bigwedge_{i\in\Delta}\gamma_{A_i})(x).$$

Let $x,y\in H$. Then

$$(\bigwedge_{i\in\Delta}\mu_{A_i})(x\cdot y)=\inf_{i\in\Delta}\{\mu_{A_i}(x\cdot y)\}\geqslant\inf_{i\in\Delta}\{\mu_{A_i}(y)\}=(\bigwedge_{i\in\Delta}\mu_{A_i})(y),$$

and

$$(\bigwedge_{i\in\Delta}\gamma_{A_i})(x\cdot y)=\sup_{i\in\Delta}\{\gamma_{A_i}(x\cdot y)\}\leqslant\sup_{i\in\Delta}\{\gamma_{A_i}(y)\}=(\bigwedge_{i\in\Delta}\gamma_{A_i})(y).$$

Let $x,y_1,y_2\in H$. Then

$$\begin{aligned}(\bigwedge_{i\in\Delta}\mu_{A_i})((y_1\cdot(y_2\cdot x))\cdot x) &= \inf_{i\in\Delta}\{\mu_{A_i}((y_1\cdot(y_2\cdot x))\cdot x)\}\\ &\geqslant \inf_{i\in\Delta}\{\min\{\mu_{A_i}(y_1),\mu_{A_i}(y_2)\}\}\\ &= \min\{\inf_{i\in\Delta}\mu_{A_i}(y_1),\inf_{i\in\Delta}\mu_{A_i}(y_2)\}\\ &= \min\{(\bigwedge_{i\in\Delta}\mu_{A_i})(y_1),(\bigwedge_{i\in\Delta}\mu_{A_i})(y_2)\},\end{aligned}$$

and

$$\begin{aligned}(\bigwedge_{i\in\Delta}\gamma_{A_i})((y_1\cdot(y_2\cdot x))\cdot x) &= \sup_{i\in\Delta}\{\gamma_{A_i}((y_1\cdot(y_2\cdot x))\cdot x)\}\\ &\leqslant \sup_{i\in\Delta}\{\max\{\gamma_{A_i}(y_1),\gamma_{A_i}(y_2)\}\}\\ &= \max\{\sup_{i\in\Delta}\gamma_{A_i}(y_1),\sup_{i\in\Delta}\gamma_{A_i}(y_2)\}\\ &= \max\{(\bigwedge_{i\in\Delta}\gamma_{A_i})(y_1),(\bigwedge_{i\in\Delta}\gamma_{A_i})(y_2)\}.\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} A_i$ is an intuitionistic fuzzy ideal of H. □

Definition 3.13 ([22])**.** An intuitionistic fuzzy set $A=(\mu_A,\gamma_A)$ in a Hilbert algebra H is said to be an *intuitionistic fuzzy deductive system* of H if the following conditions hold:

$$(\forall x\in H)\left(\begin{array}{c}\mu_A(1)\geqslant\mu_A(x)\\ \gamma_A(1)\leqslant\gamma_A(x)\end{array}\right),$$

$$(\forall x,y\in H)\left(\begin{array}{c}\mu_A(y)\geqslant\min\{\mu_A(x\cdot y),\mu_A(x)\}\\ \gamma_A(y)\leqslant\max\{\gamma_A(x\cdot y),\mu_A(x)\}\end{array}\right).$$

Proposition 3.14. *Every intuitionistic fuzzy ideal of a Hilbert algebra* H *is an intuitionistic fuzzy deductive system of* H.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of H. If $y_1 = x \cdot y$, $y_2 = x$, where $x, y \in H$, then by (1) and (2) of Lemma 2.2 and (3.3), we have

$$\mu_A(y) = \mu_A(1 \cdot y) = \mu_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \geqslant \min\{\mu_A(x \cdot y), \mu_A(x)\},$$

and

$$\gamma_A(y) = \gamma_A(1 \cdot y) = \gamma_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \leqslant \max\{\gamma_A(x \cdot y), \gamma_A(x)\}.$$

Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy deductive system of H. □

Lemma 3.15. *An intuitionistic fuzzy set* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal of* H *if and only if* μ_A *and* $\overline{\gamma}_A$ *are fuzzy ideals of* H.

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H. Then obviously μ_A is a fuzzy ideal of H. Consider for every $x, y \in H$ we have $\overline{\gamma}_A(1) = 1 - \gamma_A(1) \geqslant 1 - \gamma_A(x) = \overline{\gamma}_A(x)$. Also,

$$\begin{aligned}\overline{\gamma}_A(y) &= 1 - \gamma_A(y)\\ &\geqslant 1 - \max\{\gamma_A(x \cdot y), \gamma_A(x)\}\\ &= \min\{1 - \gamma_A(x \cdot y), 1 - \gamma_A(x)\}\\ &= \min\{\overline{\gamma}_A(x \cdot y), \overline{\gamma}_A(x)\}.\end{aligned}$$

Hence, $\overline{\gamma}_A$ is a fuzzy ideal of H.

