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Approximation approaches for rough hypersoft sets based on hesitant bipolar-valued fuzzy hypersoft relations on semigroups

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Abstract

In the hybrid context of hesitant bipolar-valued fuzzy hypersoft relations, the modern notion of extended roughness is constructed to rough approximations of hypersoft sets and fuzzy sets based on such context in this research. Then, corresponding examples are proposed, and further verified in connections between the hesitant bipolar-valued fuzzy hypersoft relations and the upper (resp., lower) rough approximations of hypersoft sets and fuzzy sets. Specifically, relationships are shown between the non-rough hypersoft sets (resp., non-rough fuzzy sets) and hesitant bipolar-valued fuzzy hypersoft reflexive relations together with hesitant bipolar-valued fuzzy hypersoft antisymmetric relations. To find the optimal multi-parameter of a hypersoft set such that the best choice exists, the notion of the set-valued measurement issues and decision-making algorithm for such objective is developed in the terms of rough set theory. Associated with the aforementioned accomplishments, the notion of novel models has been used to semigroups. Subsequently, the argumentation within relationships concerning the upper (resp., lower) rough approximations of hypersoft quasi-ideals and fuzzy quasi-ideals are proved under hypersoft homomorphism problems.

Keywords: Rough set, rough hypersoft set, rough fuzzy set, hypersoft quasi-ideal over semigroup, fuzzy quasi-ideal of semigroup, hypersoft homomorphism, hesitant bipolar-valued fuzzy hypersoft relation, decision-making method.

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1. Introduction

A philosophical standpoint of vagueness is reflected in the notion of set theory as discussed in [9, 40] by computer scientists and mathematicians. Then, discovered that the meaning of vagueness is considered as the property of sets and general sense reasoning based on natural language. Furthermore, vagueness may be camouflaged in a decision-making problem for computer science, machine learning, artificial intelligence. In the study of vagueness in classical set theory, a detailed study on properties of rough set theory can be found. The logical implication of rough set theory was originally introduced by Pawlak [39] in 1982. The notion of rough (inexact) and definable (exact) sets was introduced in approximation

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spaces based on equivalence relations, where upper and lower approximations are two crisp (precise) sets (or two basic operations in approximation spaces) depending on vague (imprecise) data. This theory can be characterized as a mathematical model in the following.

Given a non-empty universe V and an equivalence relation E on V, (V, E) is denoted as a Pawlak's approximation space, and $[v]_E$ is denoted as an equivalence class of $v \in V$ induced by E. In the following, let (V, E) be a given Pawlak's approximation space and let X be a subset of V. Upon a collection of all equivalence classes generated by all elements in V, Pawlak suggests an approximation model as follows:

$$\lceil X \rceil_E := \bigcup_{v \in V} \{[v]_E : [v]_E \cap X \neq \emptyset\}$$

is said to be an upper approximation of X within (V, E). The set

$$\lfloor X \rfloor_E := \bigcup_{v \in V} \{[v]_E : [v]_E \subseteq X\}$$

is said to be a lower approximation of X within (V, E). A difference $\lceil X \rceil_E - \lfloor X \rfloor_E$ is said to be a boundary region of X within (V, E). As introduced above, such sets are obtained the following interpretation.

(i) $\lceil X \rceil_E$ is a set of all elements, which can be possibly classified as X using E (are possibly X in view of E). In this way, a complement of $\lceil X \rceil_E$ is said to be a negative region of X within (V, E).

(ii) $\lfloor X \rfloor_E$ is a set of all elements, which can be certain classified as X using E (are certainly X in view of E). In this way, such a set is said to be a positive region of X within (V, E).

(iii) $\lceil X \rceil_E - \lfloor X \rfloor_E$ is a set of all elements, which can be classified neither as X nor as non-X using E.

In what follows, a pair $(\lceil X \rceil_E, \lfloor X \rfloor_E)$ is said to be a rough (or an inexact) set of X within (V, E) if $\lceil X \rceil_E - \lfloor X \rfloor_E$ is a non-empty set. In this way, X is said to be a rough set. X is said to be a definable (or an exact) set within (V, E) if $\lceil X \rceil_E - \lfloor X \rfloor_E$ is an empty set.

As mentioned above, observe that if the boundary region of a set is empty it means that the set is crisp. In the opposite case, the set is rough. Besides, if the boundary region of a set is non-empty it means that our information (or knowledge) about the set is not satisfactory to define the set exactly.

In Pawlak's approximation spaces, rough set theory is developed to expand notions, namely, rough fuzzy sets and rough soft sets. In 1965, fuzzy set theory was introduced by Zadeh [53]. Because of its wide applicability and also due to natural theoretical interest there had been many kinds of research on fuzzy set theory. In a fuzzy context, the notion of the roughness of fuzzy sets was proposed by Dubois and Prade [15] in 1990. A detailed study on upper and lower approximations of a fuzzy membership function can be found. In 1999, soft set theory was introduced by Russian researcher Molodtsov [31]. This theory has been applied to many different fields with great success. Especially, it is used in decision-making problems. Under the combination of rough set theory and soft set theory, the roughness of soft sets was introduced by Feng et al. [17] in 2010. In this concept, upper and lower approximations of a set of approximate elements (or alternative objects) of a soft set are studied. From the concept under decision-making problems in sense of soft set theory, the optimal parameter has one element for the best alternative. To find multi-parameter such that the best choice exists, the concept of hypersoft sets is one of many powerful tools for this finding. Such a concept is referred to as a generalization of soft sets. This generalized notion was proposed by Smarandache [46] in 2018. Moreover, many fundamental operations on hypersoft sets are introduced by Abbas et al. [1] in 2020. In particular, the notion of roughness for hypersoft sets with applications was proposed by Rahman et al. [45] in recent years. This approach is based on Pawlak's approximation spaces. Hypersoft sets are constantly researched and the results are interesting as can be seen in [8, 12, 33–35, 37].

Based on the above-mentioned study with respect to Pawlak's rough set theory, two approximation operations belong to approximation spaces based on equivalence relations. The popular extensions of this, such as arbitrary binary relations-based approximation spaces, fuzzy binary relations-based approximation spaces, soft binary relations-based approximation spaces, and fuzzy soft binary relations-based approximation spaces, are well-known references to deal with roughness problems. Definitions and results can be found in (see, e.g., [11, 21, 29, 30, 43, 44, 52]).

In extended roughness works, the concept of fuzzy binary relations (or fuzzy relations) is taken in almost all the literature to date. This concept was introduced by Zadeh [54] in 1971. It is defined as the generalization of a crisp set. Several researchers pointed out that different extensions of fuzzy relations have been carried out according to three different situations:

(i) In the type of asymmetric bipolarity, Zhang [57] introduced the notion of bipolar fuzzy sets in 1994. There are well-known references to deal with bipolar information (see, e.g., [18, 36, 59]). A bipolar fuzzy set is a pair of mappings, namely, a positive membership function and negative membership function. The positive membership degree of an element is in $+\mathrm{I} := [0, 1]$, the negative membership degree of an element is in $-\mathrm{I} := [-1, 0]$. In 2019, the notion of bipolar fuzzy relations was proposed by Lee and Hur [23] in terms of bipolar fuzzy sets, which is an extended concept of fuzzy relations under fuzzy logic.

(ii) Molodtsov's soft set theory successfully applied the soft theory in several directions. In 2001, Maji et al. [25] proposed the notion of fuzzy soft sets by embedding the ideas of fuzzy sets in terms of soft sets. Sometimes, the fuzzy soft set is referred to as a generalization of fuzzy sets. In recent years, Mattam and Gopalan [29] presented the concept of fuzzy soft binary relations (or fuzzy soft relations) in terms of fuzzy soft sets, which is used for approximations in the sense of rough set theory. The importance of the fuzzy soft relation and fuzzy soft set can be addressed into many tasks where a higher order of uncertainty is relevant, such as those in image processing [30].

(iii) As an extension of fuzzy set theory, in 2010, Torra [48] proposed the notion of hesitant fuzzy sets in which the membership degree of a given element is defined as a set of possible values in $+\mathrm{I}$. In 2014, Deepak and John [14] introduced the concept of hesitant fuzzy relations in terms of hesitant fuzzy sets. This is a form of an extended concept of fuzzy relations under the context of set-valued functions.

In 2019, the notion of hesitant bipolar-valued fuzzy sets was presented as the combination of bipolar-valued fuzzy sets and hesitant fuzzy sets, which is used in multi-attribute group decision-making. This special case was introduced by Mandal and Ranadive [27]. In recent years, Wang et al. [49] introduced the notion of hesitant bipolar-valued fuzzy soft sets. This theory is the development of hesitant bipolar-valued fuzzy sets, which further improve the accuracy of decision-making.

In growth, rough set theory can solve uncertainty problems in information and algebraic systems. Definitions and results can be found in, see, e.g., [4–7, 21, 22, 24, 38, 40–44, 50, 51, 55, 56, 58]. In particular, the notion of quasi-ideal of semigroups, introduced by Steinfeld [47] in 1956, was considered under rough set theory depending on preorder and compatible relations. In other words, the upper and lower rough approximations of quasi-ideals of semigroups were verified in crisp approximation spaces. This result was studied by Prasertpong and Buada [42]. Besides, in a fuzzy context, the quasi-ideal of semigroups is advantageous to develop characterizations in terms of fuzzy subsets of semigroups. This concept was proposed by Julatha and Siripitukdet [20] in 2017. In this study, we observe that many results in semigroups can be used to algebraic automata theory for applications related to machine learning.

In this paper, we focus on the notion of rough hypersoft sets and the concept of rough fuzzy sets in extended approximation spaces. First, we extend the concept of fuzzy relations, injecting the concept of hesitant bipolar-valued fuzzy soft sets and hypersoft sets. That is, we proposed hesitant bipolar-valued fuzzy hypersoft relations. We present in full detail how this relation can be further used for building extended approximation spaces, upper (resp., lower) approximations, and we also demonstrate that the proposed models exist rough hypersoft sets and rough fuzzy sets. To find the optimal multi-parameter of a hypersoft set such that the best choice exists, the notion of the set-valued measurement issues and decision-making algorithm for such objective is developed in the context of rough set theory. Second, we further study upper (resp., lower) approximations of hypersoft quasi-ideals over semigroups (resp., fuzzy quasi-ideals of semigroups) in approximation spaces under semigroups.

The remainder of this paper is organized as follows. In Section 2, we shall recapitulate some of the earlier definitions and results for the background of the current work. In Section 3, the contributions of the section are as follows.

(i) We introduce the concept of hesitant bipolar-valued fuzzy hypersoft relations in terms of hesitant bipolar-valued fuzzy soft sets and hypersoft sets. We extend an approximation space by the sense of hesitant bipolar-valued fuzzy hypersoft relations.

(ii) We propose the concept of upper (resp., lower) approximations of hypersoft sets and fuzzy sets in approximation spaces based on hesitant bipolar-valued fuzzy hypersoft relations. We introduce the notions of rough hypersoft sets and rough fuzzy sets induced by hesitant bipolar-valued fuzzy hypersoft relations, and corresponding examples are presented.

(iii) We establish associations between hypersoft sets (resp., fuzzy sets) and upper and lower approximations of hypersoft sets (resp., fuzzy sets) by hesitant bipolar-valued fuzzy hypersoft relations.

In Section 4, the contributions of the section are as follows.

(i) We establish associations between hypersoft quasi-ideals over semigroups (resp., fuzzy quasi-ideals of semigroups) and upper and lower approximations of hypersoft quasi-ideals over semigroups (resp., fuzzy quasi-ideals of semigroups).

(ii) We establish connections between two upper and lower approximations of hypersoft quasi-ideals over semigroups (resp., fuzzy quasi-ideals of semigroups) in the viewpoint of hypersoft semigroup homomorphism problems.

In Section 5, we contain some concluding remarks pointing to set-valued measurement issues and decision-making algorithms for decision-making problems. Besides, the work is summarized.

2. Basic notions and earlier works

In this section, we first recall some properties and definitions which will be used in subsequent sections.

Throughout this paper, K, V, and W denote non-empty sets, and $\mathcal{P}(V)$ denotes a collection of all subsets of V.

2.1. Some essential attributes in semigroups

Definition 2.1 ([13])**.** Let $*$ be a given binary operation on V. A semigroup is denoted by an algebraic system $(V, *)$, where $*$ is associative. We usually write simply V instead of $(V, *)$. In the following, if $(V, *)$ is a semigroup, then $\acute{v} * \grave{v}$ is denoted by $\acute{v}\grave{v}$ for all $\acute{v}, \grave{v} \in V$. Given two non-empty subsets X and Y of a semigroup V, the product $X * Y$ (simply XY) is defined by

$$XY = \{\acute{v}\grave{v} : \acute{v} \in X \text{ and } \grave{v} \in Y\}.$$

Definition 2.2 ([19])**.** Let V be a semigroup, and let X be a non-empty subset of V.

(i) X is said to be a subsemigroup of V if $XX \subseteq X$.

(ii) X is said to be a left ideal of V if $VX \subseteq X$.

(iii) X is said to be a right ideal of V if $XV \subseteq X$.

(iv) X is said to be an ideal of V if it is a left ideal and a right ideal of V.

Definition 2.3 ([47]). Let V be a semigroup, and let X be a non-empty subset of V. X is said to be a quasi-ideal of V if $XV \cap VX \subseteq X$.

Definition 2.4 ([19]). Let V be a semigroup. For an element $\nu \in V$, ν is said to be a regular element if there exists $\dot{\nu} \in V$ such that $\nu = \nu\dot{\nu}\nu$. V is said to be a regular semigroup if all elements of V are regular.

Proposition 2.5 ([19]). *Let V be a regular semigroup. Then $XY = X \cap Y$ for every right ideal X and every left ideal Y of V.*

2.2. Some properties of fuzzy sets

Definition 2.6 ([53]). f is said to be a fuzzy subset of V if it is a function from V to the closed unit interval $+I$. Specifically, 1_V is denoted as a fuzzy subset of V defined by $1_V(\nu) = 1$ for all $\nu \in V$, and 0_V is denoted as a fuzzy subset of V defined by $0_V(\nu) = 0$ for all $\nu \in V$.

Definition 2.7 ([53]). Let f and g be fuzzy subsets of V.

(i) $f \subseteq g$ is denoted by meaning $f(\nu) \leqslant g(\nu)$ for all $\nu \in V$.

(ii) A fuzzy set intersection of f and g is denoted by $f \cap g$, where $(f \cap g)(\nu)$ is a minimum value of $f(\nu)$ and $g(\nu)$ (simply $f(\nu) \wedge g(\nu)$) for all $\nu \in V$.

(iii) A fuzzy set union of f and g is denoted by $f \cup g$, where $(f \cup g)(\nu)$ is a maximum value of $f(\nu)$ and $g(\nu)$ (simply $f(\nu) \vee g(\nu)$) for all $\nu \in V$.

(iv) A fuzzy set complement of f is denoted by f', where f' is a function from V to $+I$ defined by $f'(\nu) = 1 - f(\nu)$ for all $\nu \in V$.

Definition 2.8 ([24]). Let f be a fuzzy subset of V and $\iota \in +I$. The set

$$V^{(f,\iota,\geqslant)} := \{\nu \in V : f(\nu) \geqslant \iota\}$$

is said to be an ι-level set of f.

Definition 2.9 ([32]). Let f and g be fuzzy subsets of a semigroup V. The product $f \circ g$ is defined by

$$(f \circ g)(\nu) = \begin{cases} \sup_{\nu = \acute{\nu}\grave{\nu}}\{\min\{f(\acute{\nu}), g(\grave{\nu})\}\}, & \text{if } \nu = \acute{\nu}\grave{\nu} \text{ for some } \acute{\nu}, \grave{\nu} \in V, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.10 ([32]). Let f be a fuzzy subset of a semigroup V.

(i) f is said to be a fuzzy ideal of V if it satisfies

$$f(\acute{\nu}\grave{\nu}) \geqslant \max\{f(\acute{\nu}), f(\grave{\nu})\}$$

for all $\acute{\nu}, \grave{\nu} \in V$.

(ii) f is said to be a fuzzy quasi-ideal of V if it satisfies

$$f \circ 1_V \cap 1_V \circ f \subseteq f.$$

Proposition 2.11 ([51]). *Let f be a fuzzy subset of a semigroup V. Then f is a fuzzy ideal of V if and only if for all $\iota \in +I$, if $V^{(f,\iota,\geqslant)}$ is non-empty, then $V^{(f,\iota,\geqslant)}$ is an ideal of V.*

Proposition 2.12 ([20]). *Let f be a fuzzy subset of a semigroup V. Then f is a fuzzy quasi-ideal of V if and only if for all* $\iota \in +I$, *if* $V^{(f,\iota,\geqslant)}$ *is non-empty, then* $V^{(f,\iota,\geqslant)}$ *is a quasi-ideal of V.*

Definition 2.13 ([54]). Based on Definition 2.6, an element in a collection of all fuzzy subsets of $V \times W$ is said to be a fuzzy relation from V to W. Given a fuzzy relation R from V to W and elements $v \in V$, $w \in W$, the value $R(v, w)$ in $+I$ is a membership grade of the relation between v and w based on R.

Definition 2.14 ([10]). Let R be a fuzzy relation from V to V. R is said to be a classical fuzzy perfect antisymmetric relation if for all $\acute{v}, \grave{v} \in V$, $R(\acute{v}, \grave{v}) > 0$ and $R(\grave{v}, \acute{v}) > 0$ imply $\acute{v} = \grave{v}$.

Definition 2.15 ([32]). Let V be a given semigroup, and let R be a fuzzy relation from V to V. R is said to be a classical fuzzy compatible relation if for all $v, \acute{v}, \grave{v} \in V$, $R(\acute{v}v, \grave{v}v) \geqslant R(\acute{v}, \grave{v})$ and $R(v\acute{v}, v\grave{v}) \geqslant R(\acute{v}, \grave{v})$.

Definition 2.16 ([57]). $f := (f^-, f^+)$ is said to be a bipolar-valued fuzzy set on V if f^- is a function from V to $-I$ and f^+ is a function from V to $+I$. Here, the bipolar fuzzy set f on V is obtained the following interpretation.

(i) A positive membership degree $f^+(v)$ denotes a satisfaction degree of the element v to the property corresponding to the bipolar fuzzy set f for all $v \in V$.

(ii) A negative membership degree $f^-(v)$ denotes a satisfaction degree of the element v to some implicit counter-property corresponding to the bipolar fuzzy set f for all $v \in V$.

Definition 2.17 ([57]). Let $f := (f^-, f^+)$ and $g := (g^-, g^+)$ be bipolar fuzzy sets on V. f is a subset of g if it satisfies

$$f^-(v) \geqslant g^-(v) \text{ and } f^+(v) \leqslant g^+(v)$$

for all $v \in V$.

Definition 2.18 ([23]). Based on Definition 2.16, an element in a collection of all bipolar fuzzy sets on $V \times W$ is said to be a bipolar fuzzy relation from V to W.

Definition 2.19 ([23]). Let $R := (R^-, R^+)$ be a bipolar fuzzy relation from V to V.

(i) R is said to be a bipolar fuzzy reflexive relation if it satisfies

$$R^-(v, v) = -1 \text{ and } R^+(v, v) = 1$$

for all $v \in V$.

(ii) R is said to be a bipolar fuzzy symmetric relation if it satisfies

$$R^-(\acute{v}, \grave{v}) = R^-(\grave{v}, \acute{v}) \text{ and } R^+(\acute{v}, \grave{v}) = R^+(\grave{v}, \acute{v})$$

for all $\acute{v}, \grave{v} \in V$.

(iii) R is said to be a bipolar fuzzy transitive relation if it satisfies

$$R^+(\acute{v}, \grave{v}) \geqslant \sup_{v \in V}\{\min\{R^+(\acute{v}, v), R^+(v, \grave{v})\}\} \quad \text{and} \quad R^-(\acute{v}, \grave{v}) \leqslant \inf_{v \in V}\{\max\{R^-(\acute{v}, v), R^-(v, \grave{v})\}\}$$

for all $\acute{v}, \grave{v} \in V$.

(iv) R is said to be a bipolar fuzzy equivalence relation if it is a bipolar fuzzy reflexive relation, a bipolar fuzzy symmetric relation and a bipolar fuzzy transitive relation.

Definition 2.20 ([27]). $f := (f^-, f^+)$ is said to be a hesitant bipolar-valued fuzzy set on V if f^- is a function from V to $\mathcal{P}(-I)$ and f^+ is a function from V to $\mathcal{P}(+I)$. For $v \in V$, $f^-(v)$ and $f^+(v)$ satisfy items (i) and (ii) in Definition 2.16.

2.3. Some essential definitions of soft sets and hypersoft sets

Definition 2.21 ([31]). Let A be a non-empty subset of K. If F is a mapping from A to $\mathcal{P}(V)$, then (F, A) is said to be a soft set over V with respect to A. As the understanding of the soft set, V is said to be a universe of all alternative objects of (F, A), and K is said to be a set of all parameters of (F, A), where parameters are attributes, characteristics or statements of alternative objects in V. For any element $a \in A$, $F(a)$ is considered as a set of a-approximate elements (or a-alternative objects) of (F, A).