Conversely, let us take μ_A and $\overline{\gamma}_A$ are fuzzy ideals of H. Then obviously for every $x \in H$, we have $\mu_A(1) \geqslant \mu_A(x)$, $1 - \gamma_A(1) = \overline{\gamma}_A(1) \geqslant \overline{\gamma}_A(x) = 1 - \gamma_A(x)$, that is, $\gamma_A(1) \leqslant \gamma_A(x)$. Moreover,

$$\mu_A(y) \geqslant \min\{\mu_A(x \cdot y), \mu_A(x)\},$$

and

$$\begin{aligned}1 - \gamma_A(y) &= \overline{\gamma}_A(y)\\ &\geqslant \min\{\overline{\gamma}_A(x \cdot y), \overline{\gamma}_A(x)\}\\ &= \min\{1 - \gamma_A(x \cdot y), 1 - \gamma_A(x)\}\\ &= 1 - \max\{\gamma_A(x \cdot y), \gamma_A(x)\}.\end{aligned}$$

Hence, $\gamma_A(y) \leqslant \max\{\gamma_A(x \cdot y), \gamma_A(x)\}$. Thus A is an intuitionistic fuzzy ideal of H. □

Theorem 3.16. *An intuitionistic fuzzy set* $A = (\mu_A, \gamma_A)$ *is an intuitionistic fuzzy ideal of* H *if and only if* $(\mu_A, \overline{\mu}_A)$ *and* $(\gamma_A, \overline{\gamma}_A)$ *are intuitionistic fuzzy ideals of* H.

Proof. If an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H, then $\mu_A = \overline{\overline{\mu}}_A$ and γ_A are intuitionistic fuzzy ideals of H from Lemma 3.15, hence $(\mu_A, \overline{\mu}_A)$ and $(\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy ideals of H.

Conversely, if $(\mu_A, \overline{\mu}_A)$ and $(\gamma_A, \overline{\gamma}_A)$ are intuitionistic fuzzy ideals of H, then μ_A and $\overline{\gamma}_A$ are ideals of H, hence the intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H. □

Theorem 3.17. *Let* A *be a nonempty subset of a Hilbert algebra* H *and* (μ_A, γ_A) *be an intuitionistic fuzzy set in* H *defined by for all* $x \in H$ *and* $\alpha_i, \beta_i \in [0, 1]$ *such that* $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$, *and* $\alpha_i + \beta_i \leqslant 1$ *for* $i = 0, 1$,

$$\mu_A(x) = \begin{cases}\alpha_0 & \text{if } x \in A,\\ \alpha_1 & \text{otherwise,}\end{cases} \quad \text{and} \quad \gamma_A(x) = \begin{cases}\beta_0 & \text{if } x \in A,\\ \beta_1 & \text{otherwise.}\end{cases}$$

Then (μ_A, γ_A) *is an intuitionistic fuzzy ideal of* H *and* $\mu_{\alpha_0} = A = \gamma_{\beta_0}$.

Proof. Assume that (μ_A, γ_A) is an intuitionistic fuzzy ideal of H. Since $\mu_A(1) \geqslant \mu_A(x)$ and $\gamma_A(1) \leqslant \gamma_A(x)$ for all $x \in H$, we have $\mu_A(1) = \alpha_1$ and $\gamma_A(1) = \beta_1$ and so $1 \in A$. Let $x \in H$ and $y \in A$. Then $\mu_A(x \cdot y) \geqslant \mu_A(y) = \alpha_1$ and then $\mu_A(x \cdot y) = \alpha_1$. Also $\gamma_A(x \cdot y) \leqslant \gamma_A(y) = \beta_1$ and then $\gamma_A(x \cdot y) = \beta_1$. Hence, $x \cdot y \in A$. For any $y_1, y_2 \in A$ and $x \in H$, we get $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\} = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \max\{\gamma_A(y_1), \gamma_A(y_2)\} = \beta_1$, which implies that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \alpha_1$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \beta_1$. It follows that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$. Therefore, A is an ideal of H.