Definition 2.22 ([26]). Let A be a non-empty subset of K. A relative whole soft set over V with respect to A is denoted by $\mathfrak{W}_{V_A} := (V_A, A)$, where V_A is a set valued-mapping given by $V_A(a) = V$ for all $a \in A$.

Definition 2.23 ([26]). Let $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively. $\mathfrak{F}$ is a soft subset of $\mathfrak{G}$, denoted by $\mathfrak{F} \Subset \mathfrak{G}$, if $A \subseteq B$ and $F(a) \subseteq G(a)$ for all $a \in A$.

Definition 2.24 ([2]). Let $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ be two soft sets over a common alternative universe with respect to non-empty subsets A and B of K, respectively. A restricted intersection of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F} \Cap \mathfrak{G}$, is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = F(c) \cap G(c)$ for all $c \in C$.

Definition 2.25 ([3]). Let $\mathfrak{F} := (F, A)$ and $\mathfrak{G} := (G, B)$ be two soft sets over a semigroup V with respect to non-empty subsets A and B of K, respectively. A restricted product of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F} \circledcirc \mathfrak{G}$, is defined as a soft set (H, C), where $C = A \cap B$ and $H(c) = (F(c))(G(c))$ for all $c \in C$.

Definition 2.26 ([3]). Let $\mathfrak{F} := (F, A)$ be a soft set over a semigroup V with respect to a non-empty subset A of K.

(i) $\mathfrak{F}$ is said to be a soft left ideal if $\mathfrak{W}_{V_A} \circledcirc \mathfrak{F} \Subset \mathfrak{F}$.

(ii) $\mathfrak{F}$ is said to be a soft right ideal if $\mathfrak{F} \circledcirc \mathfrak{W}_{V_A} \Subset \mathfrak{F}$.

(iii) $\mathfrak{F}$ is said to be a soft ideal if it is a soft left ideal and a soft right ideal.

(iv) $\mathfrak{F}$ is said to be a soft quasi-ideal if $(\mathfrak{F} \circledcirc \mathfrak{W}_{V_A}) \Cap (\mathfrak{W}_{V_A} \circledcirc \mathfrak{F}) \Subset \mathfrak{F}$.

Proposition 2.27 ([3]). *Let $\mathfrak{F} := (F, A)$ be a soft set over a semigroup V with respect to a non-empty subset A of K. $\mathfrak{F}$ is a soft left ideal (resp., a soft right ideal, a soft ideal and a soft quasi-ideal) if and only if $F(a)$ is either empty or a left ideal (resp., a right ideal, an ideal and a quasi-ideal) of V for all $a \in A$.*

Definition 2.28 ([16]). Let $\mathfrak{F} := (F, A)$ be a soft set over a semigroup V with respect to a non-empty subset A of K. $\mathfrak{F}$ is said to be a soft semigroup if $F(a)$ is, if it is non-empty, a subsemigroup of V for all $a \in A$.

Definition 2.29 ([16]). Let $\mathfrak{F} := (F, A)$ be a soft semigroup over a semigroup V with respect to a non-empty subset A of K, and let $\mathfrak{G} := (G, B)$ be a soft semigroup over a semigroup W with respect to a non-empty subset B of K. If $\Gamma : V \to W$ is an epimorphism and $\Lambda : A \to B$ is a surjective function such that $\Gamma(F(a)) = G(\Lambda(a))$ for all $a \in A$, then $(\Gamma, \Lambda)_h$ is said to be a soft homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$.

For each $n \in \mathbb{N}$, let $\prod_{i \in \mathbb{N}} K_i := K_1 \times K_2 \times K_3 \times \cdots \times K_n$ denote the n-fold Cartesian product of distinct non-empty universal sets $K_1, K_2, K_3, \ldots, K_n$, i.e.,

$$\prod_{i \in \mathbb{N}} K_i := \{k := (k_1, k_2, k_3, \ldots, k_n) : k_j \in K_j \text{ for } j = 1, 2, 3, \ldots, n\}.$$

Definition 2.30 ([46]). Let $\prod_{i \in \mathbb{N}} A_i$ be a non-empty subset of $\prod_{i \in \mathbb{N}} K_i$. If F is a mapping from $\prod_{i \in \mathbb{N}} A_i$ to $\mathcal{P}(V)$, then $(F, \prod_{i \in \mathbb{N}} A_i)$ is said to be a hypersoft set over V with respect to $\prod_{i \in \mathbb{N}} A_i$. As the understanding of the hypersoft set, the meaning of V and $\prod_{i \in \mathbb{N}} K_i$ is defined as Definition 2.21.

Definition 2.31 ([1]). Let $\prod_{i \in \mathbb{N}} A_i$ be a non-empty subset of $\prod_{i \in \mathbb{N}} K_i$.

(i) A relative null hypersoft set over V with respect to $\prod_{i\in\mathbb{N}} A_i$ is denoted by

$$\mathfrak{N}_{\emptyset_{\prod_{i\in\mathbb{N}} A_i}} := (\emptyset_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i),$$

where $\emptyset_{\prod_{i\in\mathbb{N}} A_i}$ is a set valued-mapping given by $\emptyset_{\prod_{i\in\mathbb{N}} A_i}(a) = \emptyset$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

(ii) A relative whole hypersoft set over V with respect to $\prod_{i\in\mathbb{N}} A_i$ is denoted by

$$\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} := (V_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i),$$

where $V_{\prod_{i\in\mathbb{N}} A_i}$ is a set valued-mapping given by $V_{\prod_{i\in\mathbb{N}} A_i}(a) = V$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

(iii) If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a given hypersoft set over V, then a relative complement of $\mathfrak{F}$ is denoted by $C(\mathfrak{F}) := (F^c, \prod_{i\in\mathbb{N}} A_i)$, which is a hypersoft set defined by $F^c(a) = V - F(a)$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

Definition 2.32 ([1])**.** Let $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ be two hypersoft sets over a common alternative universe with respect to non-empty subsets $\prod_{i\in\mathbb{N}} A_i$ and $\prod_{i\in\mathbb{N}} B_i$ of $\prod_{i\in\mathbb{N}} K_i$, respectively.

(i) $\mathfrak{F}$ is a hypersoft subset of $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\Subset}\mathfrak{G}$, if $\prod_{i\in\mathbb{N}} A_i \subseteq \prod_{i\in\mathbb{N}} B_i$ and $F(a) \subseteq G(a)$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

(ii) $\mathfrak{F}$ is equal to $\mathfrak{G}$ if $\mathfrak{F}\widetilde{\Subset}\mathfrak{G}$ and $\mathfrak{G}\widetilde{\Subset}\mathfrak{F}$.

Definition 2.33 ([1])**.** Let $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ be two hypersoft sets over a common alternative universe with respect to non-empty subsets $\prod_{i\in\mathbb{N}} A_i$ and $\prod_{i\in\mathbb{N}} B_i$ of $\prod_{i\in\mathbb{N}} K_i$, respectively.

(i) A restricted intersection of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\Cap}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H(c) = F(c) \cap G(c)$ for all $c \in \prod_{i\in\mathbb{N}} C_i$.

(ii) A restricted union of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\Cup}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H(c) = F(c) \cup G(c)$ for all $c \in \prod_{i\in\mathbb{N}} C_i$.

(iii) An extended intersection of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\sqcap}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cup \prod_{i\in\mathbb{N}} B_i$, and

$$H(c) = \begin{cases} F(c), & \text{if } c \in \prod_{i\in\mathbb{N}} A_i - \prod_{i\in\mathbb{N}} B_i, \\ G(c), & \text{if } c \in \prod_{i\in\mathbb{N}} B_i - \prod_{i\in\mathbb{N}} A_i, \\ F(c) \cap G(c), & \text{if } c \in \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i, \end{cases}$$

for all $c \in \prod_{i\in\mathbb{N}} C_i$.

(iv) An extended union of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\sqcup}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cup \prod_{i\in\mathbb{N}} B_i$, and

$$H(c) = \begin{cases} F(c), & \text{if } c \in \prod_{i\in\mathbb{N}} A_i - \prod_{i\in\mathbb{N}} B_i, \\ G(c), & \text{if } c \in \prod_{i\in\mathbb{N}} B_i - \prod_{i\in\mathbb{N}} A_i, \\ F(c) \cup G(c), & \text{if } c \in \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i, \end{cases}$$

for all $c \in \prod_{i\in\mathbb{N}} C_i$.

(v) A restricted difference of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\widetilde{\ominus}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H(c) = F(c) - G(c)$ for all $c \in \prod_{i\in\mathbb{N}} C_i$.

Definition 2.34 ([49])**.** Let A be a non-empty subset of K. If F is a mapping from A to a collection of all hesitant bipolar-valued fuzzy sets of V, then (F, A) is said to be a *hesitant bipolar-valued fuzzy soft set* over V with respect to A. In this way, V and K are considered as Definition 2.21.

2.4. *Variations of rough sets*

Definition 2.35 ([15]). Let (V, E) be a Pawlak's approximation space, and let f be a fuzzy subset of V. An upper rough approximation of f within (V, E) is defined by the fuzzy subset $\ulcorner f \urcorner_E$ of V, where

$$\ulcorner f \urcorner_E(\acute{v}) = \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]_E\}$$

for all $\acute{v} \in V$. A lower rough approximation of f within (V, E) is defined by the fuzzy subset $\llcorner f \lrcorner_E$ of V, where

$$\llcorner f \lrcorner_E(\acute{v}) = \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]_E\}$$

for all $\acute{v} \in V$. f is said to be a definable fuzzy set within (V, E) if $\ulcorner f \urcorner_E = \llcorner f \lrcorner_E$; otherwise f is said to be a rough fuzzy set within (V, E).

Definition 2.36 ([45]). Let (V, E) be a Pawlak's approximation space, and let $\mathfrak{F} := (F, \prod_{i \in \mathbb{N}} A_i)$ be a given hypersoft set over V. An upper rough approximation of $\mathfrak{F}$ within (V, E) is denoted by $\mathfrak{F}\rceil_E := (F\rceil_E, \prod_{i \in \mathbb{N}} A_i)$, where

$$F\rceil_E(a) = \lceil F(a) \rceil_E$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$. A lower rough approximation of $\mathfrak{F}$ within (V, E) is denoted by $\mathfrak{F}\rfloor_E := (F\rfloor_E, \prod_{i \in \mathbb{N}} A_i)$, where

$$F\rfloor_E(a) = \lfloor F(a) \rfloor_E$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$. $\mathfrak{F}$ is said to be a definable hypersoft set within (V, E) if $\mathfrak{F}\rceil_E = \mathfrak{F}\rfloor_E$; otherwise $\mathfrak{F}$ is said to be a rough hypersoft set within (V, E).