Conversely, suppose that A is an ideal of H. Since $1 \in A$, it follows that $\mu_A(1) = \alpha_1 \geqslant \mu_A(x)$ for all $x \in H$. Let $x, y \in H$. If $y \in A$, then $x \cdot y \in A$ and so $\mu_A(x \cdot y) = \alpha_1 = \mu_A(y)$, $\gamma_A(x \cdot y) = \beta_1 = \gamma_A(y)$. If $y \in H \backslash A$, then $\mu_A(y) = \alpha_2$, $\gamma_A(y) = \beta_2$, and hence $\mu_A(x \cdot y) \geqslant \alpha_2 = \mu_A(y)$ and $\gamma_A(x \cdot y) \leqslant \beta_2 = \gamma_A(y)$. Finally, let $y_1, y_2 \in H$. If $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$. Then $\mu_A(y_1) = \alpha_2$ or $\mu_A(y_2) = \alpha_2$. It follows that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \alpha_2 = \min\{\mu_A(y_1), \mu_A(y_2)\}$. Also if $y_1 \in H \backslash A$ or $y_2 \in H \backslash A$, then $\gamma_A(y_1) = \beta_2$ or $\gamma_A(y_2) = \beta_2$. It follows that $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \beta_2 = \max\{\gamma_A(y_1), \gamma_A(y_2)\}$. Assume that $y_1, y_2 \in A$. Then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in A$ and thus $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \alpha_1 = \min\{\mu_A(y_1), \mu_A(y_2)\}$ and $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \beta_1 = \max\{\gamma_A(y_1), \gamma_A(y_2)\}$. Hence, (μ_A, γ_A) is an intuitionistic fuzzy ideal of H. □

Theorem 3.18. *If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of a Hilbert algebra H, then the sets $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ are ideals of H for every $\alpha \in \mathrm{Im}(f_A) \cap \mathrm{Im}(g_A) \cap [0, 0.5]$.*

Proof. Assume that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of a Hilbert algebra H and

$$\alpha \in \mathrm{Im}(f_A) \cap \mathrm{Im}(g_A) \cap [0, 0.5].$$

Let $x \in U(\mu_A, \alpha)$. Then $\mu_A(x) \geqslant \alpha$. Since A is an intuitionistic fuzzy ideal of H, $\mu_A(1) \geqslant \mu_A(x) \geqslant \alpha$. Hence, $1 \in U(\mu_A, \alpha)$. Let $x \in L(\gamma_A, \alpha)$. Then $\gamma_A(x) \leqslant \alpha$. Since A is an intuitionistic fuzzy ideal of H, $\gamma_A(1) \leqslant \gamma_A(x) \leqslant \alpha$. Hence, $1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y \in U(\mu_A, \alpha)$. Since A is an intuitionistic fuzzy ideal of H, $\mu_A(x \cdot y) \geqslant \mu_A(y) \geqslant \alpha$. Hence, $x \cdot y \in U(\mu_A, \alpha)$. Let $x_1 \in H$ and $y_1 \in L(\gamma_A, \alpha)$. Since A is an intuitionistic fuzzy ideal of H, $\gamma_A(x_1 \cdot y_1) \leqslant \gamma_A(y_1) \leqslant \alpha$. Hence, $x_1 \cdot y_1 \in L(\gamma_A, \alpha)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, \alpha)$. Then $\mu_A(y_1) \geqslant \alpha$ and $\mu_A(y_2) \geqslant \alpha$. Since A is an intuitionistic fuzzy ideal of H, $\mu_A(y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\} \geqslant \alpha$. Hence, $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, \alpha)$. Let $x' \in H$ and $y_1', y_2' \in L(\gamma_A, \alpha)$. Then $\gamma_A(y_1') \leqslant \alpha$ and $\gamma_A(y_2') \leqslant \alpha$. Since A is an intuitionistic fuzzy ideal of H, $\gamma_A((y_1' \cdot (y_2' \cdot x')) \cdot x') \leqslant \max\{\gamma_A(y_1'), \gamma_A(y_2')\} \leqslant \alpha$. Hence, $(y_1' \cdot (y_2' \cdot x')) \cdot x' \in L(\gamma_A, \alpha)$. Thus $U(\mu_A, \alpha)$ and $L(\gamma_A, \alpha)$ of H are ideals of H for every $\alpha \in \mathrm{Im}(f_A) \cap \mathrm{Im}(g_A) \cap [0, 0.5]$. □

Corollary 3.19. *Let χ_M be the characteristic function of an intuitionistic fuzzy ideal of H. Then the intuitionistic fuzzy set $\overline{M} = (\chi_M, \overline{\chi_M})$ is an intuitionistic fuzzy ideal of H.*