Definition 2.37 ([43]). Let R be a fuzzy relation from V to W and $\varphi \in +I$. For an element $v \in V$,

$$[v]^{s}_{R,\varphi} := \{w \in W : R(v, w) \geqslant \varphi\}$$

is said to be a successor class of v with respect to φ-level based on R.

Definition 2.38 ([43]). Let R be a fuzzy relation from V to W and $\varphi \in +I$. For an element $v \in V$,

$$[v]^{cs}_{R,\varphi} := \{\dot{v} \in V : [v]^{s}_{R,\varphi} = (\dot{v})^{s}_{R,\varphi}\}$$

is said to be a core of the successor class of v with respect to φ-level based on R. $[V]^{cs}_{R,\varphi}$ is denoted as a collection of $[v]^{cs}_{R,\varphi}$ for all $v \in V$.

Definition 2.39 ([43]). If $\varphi \in +I$ and R is a fuzzy relation from V to W related to $[V]^{cs}_{R,\varphi}$, then $(V, W, [V]^{cs}_{R,\varphi})$ is said to be an approximation space based on $[V]^{cs}_{R,\varphi}$.

Definition 2.40 ([43]). Let $(V, W, [V]^{cs}_{R,\varphi})$ be an approximation space based on $[V]^{cs}_{R,\varphi}$, and let X be a non-empty subset of V. An upper approximation of X within $(V, W, [V]^{cs}_{R,\varphi})$ is denoted by $\lceil X \rceil^{cs}_{R,\varphi}$, where

$$\lceil X \rceil^{cs}_{R,\varphi} := \bigcup_{v \in V} \{[v]^{cs}_{R,\varphi} : [v]^{cs}_{R,\varphi} \cap X \neq \emptyset\}.$$

A lower approximation of X within $(V, W, [V]^{cs}_{R,\varphi})$ is denoted by $\lfloor X \rfloor^{cs}_{R,\varphi}$, where

$$\lfloor X \rfloor^{cs}_{R,\varphi} := \bigcup_{v \in V} \{[v]^{cs}_{R,\varphi} : [v]^{cs}_{R,\varphi} \subseteq X\}.$$

A boundary region of X within $(V, W, [V]^{cs}_{R,\varphi})$ is defined by $\lceil X \rceil^{cs}_{R,\varphi} - \lfloor X \rfloor^{cs}_{R,\varphi}$. We say that $(\lceil X \rceil^{cs}_{R,\varphi}, \lfloor X \rfloor^{cs}_{R,\varphi})$ is a rough set of X within $(V, W, [V]^{cs}_{R,\varphi})$ if $\lceil X \rceil^{cs}_{R,\varphi} - \lfloor X \rfloor^{cs}_{R,\varphi}$ is a non-empty set. X is said to be a definable set within $(V, W, [V]^{cs}_{R,\varphi})$ if $\lceil X \rceil^{cs}_{R,\varphi} - \lfloor X \rfloor^{cs}_{R,\varphi}$ is an empty set.

3. Rough hypersoft sets and rough fuzzy sets via hesitant bipolar-valued fuzzy hypersoft relations

We shall now develop rough approximation models of rough hypersoft sets and rough fuzzy sets based on hesitant bipolar-valued fuzzy hypersoft relations. After defining a novel approximation space based on hesitant bipolar-valued fuzzy hypersoft relations, we provide some propositions associated with upper and lower rough approximations of hypersoft sets and fuzzy sets.

Throughout the entire remainder, $\prod_{i\in\mathbb{N}} A_i$ and $\prod_{i\in\mathbb{N}} B_i$ are two non-empty subsets of $\prod_{i\in\mathbb{N}} K_i$ such that $\prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ is non-empty.

To adapt for the notion of hypersoft sets on Definition 2.30 and hesitant bipolar-valued fuzzy soft sets on Definition 2.34 via Definition 2.17, an inclusion relation of two hesitant bipolar-valued fuzzy hypersoft sets $(F, \prod_{i\in\mathbb{N}} A_i)$ and $(G, \prod_{i\in\mathbb{N}} B_i)$ over V is denoted by $(F, \prod_{i\in\mathbb{N}} A_i) \subseteq_{ir} (G, \prod_{i\in\mathbb{N}} B_i)$, where

(i) $\prod_{i\in\mathbb{N}} A_i \subseteq \prod_{i\in\mathbb{N}} B_i$;

(ii) $(F(a))^-(v) \supseteq (G(a))^-(v)$ and $(F(a))^+(v) \subseteq (G(a))^+(v)$ for all $a \in \prod_{i\in\mathbb{N}} A_i, v \in V$.

In addition, given two elements $(\varphi, \chi), (\psi, \omega) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, we define set-valued relations on $\mathcal{P}(-I) \times \mathcal{P}(+I)$ of (φ, χ) and (ψ, ω) as follows:

(i) $(\varphi, \chi) = (\psi, \omega)$ if $\varphi = \psi$ and $\chi = \omega$;

(ii) $(\varphi, \chi) \subseteq_{sr} (\psi, \omega)$ if $\varphi \supseteq \psi$ and $\chi \subseteq \omega$.

To approximation methodology in this section, the concept of hesitant bipolar-valued fuzzy hypersoft relations based on Definitions 2.30 and 2.34 is defined as the statement that if R is a mapping from $\prod_{i\in\mathbb{N}} A_i$ to a collection of all hesitant bipolar-valued fuzzy sets of $V \times W$, then $(R, \prod_{i\in\mathbb{N}} A_i)$ is called a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$. On the consideration of the single V in Definitions 2.14 and 2.19, to adapt for a hesitant bipolar-valued fuzzy hypersoft relation $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ over $V \times V$, we define the characters of $\mathfrak{R}$ as follows.

(i) $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft reflexive relation if it satisfies

- $(R(a))^+(v, v) = +I$ for all $a \in \prod_{i\in\mathbb{N}} A_i, v \in V$;
- $(R(a))^-(v, v) = \emptyset$ for all $a \in \prod_{i\in\mathbb{N}} A_i, v \in V$.

(ii) $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft symmetric relation if it satisfies

- $(R(a))^+(\acute{v}, \grave{v}) = (R(a))^+(\grave{v}, \acute{v})$ for all $a \in \prod_{i\in\mathbb{N}} A_i, \acute{v}, \grave{v} \in V$;
- $(R(a))^-(\acute{v}, \grave{v}) = (R(a))^-(\grave{v}, \acute{v})$ for all $a \in \prod_{i\in\mathbb{N}} A_i, \acute{v}, \grave{v} \in V$.

(iii) $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft transitive relation if it satisfies

- $\bigcup_{v\in V}((R(a))^+(\acute{v}, v) \cap (R(a))^+(v, \grave{v})) \subseteq (R(a))^+(\acute{v}, \grave{v})$ for all $a \in \prod_{i\in\mathbb{N}} A_i, \acute{v}, \grave{v} \in V$;
- $\bigcap_{v\in V}((R(a))^-(\acute{v}, v) \cup (R(a))^-(v, \grave{v})) \supseteq (R(a))^-(\acute{v}, \grave{v})$ for all $a \in \prod_{i\in\mathbb{N}} A_i, \acute{v}, \grave{v} \in V$.

(iv) $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation if it satisfies the property that for all $a \in \prod_{i\in\mathbb{N}} A_i, \acute{v}, \grave{v} \in V$,

- $(R(a))^+(\acute{v}, \grave{v}) \supset \emptyset$ and $(R(a))^+(\grave{v}, \acute{v}) \supset \emptyset$ imply $\acute{v} = \grave{v}$;
- $(R(a))^-(\acute{v}, \grave{v}) \subset -I$ and $(R(a))^-(\grave{v}, \acute{v}) \subset -I$ imply $\acute{v} = \grave{v}$.

(v) $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft equivalence relation if it is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft symmetric relation and a hesitant bipolar-valued fuzzy hypersoft transitive relation.

For the example of these properties, we use Definition 2.20 in a common consideration. We define two hesitant bipolar-valued fuzzy sets $f := (f^-, f^+)$ and $g := (g^-, g^+)$ in a collection of all hesitant bipolar-valued fuzzy sets of $V \times V$ by square matrix representations f^-_M, f^+_M, g^-_M, and g^+_M as follows:

$$f^+_M := \left[\dot{v}_{ij} := f^+(v_i, v_j)\right], g^+_M := \left[\ddot{v}_{ij} := g^+(v_i, v_j)\right] \in M_n(\mathcal{P}(+I))$$

and

$$f^-_M := \left[\dddot{v}_{ij} := f^-(v_i, v_j)\right], g^-_M := \left[\ddddot{v}_{ij} := g^-(v_i, v_j)\right] \in M_n(\mathcal{P}(-I)),$$

where

$$\dot{v}_{ij} = \begin{cases} +I, & \text{if } i = j, \\ \emptyset, & \text{if } i \neq j, \end{cases} \quad \ddot{v}_{ij} = \begin{cases} \emptyset, & \text{if } i = j, \\ +I, & \text{if } i \neq j, \end{cases} \quad \dddot{v}_{ij} = \begin{cases} \emptyset, & \text{if } i = j, \\ -I, & \text{if } i \neq j, \end{cases} \quad \ddddot{v}_{ij} = \begin{cases} -I, & \text{if } i = j, \\ \emptyset, & \text{if } i \neq j. \end{cases}$$

If R is a set-valued function from A to the collection of all hesitant bipolar-valued fuzzy sets of $V \times V$ defined by

$$R(a) = f$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$, then we see that $(R, \prod_{i \in \mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft symmetric relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation. For all sets of parameters-based alternative objects of the hypersoft set $(R, \prod_{i \in \mathbb{N}} A_i)$, we observe that during the evaluating process of each relationship between two elements of V in this corresponding example, however, these possible memberships maybe not only crisp values in $-I$ and $+I$, but also interval values.

Definition 3.1. Let $\mathfrak{R} := (R, \prod_{i \in \mathbb{N}} A_i)$ be a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$. For an element $v \in V$, we call

$$[v]^s_{\mathfrak{R},(\varphi,\chi)} := \{w \in W : (R(a))^-(v, w) \subseteq \varphi \text{ and } (R(a))^+(v, w) \supseteq \chi \text{ for all } a \in \prod_{i \in \mathbb{N}} A_i\}$$

a successor class of v with respect to (φ, χ)-inclusion based on $\mathfrak{R}$. Here, $[V]^s_{\mathfrak{R},(\varphi,\chi)}$ represents a collection of $[v]^s_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.