Theorem 3.20. *An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_A, t)$ and $L(g, s)$ are either empty or ideals of H.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of H and let $s, t \in [0, 1]$ be such that $U(\mu_A, t)$ and $L(\gamma_A, s)$ are nonempty sets of H. It is clear that $1 \in U(\mu_A, t) \cap L(\gamma_A, s)$ since $\mu_A(1) \geqslant t$ and $\gamma_A(1) \leqslant s$. Let $x \in H$ and $y \in U(\mu_A, t)$. Then $\mu_A(y) \geqslant t$. It follows that $\mu_A(x \cdot y) \geqslant \mu_A(y) \geqslant t$ so that $x \cdot y \in U(\mu_A, t)$. Let $x \in H$ and $y_1, y_2 \in U(\mu_A, t)$. Then $\mu_A(y_1) \geqslant t$ and $\mu_A(y_2) \geqslant t$. Hence,

$$\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_a(y_2)\} \geqslant t,$$

so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\mu_A, t)$. Hence, $U(\mu_A, t)$ is an ideal of H. Let $x \in H$ and $y \in L(\gamma_A, s)$. Then $\gamma_A(y) \leqslant s$. It follows that $\gamma_A(x \cdot y) \leqslant \gamma_A(y) \leqslant s$ so that $x \cdot y \in L(\gamma_A, s)$. Let $x \in H$ and $y_1, y_2 \in L(\gamma_A, s)$. Then $\gamma_A(y_1) \leqslant s$ and $\gamma_A(y_2) \leqslant s$. Hence, $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \max\{\gamma_A(y_1), \gamma_A(y_2)\} \leqslant s$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\gamma_A, s)$. Hence, $L(\gamma_A, s)$ is an ideal of H.

Assume now that every nonempty sets $U(\mu_A, t)$ and $L(\gamma_A, s)$ are ideals of H. If $\mu_A(1) \geqslant \mu_A(x)$ is not true for all $x \in H$, then there exists $x_0 \in H$ such that $\mu_A(1) < \mu_A(x_0)$. But in this case for

$$s = \frac{1}{2}(\mu_A(1) + \mu_A(x_0)).$$

Then $x_0 \in U(\mu_A, s)$, that is, $U(\mu_A, s) \neq \emptyset$. Since by the assumption, $U(\mu_A, s)$ is an ideal of H, then $\mu_A(1) \geqslant s$, which is impossible. Hence, $\mu_A(1) \geqslant \mu_A(x)$. If $\gamma_A(1) \leqslant \gamma_A(x)$ is not true, then there exists $y_0 \in H$ such that $\gamma_A(1) < \gamma_A(y_0)$. But in this case for $s_0 = \frac{1}{2}(\gamma_A(1) + \gamma_A(y_0))$. Then $y_0 \in L(\gamma_A, s_0)$, that is, $L(\gamma_A, s_0) \neq \emptyset$. Since by the assumption, $L(\gamma_A, s_0)$ is an ideal of H, then $\gamma_A(1) \leqslant s_0$, which is impossible. Hence, $\gamma_A(1) \leqslant \gamma_A(x)$. If $\mu_A(x \cdot y) \geqslant \mu_A(y)$ is not true for all $x, y \in H$, then there exists $x_0, y_0 \in H$ such that $\mu_A(x_0 \cdot y_0) < \mu_A(y_0)$. Let $t = \frac{1}{2}(\mu_A(x_0 \cdot y_0) + \mu_A(y_0))$. Then $t \in [0, 1]$ and $\mu_A(x_0 \cdot y_0) < t < \mu_A(y_0)$, which prove that $y_0 \in U(\mu_A, t)$. Since $U(\mu_A, t)$ is an ideal of H, $x_0 \cdot y_0 \in U(\mu_A, t)$. Hence, $\mu_A(x_0 \cdot y_0) \geqslant t$, a contradiction. Thus $\mu_A(x \cdot y) \geqslant \mu_A(y)$ is true for all $x, y \in H$. If $\gamma_A(x \cdot y) \leqslant \gamma_A(y)$ is not true for all $x, y \in H$, then there exists $x_0, y_0 \in H$ such that $\gamma_A(x_0 \cdot y_0) > \gamma_A(y_0)$. Let $t_0 = \frac{1}{2}(\gamma_A(x_0 \cdot y_0) + \gamma_A(y_0))$. Then $t_0 \in [0, 1]$ and $\gamma_A(x_0 \cdot y_0) > t > \gamma_A(y_0)$, which prove that $y_0 \in L(\gamma_A, t_0)$. Since $L(\gamma_A, t_0)$ is an ideal of H, $x_0 \cdot y_0 \in L(\gamma_A, t_0)$. Hence, $\gamma_A(x_0 \cdot y_0) \leqslant t_0$, a contradiction. Thus $\gamma_A(x \cdot y) \leqslant \gamma_A(y)$ is true for all $x, y \in H$. Suppose that $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ such that $\mu_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) < \min\{\mu_A(u_0), \mu_A(v_0)\}$. Taking $p = \frac{1}{2}(\mu_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) + \min\{\mu_A(u_0), \mu_A(v_0)\})$. Then we have

$$\mu_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) < p < \min\{\mu_A(u_0), \mu_A(v_0)\},$$

which prove that $u_0, v_0 \in U(\mu_A, p)$. Since $U(\mu_A, p)$ is an ideal of H, $(u_0 \cdot (v_0 \cdot x_0))x_0 \in U(\mu_A, p)$, a contradiction. Thus $\mu_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \geqslant \min\{\mu_A(y_1), \mu_A(y_2)\}$ is true for all $x, y_1, y_2 \in H$. Suppose that $\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \max\{\gamma_A(y_1), \gamma_A(y_2)\}$ is not true for all $x, y_1, y_2 \in H$. Then there exist $u_0, v_0, x_0 \in H$ such that $\gamma_A((u_0 \cdot (v_0 \cdot x_0) \cdot x_0) > \max\{\gamma_A(u_0), \gamma_A(v_0)\}$. Taking

$$p_0 = \frac{1}{2}(\gamma_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) + \max\{\gamma_A(u_0), \gamma_A(v_0)\}).$$

Then we have $\gamma_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) > p_0 > \max\{\gamma_A(u_0), \gamma_A(v_0)\}$, which prove that $u_0, v_0 \in L(\gamma_A, p_0)$. Since $L(\gamma_A, p_0)$ is an ideal of H, $(u_0 \cdot (v_0 \cdot x_0)) \cdot x_0 \in L(\gamma_A, p_0)$, a contradiction. Thus

$$\gamma_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \leqslant \max\{\gamma_A(y_1), \gamma_A(y_2)\},$$

is true for all $x, y_1, y_2 \in H$. Hence, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of H. □

Theorem 3.21. *Let* $\{I_t : t \in \Delta \subset [0, 1]\}$ *be a collection of ideals of* H *such that* $H = \bigcup_{t \in \Delta} I_t$ *and for all* $s, t \in \Delta$, $s > t$ *if and only if* $I_s \subset I_t$. *Then an intuitionistic fuzzy* $A = (\mu_A, \gamma_A)$ *in* H *is defined by* $\mu_A(x) = \sup\{t \in \Delta : x \in I_t\}$ *and* $\gamma_A(x) = \inf\{t \in \Delta : x \in I_t\}$ *for all* $x \in X$ *as an intuitionistic fuzzy ideal of* H.

Proof. According to Theorem 3.20, it is sufficient to show that the nonempty sets $U(\mu_A, t)$ and $L(\gamma_A, t)$ are ideals of H. In order to prove that $U(\mu_A, t)$ is an ideal of H, we divide the proof into the following two cases:

(1) $t = \sup\{q \in \Delta : q < t\}$;

(2) $t \neq \sup\{q \in \Delta : q < t\}$.

The case (1) implies that $x \in U(\mu_A, t) \Leftrightarrow x \in I_q, \forall q < t \Leftrightarrow x \in \bigcap_{q<t} I_q$, so that $U(\mu_A, t) = \bigcap_{q<t} I_q$, which is an ideal of H. For the case (2), we claim that $U(\mu_A, t) = \bigcup_{q \geqslant t} I_q$. If $x \in \bigcup_{q \geqslant t} I_q$, then $x \in I_q$ for some $q \geqslant t$. It follows that $\mu_A(x) \geqslant q \geqslant t$, so that $x \in U(\mu_A, t)$. This shows that $\bigcup_{q \geqslant t} I_q \subset U(\mu_A, t)$. Now, assume that $x \notin \bigcup_{q \geqslant t} I_q$. Then $x \notin I_q$ for all $q \geqslant t$. Since $t = \sup\{q \in \Delta : q < t\}$, there exists $\epsilon > 0$ such that $(t - \epsilon, t) \cap \Delta = \emptyset$. Hence, $x \notin I_q$ for all $q > t - \epsilon$, which means that $x \in I_q$, then $q \leqslant t - \epsilon$. Thus $\mu_A(x) \leqslant t - \epsilon < t$ and so $x \notin U(\mu_A, t)$. Therefore, $U(\mu_A, t) \subset \bigcup_{q \geqslant t} I_q$ and thus $U(\mu_A, t) = \bigcup_{q \geqslant t} I_q$, which is an ideal of H. Next we prove that $L(\gamma_A, t)$ is an ideal of H. We consider the following two cases:

(3) $s = \inf\{r \in \Delta : s < r\}$;

(4) $s \neq \inf\{r \in \Delta : s < r\}$.