Proposition 3.2. *If $\mathfrak{R} := (R, \prod_{i \in \mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation over $V \times V$ and the pair $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $v \in [v]^s_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.*

Proof. Suppose that $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation over $V \times V$ and the pair $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$. Let $a \in \prod_{i \in \mathbb{N}} A_i$ and $v \in V$. Then, we observe that

$$(R(a))^-(v, v) = \emptyset \subseteq \varphi \text{ and } (R(a))^+(v, v) = +I \supseteq \chi.$$

Thus $v \in [v]^s_{\mathfrak{R},(\varphi,\chi)}$. □

Definition 3.3. Let $\mathfrak{R} := (R, \prod_{i \in \mathbb{N}} A_i)$ be a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$. For an element $v \in V$, we call

$$[v]^{cs}_{\mathfrak{R},(\varphi,\chi)} := \{\dot{v} \in V : [v]^s_{\mathfrak{R},(\varphi,\chi)} = [\dot{v}]^s_{\mathfrak{R},(\varphi,\chi)}\}$$

a core of the successor class of v with respect to (φ, χ)-inclusion based on $\mathfrak{R}$. We generally denote by $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ a collection of $[v]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.

Due to Definition 3.3, the following two statements hold.

Proposition 3.4. *If $\mathfrak{R} := (R, \prod_{i \in \mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $v \in [v]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.*

Proposition 3.5. *If $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then the following two arguments are equivalent.*

(i) $\acute{v} \in [\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *for all* $\acute{v}, \grave{v} \in V$.

(ii) $[\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = [\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *for all* $\acute{v}, \grave{v} \in V$.

Remark 3.6. By Propositions 3.4 and 3.5, it is easy to show that if $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is the partition of V.

Proposition 3.7. *If $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation over $V \times V$ and the pair $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $[v]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v]^{s}_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.*

Proof. Suppose $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation over $V \times V$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$. Let $v_1 \in V$, and suppose $v_2 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Thus, by Proposition 3.2, we see that $v_2 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Therefore $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. □

Proposition 3.8. *If $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft symmetric relation and a hesitant bipolar-valued fuzzy hypersoft transitive relation over $V \times V$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $[v]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$.*

Proof. Suppose that $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft symmetric relation and a hesitant bipolar-valued fuzzy hypersoft transitive relation over $V \times V$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$. Let $v_1 \in V$ be given, and let $v_2 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Then $(R(a))^-(v_1, v_2) \subseteq \varphi$ and $(R(a))^+(v_1, v_2) \supseteq \chi$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft symmetric relation, we have $(R(a))^-(v_2, v_1) \subseteq \varphi$ and $(R(a))^+(v_2, v_1) \supseteq \chi$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. We shall prove that $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Let $v_3 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Then $(R(a))^-(v_1, v_3) \subseteq \varphi$ and $(R(a))^+(v_1, v_3) \supseteq \chi$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft transitive relation, we have

$$(R(a))^-(v_2, v_3) \subseteq \bigcap_{v\in V} ((R(a))^-(v_2, v) \cup (R(a))^-(v, v_3)) \subseteq (R(a))^-(v_2, v_1) \cup (R(a))^-(v_1, v_3) \subseteq \varphi \cup \varphi = \varphi$$

and

$$(R(a))^+(v_2, v_3) \supseteq \bigcup_{v\in V} ((R(a))^+(v_2, v) \cap (R(a))^+(v, v_3)) \supseteq (R(a))^+(v_2, v_1) \cap (R(a))^+(v_1, v_3) \supseteq \chi \cap \chi = \chi$$

for all $a \in \prod_{i\in\mathbb{N}} A_i$. Whence $v_3 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$, which yields $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Conversely, we can prove that $[v_2]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Hence $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Thus $v_2 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Therefore $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. □

Proposition 3.9. *If $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft equivalence relation over $V \times V$ and $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$, then $[v]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $v \in V$. Moreover, $[V]^{s}_{\mathfrak{R},(\varphi,\chi)}$ is the partition of V.*

Proof. By Remark 3.6 and Propositions 3.7 and 3.8, this proposition immediately yields. □

Proposition 3.10. *If $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation over $V \times V$ and $(\varphi, \chi) \in \mathcal{P}(-I)\backslash\{-I\} \times \mathcal{P}(+I)\backslash\{\emptyset\}$, then the following statements are equivalent.*

(i) $\acute{v} = \grave{v}$ *for all* $\acute{v}, \grave{v} \in V$.

(ii) $[\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = [\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *for all* $\acute{v}, \grave{v} \in V$.

(iii) $\acute{v} \in [\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *for all* $\acute{v}, \grave{v} \in V$.

Proof. It is clear that(i) implies (ii). Due to Proposition 3.5, we obtain that (ii) implies (iii). In order to prove that (iii) implies (i), we let $v_1, v_2 \in V$ be such that $v_1 \in [v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then, we have $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, by Proposition 3.2, we have $v_1 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Then $v_1 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Thus, we observe that

$$(R(a))^{-}(v_2, v_1) \subseteq \varphi \subset -I, (R(a))^{+}(v_2, v_1) \supseteq \chi \supset \emptyset$$

and

$$(R(a))^{-}(v_1, v_2) \subseteq \varphi \subset -I, (R(a))^{+}(v_1, v_2) \supseteq \chi \supset \emptyset$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation, we obtain that $v_1 = v_2$. □

In the presence of research now, the notion of upper and lower rough approximations of hypersoft sets and fuzzy sets are studied under hesitant bipolar-valued fuzzy hypersoft relations.

Definition 3.11. If $(\varphi, \chi) \in \mathcal{P}(-I) \times \mathcal{P}(+I)$ and $\mathfrak{R} := (R, \prod_{i \in \mathbb{N}} K_i)$ is a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ related to $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, then the triple $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is called an approximation space based on $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. In this way, we say that $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is an approximation space type I.

Based on the context of set-valued functions and fuzzy logic of hesitant bipolar-valued fuzzy hypersoft relations, we observe that such space can be considered as an extended approximation space of the approximation space in Definition 2.39.

Definition 3.12. Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type I, and let $\mathfrak{F} := (F, \prod_{i \in \mathbb{N}} A_i)$ be a hypersoft set over V. An upper rough approximation of $\mathfrak{F}$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is denoted by $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} := (F]^{cs}_{\mathfrak{R},(\varphi,\chi)}, \prod_{i \in \mathbb{N}} A_i)$, where

$$F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) := \bigcup_{v \in V} \{[v]^{cs}_{\mathfrak{R},(\varphi,\chi)} : [v]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset\} \tag{3.1}$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$. A lower rough approximation of $\mathfrak{F}$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is denoted by $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = (F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}, \prod_{i \in \mathbb{N}} A_i)$, where

$$F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) := \bigcup_{v \in V} \{[v]^{cs}_{\mathfrak{R},(\varphi,\chi)} : [v]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)\} \tag{3.2}$$

for all $a \in \prod_{i \in \mathbb{N}} A_i$. Here, a boundary region of the hypersoft set $\mathfrak{F}$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is denoted by $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} := (F]^{cs}_{\mathfrak{R},(\varphi,\chi)}, \prod_{i \in \mathbb{N}} A_i)$, where

$$(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}, \prod_{i \in \mathbb{N}} A_i) = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\ominus} \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

As introduced above, such sets are obtained the following interpretation.

(i) $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is a set of all elements, which can be possibly classified as $F(a)$ using $\mathfrak{R}$ (are possibly $F(a)$ in view of $\mathfrak{R}$) for all $a \in \prod_{i \in \mathbb{N}} A_i$. In this way, a complement of $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is said to be a negative region of $F(a)$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ for all $a \in \prod_{i \in \mathbb{N}} A_i$.

(ii) $F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is a set of all elements, which can be certain classified as $F(a)$ using $\mathfrak{R}$ (are certainly $F(a)$ in view of $\mathfrak{R}$) for all $a \in \prod_{i \in \mathbb{N}} A_i$. In this way, such the set is said to be a positive region of $F(a)$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ for all $a \in \prod_{i \in \mathbb{N}} A_i$.

(iii) $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is a set of all elements, which can be classified neither as $F(a)$ nor as non $F(a)$ using $\mathfrak{R}$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

In what follows, for all $a \in \prod_{i\in\mathbb{N}} A_i$, if $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \neq \emptyset$, then $(F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a), F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$ is called a rough (or an inexact) set of $F(a)$ within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$, and we call $F(a)$ a rough set. For all $a \in \prod_{i\in\mathbb{N}} A_i$, if $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) = \emptyset$, then $F(a)$ is called a definable (or an exact) set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$. The hypersoft set $\mathfrak{F}$ is called a definable hypersoft set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ if $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{N}_{\emptyset_A}$; otherwise $\mathfrak{F}$ is called a rough hypersoft set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

We are now ready for the presentation of a corresponding example.

Example 3.13. Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),([-1,-0.7),[0.5,1])})$ be an approximation space type I, where $V = \{v_n := n \ : \ n \text{ is a natural number}\}, W = \{w_n := n \ : \ n \text{ is an integer}\}$, and $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft relation over $V \times W$ defined by

$$(R(k))^+(v, w) = \begin{cases} (0.2, 1,] & \text{if } 3|v-w, \\ [0.05, 0.1), & \text{if } 3 \nmid v-w, \end{cases}$$

and

$$(R(k))^-(v, w) = \begin{cases} (-0.9, -0.8], & \text{if } 3|v-w, \\ [-0.7, -0.1], & \text{if } 3 \nmid v-w, \end{cases}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$, $(v, w) \in V \times W$. We observe that if n is a natural number, then

$$[v_{3n-2}]^s_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{w_{3i-2} : i \text{ is an integer}\},$$
$$[v_{3n-1}]^s_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{w_{3i-1} : i \text{ is an integer}\},$$

and

$$[v_{3n}]^s_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{w_{3i} : i \text{ is an integer}\},$$

which yields

$$[v_{3n-2}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{v_{3i-2} : i \text{ is a natural number}\},$$
$$[v_{3n-1}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{v_{3i-1} : i \text{ is a natural number}\},$$

and

$$[v_{3n}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])} = \{v_{3i} : i \text{ is a natural number}\}.$$

If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V defined by

$$F(a) = \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number with } i \geqslant 200\}$$

for all $a \in \prod_{i\in\mathbb{N}} A_i$, then we observe that

$$F\rceil^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a) = \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number}\},$$
$$F\rfloor^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a) = \{v_{3i} : i \text{ is a natural number}\},$$

and

$$F]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a) = \{v_{3i-2} : i \text{ is a natural number}\}$$

for all $a \in \prod_{i\in\mathbb{N}} A_i$. This shows that $\mathfrak{F}$ is a rough hypersoft set within $(V, W, [V]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])})$.