For the case (3), we have $x \in L(\gamma_A, s) \Leftrightarrow x \in I_r, \forall s < r \Leftrightarrow x \in \bigcap_{s<r} I_r$ and hence $L(\gamma_A, s) = \bigcap_{s<r} I_r$, which is an ideal of H. For the case (4), there exists $\epsilon > 0$ such that $(s, s+\epsilon) \cap \Delta = \emptyset$. We will show that $L(\gamma_A, s) = \bigcup_{s\geqslant r} I_r$. If $x \in \bigcup_{s\geqslant r} I_r$, then $x \in I_r$ for some $r \leqslant s$. It follows that $\gamma_A(x) \leqslant r \leqslant s$, so that $x \in L(\gamma_A, s)$. Hence, $\bigcup_{s\geqslant r} I_r \subset L(\gamma_A, s)$. Conversely, if $x \notin \bigcup_{s\geqslant r} I_r$, then $x \notin I_r$ for all $r \leqslant s$, which implies that $x \notin I_r$ for all $r < s + \epsilon$, that is, if $x \in I_r$, then $r \geqslant s + \epsilon$. Thus $\gamma_A(x) \geqslant s + \epsilon > s$, that is, $x \notin L(\gamma_A, s)$. Therefore, $L(\gamma_A, s) \subset \bigcup_{s\geqslant r} I_r$ and consequently $L(\gamma_A, s) = \bigcup_{s\geqslant r} I_r$, which is an ideal of H. □

A mapping $f : X \to Y$ of Hilbert algebras is called a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras. For any intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in Y, we define a new intuitionistic fuzzy set $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ in X by

$$\mu_{f^{-1}(A)}(x) = \mu_A(f(x)),\ \gamma_{f^{-1}(A)}(x) = \gamma_A(f(x)), \quad \forall x \in X.$$

Theorem 3.22. *Let $f : X \to Y$ be a homomorphism of Hilbert algebras and $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in Y. If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy ideal of Y, then $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ is an intuitionistic fuzzy ideal of X.*

Proof. Since f is a homomorphism of X into Y, then $f(1) = 1 \in Y$ and, by the assumption,

$$\mu_A(f(1)) = \mu_A(1) \geqslant \mu_A(y),$$

for every $y \in Y$. In particular, $\mu_B(f(1)) \geqslant \mu_B(f(x))$ for $x \in X$. Hence,

$$\mu_{f^{-1}(A)}(1) \geqslant \mu_{f^{-1}(A)}(x).$$

Also $\gamma_A(f(1)) = \gamma_A(1) \leqslant \gamma_A(y)$ for every $y \in Y$. In particular, $\gamma_B(f(1)) \leqslant \gamma_B(f(x))$ for $x \in X$. Hence, $\gamma_{f^{-1}(A)}(1) \leqslant \gamma_{f^{-1}(A)}(x)$, which proves (3.1). Now, let $x, y \in X$. Then, by the assumption,

$$\mu_{f^{-1}(A)}(x \cdot y) = \mu_A(f(x \cdot y)) = \mu_A(f(x) \cdot f(y) \geqslant \mu_A(f(y)) = \mu_{f^{-1}(A)}(y),$$

and

$$\gamma_{f^{-1}(A)}(x \cdot y) = \gamma_A(f(x \cdot y)) = \gamma_A(f(x) \cdot f(y) \leqslant \gamma_A(f(y)) = \gamma_{f^{-1}(A)}(y).$$

For any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then

$$\mu_A(x) = \mu_A(f(a)) = f(\mu_A)(a) \geqslant f(\mu_A)(1) = \mu_A(f(1)) = \mu_A(1),$$

and

$$\gamma_A(x) = \gamma_A(f(a)) = g(\gamma_A)(a) \leqslant g(\gamma_A)(1) = \gamma_A(f(1)) = \gamma_A(1),$$

which proves (3.2). Let $x, y_1, y_2 \in X$. Then by assumption,

$$\begin{aligned}
\mu_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x)) &= \mu_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= \mu_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= \mu_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= \mu_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&\geqslant \min\{\mu_A(f(y_1)), \mu_A(f(y_2))\} \\
&= \min\{\mu_{f^{-1}(A)}(y_1), \mu_{f^{-1}(A)}(y_2)\},
\end{aligned}$$

and

$$\begin{aligned}\gamma_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x)) &= \gamma_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\ &= \gamma_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\ &= \gamma_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\ &= \gamma_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &\leqslant \max\{\gamma_A(f(y_1)), \gamma_A(f(y_2))\} \\ &= \max\{\gamma_{f^{-1}(A)}(y_1), \gamma_{f^{-1}(A)}(y_2)\},\end{aligned}$$

which proves (3.3). Hence, $f^{-1}(A) = (\mu_{f^{-1}(A)}, \gamma_{f^{-1}(A)})$ is an intuitionistic fuzzy ideal of X. □