Remark 3.14. Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I, and let $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ be a hypersoft set over V. Then $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} \mathfrak{F} \widetilde{\Subset} \mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Observe that Equations (3.1) and (3.2) in Definition 3.12 can also be presented by means of the following proposition.

Proposition 3.15. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *be a hypersoft set over* V*. Based on Equations* (3.1) *and* (3.2) *in Definition* 3.12*, we have the following statements:*

(i) $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) = \{v \in V : [v]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset\}$ *for all* $a \in \prod_{i\in\mathbb{N}} A_i$;

(ii) $F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) = \{v \in V : [v]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)\}$ *for all* $a \in \prod_{i\in\mathbb{N}} A_i$.

Directly from Definition 3.12, we can obtain the following three propositions below.

Proposition 3.16. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *be a hypersoft set over* V*. Then, we have the following statements.*

(i) *If* $\mathfrak{F} = \mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}$*, then* $\mathfrak{F}$ *is equal to* $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *and* $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$*. Moreover, the hypersoft set* $\mathfrak{F}$ *is a definable hypersoft set within* $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

(ii) *If* $\mathfrak{F} = \mathfrak{N}_{\emptyset_{\prod_{i\in\mathbb{N}} A_i}}$*, then* $\mathfrak{F}$ *is equal to* $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *and* $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$*. Moreover, the hypersoft set* $\mathfrak{F}$ *is a definable hypersoft set within* $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

(iii) $(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})]^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(iv) $(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(v) $(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(vi) $(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})]^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(vii) $C(\mathfrak{F})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = C(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

(viii) $C(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})]^{cs}_{\mathfrak{R},(\varphi,\chi)} = C(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

(ix) $C(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = C(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

(x) $C(C(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})) = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(xi) $C(C(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})) = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proposition 3.17. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *and* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *be hypersoft sets over* V*. Then, we have the following statements:*

(i) $(\mathfrak{F}\tilde{\Cap}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Subset} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Cap} \mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $(\mathfrak{F}\tilde{\Cap}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Cap} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iii) $(\mathfrak{F}\tilde{\sqcap}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Subset} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\sqcap} \mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iv) $(\mathfrak{F}\tilde{\sqcap}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\sqcap} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(v) $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Cup} \mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = (\mathfrak{F}\tilde{\Cup}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(vi) $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Cup} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Subset} (\mathfrak{F}\tilde{\Cup}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(vii) $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\sqcup} \mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = (\mathfrak{F}\tilde{\sqcup}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(viii) $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\sqcup}\mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}(\mathfrak{F}\widetilde{\sqcup}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proposition 3.18. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *and* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *be hypersoft sets over* V. *If* $\mathfrak{F}\widetilde{\Subset}\mathfrak{G}$, *then the following statements hold:*

(i) $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}\mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proposition 3.19. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(V, V, [V]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\psi,\omega)})$ *be approximation spaces type I with the property that the inclusion relation of the hesitant bipolar-valued fuzzy hypersoft reflexive relation* $\mathfrak{R}$ *and the hesitant bipolar-valued fuzzy hypersoft transitive relation* $\mathfrak{S}$ *is* $\mathfrak{R} \subseteq_{ir} \mathfrak{S}$, *and* $(\psi, \omega) \subseteq_{sr} (\varphi, \chi)$. *If* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *is a hypersoft set over* V, *then* $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}\mathfrak{F}\rceil^{cs}_{S,(\psi,\omega)}$ *and* $\mathfrak{F}\rfloor^{cs}_{S,(\psi,\omega)}\widetilde{\Subset}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proof. Assume that $\mathfrak{F}$ is a hypersoft set over V and $a \in \prod_{i\in\mathbb{N}} A_i$. Let $v_1 \in F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Then $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset$. Thus, there exists $v_2 \in V$ such that $v_2 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in F(a)$. Hence, we get that $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, by Proposition 3.2, we obtain $v_1 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Hence $v_1 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Thus, we observe that

$$(S(k))^-(v_2, v_1) \subseteq (R(k))^-(v_2, v_1) \subseteq \varphi \subseteq \psi,$$
$$(S(k))^+(v_2, v_1) \supseteq (R(k))^+(v_2, v_1) \supseteq \chi \supseteq \omega,$$
$$(S(k))^-(v_1, v_2) \subseteq (R(k))^-(v_1, v_2) \subseteq \varphi \subseteq \psi,$$
$$(S(k))^+(v_1, v_2) \supseteq (R(k))^+(v_1, v_2) \supseteq \chi \supseteq \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. We shall prove that $[v_1]^{s}_{S,(\psi,\omega)} = [v_2]^{s}_{S,(\psi,\omega)}$. We let $v_3 \in [v_2]^{s}_{S,(\psi,\omega)}$. Then

$$(S(k))^-(v_2, v_3) \subseteq \psi \text{ and } (S(k))^+(v_2, v_3) \supseteq \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. By the fact that $\mathfrak{S}$ is a hesitant bipolar-valued fuzzy hypersoft transitive relation, it is true that

$$(S(k))^-(v_1, v_3) \subseteq \bigcap_{v\in V}((S(k))^-(v_1, v) \cup (S(k))^-(v, v_3)) \subseteq (S(k))^-(v_1, v_2) \cup (S(k))^-(v_2, v_3) \subseteq \psi \cup \psi = \psi$$

and

$$(S(k))^+(v_1, v_3) \supseteq \bigcup_{v\in V}((S(k))^+(v_1, v) \cap (S(k))^+(v, v_3)) \supseteq (S(k))^+(v_1, v_2) \cap (S(k))^+(v_2, v_3) \supseteq \omega \cap \omega = \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Hence, we get that $v_3 \in [v_1]^{s}_{S,(\psi,\omega)}$, which yields $[v_2]^{s}_{S,(\psi,\omega)} \subseteq [v_1]^{s}_{S,(\psi,\omega)}$. Conversely, we can verify that $[v_1]^{s}_{S,(\psi,\omega)} \subseteq [v_2]^{s}_{S,(\psi,\omega)}$. Thus $[v_1]^{s}_{S,(\psi,\omega)} = [v_2]^{s}_{S,(\psi,\omega)}$. Whence $v_2 \in [v_1]^{cs}_{S,(\psi,\omega)}$. Thus $v_2 \in [v_1]^{cs}_{S,(\psi,\omega)} \cap F(a)$. Hence $[v_1]^{cs}_{S,(\psi,\omega)} \cap F(a) \neq \emptyset$, then $v_1 \in F\rceil^{cs}_{S,(\psi,\omega)}(a)$. Therefore $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \subseteq F\rceil^{cs}_{S,(\psi,\omega)}(a)$. It follows that $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}\mathfrak{F}\rceil^{cs}_{S,(\psi,\omega)}$. Next, we let $v_4 \in F\rfloor^{cs}_{S,(\psi,\omega)}(a)$. Then, we have $[v_4]^{cs}_{S,(\psi,\omega)} \subseteq F(a)$. Observe that it suffices to prove that $[v_4]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_4]^{cs}_{S,(\psi,\omega)}$. Suppose $v_5 \in [v_4]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $[v_4]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_5]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and by Proposition 3.2, we get $v_4 \in [v_4]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_5 \in [v_5]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Thus, we have $v_4 \in [v_5]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_5 \in [v_4]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Since $(\varphi, \chi) \supseteq (\psi, \omega)$ and $\mathfrak{R}$ is a subset of S, we observe that

$$(S(k))^-(v_5, v_4) \subseteq (R(k))^-(v_5, v_4) \subseteq \varphi \subseteq \psi,$$

$$(S(k))^+(v_5,v_4) \supseteq (R(k))^+(v_5,v_4) \supseteq \chi \supseteq \omega,$$

$$(S(k))^-(v_4,v_5) \subseteq (R(k))^-(v_4,v_5) \subseteq \varphi \subseteq \psi,$$

$$(S(k))^+(v_4,v_5) \supseteq (R(k))^+(v_4,v_5) \supseteq \chi \supseteq \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. We shall show that $[v_4]^s_{S,(\psi,\omega)} = [v_5]^s_{S,(\psi,\omega)}$. We assume $v_6 \in [v_5]^s_{S,(\psi,\omega)}$. Observe that

$$(S(k))^-(v_5,v_6) \subseteq \psi \text{ and } (S(k))^+(v_5,v_6) \supseteq \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Since $\mathfrak{S}$ is a hesitant bipolar-valued fuzzy hypersoft transitive relation, we observe that

$$(S(k))^-(v_4,v_6) \subseteq \bigcap_{v\in V}((S(k))^-(v_4,v) \cup (S(k))^-(v,v_6)) \subseteq (S(k))^-(v_4,v_5) \cup (S(k))^-(v_5,v_6) \subseteq \psi \cup \psi = \psi$$

and

$$(S(k))^+(v_4,v_6) \supseteq \bigcup_{v\in V}((S(k))^+(v_4,v) \cap (S(k))^+(v,v_6)) \supseteq (S(k))^+(v_4,v_5) \cap (S(k))^+(v_5,v_6) \supseteq \omega \cap \omega = \omega$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. We get that $v_6 \in [v_4]^s_{S,(\psi,\omega)}$. Hence $[v_5]^s_{S,(\psi,\omega)} \subseteq [v_4]^s_{S,(\psi,\omega)}$. Conversely, we can find that $[v_4]^s_{S,(\psi,\omega)} \subseteq [v_5]^s_{S,(\psi,\omega)}$. Therefore $[v_4]^s_{S,(\psi,\omega)} = [v_5]^s_{S,(\psi,\omega)}$. Wherefore $v_5 \in [v_4]^{cs}_{S,(\psi,\omega)}$. We see that $[v_4]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_4]^{cs}_{S,(\psi,\omega)} \subseteq F(a)$. Then $v_4 \in F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Whence $F\rfloor^{cs}_{S,(\psi,\omega)}(a) \subseteq F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. We deduce that $\mathfrak{F}\rfloor^{cs}_{S,(\psi,\omega)} \widetilde{\Subset} \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$. The proof is complete. □

Proposition 3.20. *Let $(V,V,[V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I with the property that $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation over $V \times V$, and $(\varphi,\chi) \in \mathcal{P}(-I)\backslash\{-I\} \times \mathcal{P}(+I)\backslash\{\emptyset\}$. If $\mathfrak{F} := (F,\prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then $\mathfrak{F}$ is a definable hypersoft set within $(V,V,[V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.*

Proof. Assume that $\mathfrak{F}$ is a hypersoft set over V. Then, by Remark 3.14, we get $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} \mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Let a be an element in $\prod_{i\in\mathbb{N}} A_i$, and let $v_1 \in F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Then $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset$. Thus, there exists $v_2 \in V$ such that $v_2 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in F(a)$. By Proposition 3.10, we have $v_1 = v_2$. We must prove that $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)$. Let $v_3 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then, by Proposition 3.10, we have $v_1 = v_3$. Hence $v_3 \in F(a)$, which implies that $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)$. Thus $v_1 \in F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Whence $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \subseteq F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Therefore, we get $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Thus $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is equal to $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$. As a consequence, $\mathfrak{F}$ is a definable hypersoft set within $(V,[V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$. □

As mentioned above, we shall present Example 3.21 below.