4. Equivalence relations on intuitionistic fuzzy ideals of Hilbert algebras

Let $\mathscr{I}(H)$ be the family of all intuitionistic fuzzy ideals of a Hilbert algebra H and let $t \in [0,1]$. Define binary relations U^t and L^t on $\mathscr{I}(H)$ as follows:

$$(A, B) \in U^t \;\Leftrightarrow\; U(\mu_A, t) = U(\mu_B, t),$$

and

$$(A, B) \in L^t \;\Leftrightarrow\; L(\gamma_A, t) = L(\gamma_B, t),$$

respectively, for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ in $\mathscr{I}(H)$. Then clearly U^t and L^t are equivalence relations on $\mathscr{I}(H)$. For any $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$, let $[A]_{U^t}$ (resp., $[A]_{L^t}$) denote the equivalence class of A modulo U^t (resp., L^t), and denote by $\mathscr{I}(H)/U^t$ (resp., $\mathscr{I}(H)/L^t$) the system of all equivalence classes modulo U^t (resp., L^t), so

$$\mathscr{I}(H)/U^t := \{[A]_{U^t} : A = (\mu_A, \gamma_A) \in \mathscr{I}(H)\},$$

and

$$\mathscr{I}(H)/L^t = \{[A]_{L^t} : A = (\mu_A, \gamma_A) \in \mathscr{I}(H)\},$$

respectively. Now, let I(H) denote the family of all ideals of H and let $t \in [0,1]$. Define maps f_t and g_t from $\mathscr{I}(H)$ to $I(H) \cup \{\emptyset\}$ by $f_t(A) = U(\mu_A, t)$ and $g_t(A) = L(\gamma_A, t)$, respectively, for all $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$. Then f_t and g_t are clearly well defined.

Theorem 4.1. *For any $t \in (0,1)$, the maps f_t and g_t are surjective from $\mathscr{I}(H)$ to $I(H) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$. Note that $\mathbf{\overline{0}} = (\mathbf{\overline{0}}, \mathbf{\overline{1}})$ is in $\mathscr{I}(H)$, where $\mathbf{\overline{0}}$ and $\mathbf{\overline{1}}$ are fuzzy sets in H defined by $\mathbf{\overline{0}}(x) = 0$ and $\mathbf{\overline{1}}(x) = 1$ for all $x \in H$. Obviously $f_t(\mathbf{\overline{0}}) = U(\mathbf{\overline{0}}, t) = \emptyset = L(\mathbf{\overline{0}}, t) = g_t(\mathbf{\overline{0}})$. Let $G(\neq \emptyset) \in I(H)$. For $\overline{G} = (\chi_G, \overline{\chi_G}) \in \mathscr{I}(H)$, we have $f_t(\overline{G}) = U(\chi_G, t) = G$ and $g_t(\overline{G}) = L(\overline{\chi_G}; t) = G$. Hence, f_t and g_t are surjective. □

Theorem 4.2. *The quotient sets $\mathscr{I}(H)/U^t$ and $\mathscr{I}(H)/L^t$ are equipotent to $I(H) \cup \{\emptyset\}$ for every $t \in (0,1)$.*

Proof. For $t \in (0,1)$, let f_t^* (resp., g_t^*) be a map from $\mathscr{I}(H)/U^t$ (resp., $\mathscr{I}(H)/L^t$) to $I(H) \cup \{\emptyset\}$ defined by $f_t^*([A]_{U^t}) = f_t(A)$ (resp., $g_t^*([A]_{L^t}) = g_t(A)$) for all $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$. If $U(\mu_A, t) = U(\mu_B, t)$ and $L(\gamma_A, t) = L(\gamma_B, t)$ for $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B) \in \mathscr{I}(H)$, then $(A, B) \in U^t$ and $(A, B) \in L^t$, hence $[A]_{U^t} = [B]_{U^t}$ and $[A]_{L^t} = [B]_{L^t}$. Therefore, the maps f_t^* and g_t^* are injective. Now, let $G(\neq \emptyset) \in I(H)$. For $\overline{G} = (\chi_G, \overline{\chi_G}) \in \mathscr{I}(H)$, we have

$$f_t^*([\overline{G}]_{U^t} = f_t(\overline{G}) = U(\chi_G, t) = G,$$

and

$$g_t^*([\overline{G}]_{L^t} = g_t(\overline{G}) = L(\overline{\chi_G}, t) = G.$$

Finally, for $\mathbf{\overline{0}} = (\mathbf{\overline{0}}, \mathbf{\overline{1}}) \in \mathscr{I}(H)$, we get

$$f_t^*([\mathbf{\overline{0}}]_{U^t} = f_t(\mathbf{\overline{0}}) = U(\mathbf{\overline{0}}, t) = \emptyset,$$

and

$$g_t^*([\overline{\mathbf{0}}]_{L^t} = g_t(\overline{\mathbf{0}}) = L(\overline{\mathbf{0}}, t) = \emptyset.$$

This shows that f_t^* and g_t^* are surjective. This completes the proof. □

For any $t \in [0,1]$, we define another relation R^t on $\mathscr{I}(H)$ as follows:

$$(A, B) \in R^t \Leftrightarrow U(\mu_A, t) \cap L(\gamma_A, t) = U(\mu_B, t) \cap L(\gamma_B, t),$$

for any $A = (\mu_A, \gamma_A), B = (\mu_B, \gamma_B) \in \mathscr{I}(H)$. Then the relation R^t is also an equivalence relation on $\mathscr{I}(H)$.