Example 3.21. Let $(V,V,[V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\emptyset,+I)})$ be a given approximation space type I, where $V = \{v_n := n^3+1 : n \text{ is a natural number}\}$ and $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation over $V \times V$ defined by

$$(R(k))^+(\acute{v},\grave{v}) = \begin{cases} +I, & \text{if } \acute{v} \leqslant \grave{v}, \\ \emptyset, & \text{if } \acute{v} > \grave{v}, \end{cases} \quad \text{and} \quad (R(k))^-(\acute{v},\grave{v}) = \begin{cases} \emptyset, & \text{if } \acute{v} \leqslant \grave{v}, \\ -I, & \text{if } \acute{v} > \grave{v}, \end{cases}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i, \acute{v},\grave{v} \in V$. We observe that if n is a natural number, then $[v_n]^{cs}_{\mathfrak{R},(\emptyset,+I)} = \{v_n\}$. It is true that if $\mathfrak{F} := (F,\prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then it is easy to see that $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\emptyset,+I)}$, $\mathfrak{F}$ and $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\emptyset,+I)}$ are identical. This implies that $\mathfrak{F}$ is a definable hypersoft set within $(V,V,[V]^{cs}_{\mathfrak{R},(\emptyset,+I)})$.

Definition 3.22. Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type I. Let f be a fuzzy subset of V. An upper rough approximation of f within the triple $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is defined by the fuzzy subset $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ of V, where

$$\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{v}) = \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\}$$

for all $\acute{v} \in V$. A lower rough approximation of f within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is defined by the fuzzy subset $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ of V, where

$$\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{v}) = \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\}$$

for all $\acute{v} \in V$. The fuzzy subset f is called a definable fuzzy set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ if $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is equal to $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$; otherwise f is called a rough fuzzy set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

Now, we consider the following example.

Example 3.23. Based on $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),([-1,-0.7),[0.5,1])})$ in Example 3.13, let f be a fuzzy subset of V defined by

$$f(v) = 1 - \frac{1}{v}$$

for all $v \in V$. Observe that if n is a natural number, then $\ulcorner f \urcorner^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(v_n) = 1$ and

$$\llcorner f \lrcorner^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(v_n) = \begin{cases} 0, & \text{if } v_n \in [v_{3n-2}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}, \\ \frac{1}{2}, & \text{if } v_n \in [v_{3n-1}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}, \\ \frac{2}{3}, & \text{if } v_n \in [v_{3n}]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}. \end{cases}$$

Therefore, it is easy to see that f is a rough fuzzy set within $(V, W, [V]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])})$.

Remark 3.24. Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type I. If f is a fuzzy subset of V, then we observe that $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq f \subseteq \ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

From the model of Definition 3.22, we have some basic properties as the following three propositions.

Proposition 3.25. *Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type I. If f is a fuzzy subset of V, then we have the following statements.*

(i) *If $f = 1_V$, then f is equal to $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Moreover, the fuzzy subset f is a definable fuzzy set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.*

(ii) *If $f = 0_V$, then f is equal to $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Moreover, the fuzzy subset f is a definable fuzzy set within $(V, W, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.*

(iii) $\ulcorner (\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}) \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(iv) $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \llcorner (\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}) \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(v) $f \subseteq \llcorner (\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}) \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

(vi) $\ulcorner (\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}) \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq f$.

(vii) $\llcorner f' \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} = (\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)})'$.

(viii) $\ulcorner f' \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} = (\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)})'$.

Proposition 3.26. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* f *and* g *be fuzzy subsets of* V*. Then, we have the following statements:*

(i) $\ulcorner f \cap g \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap \ulcorner g \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $\llcorner f \cap g \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} = \llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap \llcorner g \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iii) $\ulcorner f \cup g \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} = \ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \cup \ulcorner g \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iv) $\llcorner f \cup g \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \supseteq \llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \cup \llcorner g \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proposition 3.27. *Let* $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be an approximation space type I, and let* f *and* g *be fuzzy subsets of* V*. If* $f \subseteq g$*, then we have the following statements:*

(i) $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \ulcorner g \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \llcorner g \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proposition 3.28. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(V, V, [V]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\psi,\omega)})$ *be two given approximation spaces type I with the property that* $\mathfrak{R}$ *and* $\mathfrak{S}$ *are hesitant bipolar-valued fuzzy hypersoft equivalence relations,* $\mathfrak{R} \subseteq_{ir} \mathfrak{S}$ *and* $(\psi, \omega) \subseteq_{sr} (\varphi, \chi)$*. If* f *is a fuzzy subset of* V*, then* $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \ulcorner f \urcorner^{cs}_{S,(\psi,\omega)}$ *and* $\llcorner f \lrcorner^{cs}_{S,(\psi,\omega)} \subseteq \llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proof. Suppose f is a fuzzy subset of V. Let $\acute{v} \in V$. Then, by Proposition 3.9, we have

$$\begin{aligned}
\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{v}) &= \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\} \\
&= \sup_{\grave{v}\in V}\{f(\grave{v}) : (R(k))^{-}(\acute{v},\grave{v}) \subseteq \varphi \text{ and } (R(k))^{+}(\acute{v},\grave{v}) \supseteq \chi\} \\
&\leqslant \sup_{\grave{v}\in V}\{f(\grave{v}) : (S(k))^{-}(\acute{v},\grave{v}) \subseteq \psi \text{ and } (S(k))^{+}(\acute{v},\grave{v}) \supseteq \omega\} \\
&= \sup\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{S,(\psi,\omega)}\} \\
&= \ulcorner f \urcorner^{cs}_{S,(\psi,\omega)}(\acute{v}).
\end{aligned}$$

Thus $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq \ulcorner f \urcorner^{cs}_{S,(\psi,\omega)}$. Now

$$\begin{aligned}
\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{v}) &= \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\} \\
&= \inf_{\grave{v}\in V}\{f(\grave{v}) : (R(k))^{-}(\acute{v},\grave{v}) \subseteq \varphi \text{ and } (R(k))^{+}(\acute{v},\grave{v}) \supseteq \chi\} \\
&\geqslant \inf_{\grave{v}\in V}\{f(\grave{v}) : (S(k))^{-}(\acute{v},\grave{v}) \subseteq \psi \text{ and } (S(k))^{+}(\acute{v},\grave{v}) \supseteq \omega\} \\
&= \inf\{f(\grave{v}) : \grave{v} \in [\acute{v}]^{cs}_{S,(\psi,\omega)}\} \\
&= \llcorner f \lrcorner^{cs}_{S,(\psi,\omega)}(\acute{v}).
\end{aligned}$$

Therefore $\llcorner f \lrcorner^{cs}_{S,(\psi,\omega)} \subseteq \llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$. □

Based on Proposition 3.10, it is obvious that the following proposition can be gotten.

Proposition 3.29. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be an approximation space type I with the property that* $\mathfrak{R}$ *is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation over* $V \times V$*, and* $(\varphi, \chi) \in \mathcal{P}(-I)\backslash\{-I\} \times \mathcal{P}(+I)\backslash\{\emptyset\}$*. If* f *is a fuzzy subset of* V*, then* f *is a definable fuzzy set within* $(V, V, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$.

Definition 3.30. Let f be a fuzzy subset of V and $\iota \in +I$. A $(f, \iota, \geqslant)$-relative whole hypersoft set over V with respect to A is denoted by $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$, where $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}$ is a set valued-mapping given by $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) = V^{(f,\iota,\geqslant)}$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

Proposition 3.31. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type I, and let* f *be a fuzzy subset of* V *and* $\iota \in +I$. *Then, we have the following statements:*

(i) $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)} = (V^{(\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$;

(ii) $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = (V^{(\llcorner f\lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$.

Proof.

(i) Let $a \in \prod_{i\in\mathbb{N}} A_i$. Then, we consider the following.

$$\begin{aligned} v_1 \in V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}\rceil^{cs}_{R,(\varphi,\chi)}(a) &\Leftrightarrow [v_1]^{cs}_{R,(\varphi,\chi)} \cap V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \neq \emptyset \\ &\Leftrightarrow [v_1]^{cs}_{R,(\varphi,\chi)} \cap V^{(f,\iota,\geqslant)} \neq \emptyset \\ &\Leftrightarrow f(v_2) \geqslant \iota \text{ for some } v_2 \in [v_1]^{cs}_{R,(\varphi,\chi)} \\ &\Leftrightarrow \sup\{f(v_2) : v_2 \in [v_1]^{cs}_{R,(\varphi,\chi)}\} \geqslant \iota \\ &\Leftrightarrow \ulcorner f\urcorner^{cs}_{R,(\varphi,\chi)}(v_1) \geqslant \iota \\ &\Leftrightarrow v_1 \in V^{(\ulcorner f\urcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)} \\ &\Leftrightarrow v_1 \in V^{(\ulcorner f\urcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a). \end{aligned}$$

Hence $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}\rceil^{cs}_{R,(\varphi,\chi)}(a) = V^{(\ulcorner f\urcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$. Therefore

$$(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rceil^{cs}_{R,(\varphi,\chi)} = (V^{(\ulcorner f\urcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i).$$

(ii) Let $a \in \prod_{i\in\mathbb{N}} A_i$. Then, we consider the following.

$$\begin{aligned} v_1 \in V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}\rfloor^{cs}_{R,(\varphi,\chi)}(a) &\Leftrightarrow [v_1]^{cs}_{R,(\varphi,\chi)} \subseteq V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \\ &\Leftrightarrow [v_1]^{cs}_{R,(\varphi,\chi)} \subseteq V^{(f,\iota,\geqslant)} \\ &\Leftrightarrow f(v_2) \geqslant \iota \text{ for all } v_2 \in [v_1]^{cs}_{R,(\varphi,\chi)} \\ &\Leftrightarrow \inf\{f(v_2) : v_2 \in [v_1]^{cs}_{R,(\varphi,\chi)}\} \geqslant \iota \\ &\Leftrightarrow \llcorner f\lrcorner^{cs}_{R,(\varphi,\chi)}(v_1) \geqslant \iota \\ &\Leftrightarrow v_1 \in V^{(\llcorner f\lrcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)} \\ &\Leftrightarrow v_1 \in V^{(\llcorner f\lrcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a). \end{aligned}$$

Therefore $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}\rfloor^{cs}_{R,(\varphi,\chi)}(a) = V^{(\llcorner f\lrcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$. Thus

$$(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rfloor^{cs}_{R,(\varphi,\chi)} = (V^{(\llcorner f\lrcorner^{cs}_{R,(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i).$$

□

4. Rough approximations for hypersoft quasi-ideals and fuzzy quasi-ideals of semigroups

In this section, we apply the definitions given above to the more specific set of semigroups. We focus to consider the upper and lower rough approximations of hypersoft quasi-ideals and fuzzy quasi-ideals of semigroups. We provide some properties under hypersoft homomorphism problems.