Theorem 4.3. *For any $t \in (0,1)$, the map $\varphi_t : \mathscr{I}(H) \to I(H) \cup \{\emptyset\}$ is defined by $\varphi_t(A) = f_t(A) \cap g_t(A)$ for each $A = (\mu_A, \gamma_A) \in \mathscr{I}(H)$ as surjective.*

Proof. Let $t \in (0,1)$. For $\mathbf{0} = (\overline{\mathbf{0}}, \overline{\mathbf{1}}) \in \mathscr{I}(H)$,

$$\varphi_t(\overline{\mathbf{0}}) = f_t(\overline{\mathbf{0}}) \cap g_t(\overline{\mathbf{0}}) = U(\overline{\mathbf{0}}, t) \cap L(\overline{\mathbf{0}}, t) = \emptyset.$$

For any $H \in \mathscr{I}(H)$, there exists $\overline{H} = (\chi_H, \overline{\chi_H}) \in \mathscr{I}(H)$ such that

$$\varphi_t(\overline{H}) = f_t(\overline{H}) \cap g_t(\overline{H}) = U(\chi_H, t) \cap L(\overline{\chi_H}, t) = H.$$

This completes the proof. □

Theorem 4.4. *For any $t \in (0,1)$, the quotient set $\mathscr{I}(H)/R^t$ is equipotent to $I(H) \cup \{\emptyset\}$.*

Proof. Let $t \in (0,1)$ and let $\varphi_t^* : \mathscr{I}(H)/R^t \to I(H) \cup \{\emptyset\}$ be a map defined by $\varphi_t^*([A]_{R^t}) = \varphi_t(A)$ for all $[A]_{R^t} \in \mathscr{I}(H)/R^t$. If $\varphi_t^*([A]_{R^t}) = \varphi_t^*([B]_{R^t})$ for any $[A]_{R^t}, [B]_{R^t} \in \mathscr{I}(H)/R^t$, then

$$f_t(A) \cap g_t(A) = f_t(B) \cap g_t(B),$$

that is, $U(\mu_A, t) \cap L(\gamma_A, t) = U(\mu_B, t) \cap L(\gamma_B, t)$, hence $(A, B) \in R^t$. It follows that $[A]_{R^t} = [B]_{R^t}$ so that φ_t^* is injective. For $\mathbf{0} = (\overline{\mathbf{0}}, \overline{\mathbf{1}}) \in \mathscr{I}(H)$,

$$\varphi_t^*([\overline{\mathbf{0}}]_{R^t}) = \varphi_t(\overline{\mathbf{0}}) = f_t(\overline{\mathbf{0}}) \cap g_t(\overline{\mathbf{0}}) = U(\overline{\mathbf{0}}, t) \cap L(\overline{\mathbf{1}}, t) = \emptyset.$$

If $H \in \mathscr{I}(H)$, then for $\overline{H} = (\chi_H, \overline{\chi_H}) \in \mathscr{I}(H)$, we have

$$\varphi_t^*([\overline{H}]_{R^t}) = \varphi_t(\overline{H}) = f_t(\overline{H}) \cap g_t(\overline{H}) = U(\chi_H, t) \cap L(\overline{\chi_H}, t) = H.$$

Hence, φ_t^* is surjective, this completes the proof. □

5. Conclusions and future works

We have introduced and studied the concepts of intuitionistic fuzzy subalgebras and intuitionistic fuzzy ideals in Hilbert algebras and investigated some of their properties. We also studied inverse images of homomorphisms under intuitionistic fuzzy ideals. Finally, we have defined and studied some equivalence relations on the class of all intuitionistic fuzzy ideals.

The research topics of interest by our research team being studied in Hilbert algebras are as follows:

(1) to study int-soft ideals over the soft sets in Hilbert algebras based on the concept of Muhiuddin and Mahboob [18];

(2) to study $\mathcal{N}$-ideals theory in Hilbert algebras based on $\mathcal{N}$-structures using the concept of Muhiuddin et al. [2, 17];

(3) to introduce the concept of bipolar (λ, δ)-fuzzy subalgebras and bipolar (λ, δ)-fuzzy ideals based on the concept of Ansari et al. [3, 20].

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