For the remainder of this section, V and W stand for a semigroup. Applying Definitions 2.15 to progress under the concept of set-valued functions, we shall propose the concept of hesitant bipolar-valued fuzzy hypersoft compatible relations as follows.

Definition 4.1. Let $\mathfrak{R} := (R, \prod_{i\in\mathbb{N}} A_i)$ be a hesitant bipolar-valued fuzzy hypersoft relation over $V \times V$. $\mathfrak{R}$ is called a hesitant bipolar-valued fuzzy hypersoft compatible relation if it satisfies

$$((R(a))^+(\acute{v}v, \grave{v}v) \supseteq ((R(a))^+(\acute{v}, \grave{v}), \qquad ((R(a))^+(v\acute{v}, v\grave{v}) \supseteq ((R(a))^+(\acute{v}, \grave{v}),$$
$$((R(a))^-(\acute{v}v, \grave{v}v) \subseteq ((R(a))^-(\acute{v}, \grave{v}), \qquad ((R(a))^-(v\acute{v}, v\grave{v}) \subseteq ((R(a))^-(\acute{v}, \grave{v})$$

for all $a \in \prod_{i\in\mathbb{N}} A_i, v, \acute{v}, \grave{v} \in V$.

Definition 4.2. Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I. $(V, V, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is called an approximation space type II if $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft compatible relation.

Proposition 4.3. *If* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *is a given approximation space type II, then* $([\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)})([\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ *is a subset of* $[\acute{v}\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *for all* $\acute{v}, \grave{v} \in V$.

Proof. Let $v_1, v_2 \in V$, and let $v_3 \in ([v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)})([v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)})$. Then, there exist $v_4 \in [v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $v_5 \in [v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ such that $v_3 = v_4v_5$. Observe that $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_4]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $[v_2]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_5]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Suppose $v_6 \in [v_1v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Then

$$(R(k))^-(v_1v_2, v_6) \subseteq \varphi \text{ and } (R(k))^+(v_1v_2, v_6) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and using Proposition 3.2, we obtain that $v_1 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. It follows that $v_1 \in [v_4]^{s}_{\mathfrak{R},(\varphi,\chi)}$ and $v_2 \in [v_5]^{s}_{\mathfrak{R},(\varphi,\chi)}$. We see that

$$(R(k))^-(v_4, v_1) \subseteq \varphi, \quad (R(k))^+(v_4, v_1) \supseteq \chi, \quad (R(k))^-(v_5, v_2) \subseteq \varphi, \quad (R(k))^+(v_5, v_2) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft transitive relation and a hesitant bipolar-valued fuzzy hypersoft compatible relation, we observe that

$$\begin{aligned}(R(k))^-(v_4v_5, v_1v_2) &\subseteq \bigcap_{v\in V}((R(k))^-(v_4v_5, v) \cup (R(k))^-(v, v_1v_2))\\ &\subseteq (R(k))^-(v_4v_5, v_1v_5) \cup (R(k))^-(v_1v_5, v_1v_2)\\ &\subseteq (R(k))^-(v_4, v_1) \cup (R(k))^-(v_5, v_2) \subseteq \varphi \cup \varphi = \varphi\end{aligned}$$

and

$$\begin{aligned}(R(k))^+(v_4v_5, v_1v_2) &\supseteq \bigcup_{v\in V}((R(k))^+(v_4v_5, v) \cap (R(k))^+(v, v_1v_2))\\ &\supseteq (R(k))^+(v_4v_5, v_1v_5) \cap (R(k))^+(v_1v_5, v_1v_2)\\ &\supseteq (R(k))^+(v_4, v_1) \cap (R(k))^+(v_5, v_2) \supseteq \chi \cap \chi = \chi\end{aligned}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Since $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft transitive relation, we observe that

$$\begin{aligned}(R(k))^-(v_4v_5, v_6) &\subseteq \bigcap_{v\in V}((R(k))^-(v_4v_5, v) \cup (R(k))^-(v, v_6))\\ &\subseteq (R(k))^-(v_4v_5, v_1v_2) \cup (R(k))^-(v_1v_2, v_6) \subseteq \varphi \cup \varphi = \varphi\end{aligned}$$

and

$$\begin{aligned}(R(k))^+(v_4v_5, v_6) &\supseteq \bigcup_{v\in V}((R(k))^+(v_4v_5, v) \cap (R(k))^+(v, v_6))\\ &\supseteq (R(k))^+(v_4v_5, v_1v_2) \cap (R(k))^+(v_1v_2, v_6) \supseteq \chi \cap \chi = \chi\end{aligned}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. We obtain that $v_6 \in [v_4v_5]^s_{\mathfrak{R},(\varphi,\chi)}$. It is true that $[v_1v_2]^s_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_4v_5]^s_{\mathfrak{R},(\varphi,\chi)}$. On the other hand, we can show that $[v_4v_5]^s_{\mathfrak{R},(\varphi,\chi)} \subseteq [v_1v_2]^s_{\mathfrak{R},(\varphi,\chi)}$. Thus $[v_1v_2]^s_{\mathfrak{R},(\varphi,\chi)} = [v_4v_5]^s_{\mathfrak{R},(\varphi,\chi)}$. Hence $v_3 = v_4v_5 \in [v_1v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. This verifies that $([v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)})([v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}) \subseteq [v_1v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. □

According to Proposition 4.3, we indicate that it does not hold in general for an equality case. In what follows, we shall consider the following example.

Let the triple $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\emptyset,+I)})$ be a given approximation space type II, where $V = \{v_n := n \ : \ n \text{ is a natural number}\}$ is a semigroup under the usual addition $+$, and $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation and a hesitant bipolar-valued fuzzy hypersoft compatible relation over $V \times V$ defined by

$$(R(k))^+(\acute{v}, \grave{v}) = \begin{cases} +I, & \text{if } \acute{v} = \grave{v},\\ \emptyset, & \text{if } \acute{v} \neq \grave{v},\end{cases} \quad \text{and} \quad (R(k))^-(\acute{v}, \grave{v}) = \begin{cases} \emptyset, & \text{if } \acute{v} = \grave{v},\\ -I, & \text{if } \acute{v} \neq \grave{v}\end{cases}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i, \acute{v}, \grave{v} \in V$. Observe that if n is a natural number, then $[v_n]^{cs}_{\mathfrak{R},(\emptyset,+I)} = \{n\}$. Thus, we get that $[\acute{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)} + [\grave{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)} = [\acute{v} + \grave{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)}$ for all $\acute{v}, \grave{v} \in V$. Indeed, assume m and n are natural numbers. Then

$$[v_m]^{cs}_{\mathfrak{R},(\emptyset,+I)} + [v_n]^{cs}_{\mathfrak{R},(\emptyset,+I)} = \{m\} + \{n\} = \{m + n\} = [m + n]^{cs}_{\mathfrak{R},(\emptyset,+I)} = [v_m + v_n]^{cs}_{\mathfrak{R},(\emptyset,+I)}.$$

Thus, we observe that the example can be considered as a specific case of Proposition 4.3, i.e., $[\acute{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)} + [\grave{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)} = [\acute{v} + \grave{v}]^{cs}_{\mathfrak{R},(\emptyset,+I)}$ for all $\acute{v}, \grave{v} \in V$. Therefore, this example leads to the following definition.

Definition 4.4. Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type II. $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is called a complete collection induced by $\mathfrak{R}$ if $([\acute{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)})([\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}) = [\acute{v}\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $\acute{v}, \grave{v} \in V$. In this way, we say that $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft complete relation. Moreover, $(V, V, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ is called an approximation space type III if $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft complete relation.

For hypersoft sets $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ over V, a restricted product of $\mathfrak{F}$ and $\mathfrak{G}$, denoted by $\mathfrak{F}\tilde{\circledcirc}\mathfrak{G}$, is defined as a hypersoft set $(H, \prod_{i\in\mathbb{N}} C_i)$, where $\prod_{i\in\mathbb{N}} C_i = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H(c) = (F(c))(G(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_i$.

Proposition 4.5. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type II. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ are hypersoft sets over V, then $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\tilde{\circledcirc}\mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\tilde{\Subset}(\mathfrak{F}\tilde{\circledcirc}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$.*

Proof. Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are hypersoft sets over V. Let $\mathfrak{H}_1 := (H_1, \prod_{i\in\mathbb{N}} C_{1_i}) = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\tilde{\circledcirc}\mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_1(c) = (F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))(G]^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_{1_i}$. We let $\mathfrak{H}_2 := (H_2, \prod_{i\in\mathbb{N}} C_{2_i}) = \mathfrak{F}\tilde{\circledcirc}\mathfrak{G}$. Then $\prod_{i\in\mathbb{N}} C_{2_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_2(c) = (F(c))(G(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_{2_i}$. Next, we shall prove that $\mathfrak{H}_1\tilde{\Subset}\mathfrak{H}_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Obviously $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} C_{2_i}$. Let $\acute{c}$

be an element in $\prod_{i\in\mathbb{N}} C_{1_i}$, and let $\nu_1 \in H_1(\acute{c})$. Then $\nu_1 \in (F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}))(G]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}))$. Thus, there exist $\nu_2 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ and $\nu_3 \in G]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ such that $\nu_1 = \nu_2\nu_3$. Hence, we get that $[\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(\acute{c}) \neq \emptyset$ and $[\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap G(\acute{c}) \neq \emptyset$. Thus, there exist $\nu_4, \nu_5 \in V$ such that $\nu_4 \in [\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(\acute{c})$ and $\nu_5 \in [\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap G(\acute{c})$. By Proposition 4.3, we obtain that

$$\nu_4\nu_5 \in ([\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)})([\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)}) \subseteq [\nu_2\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

Observe that $\nu_4\nu_5 \in (F(\acute{c}))(G(\acute{c}))$. Hence

$$[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap H_2(\acute{c}) = [\nu_2\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap (F(\acute{c}))(G(\acute{c})) \neq \emptyset.$$

Thus $\nu_1 \in H_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$. Whence $H_1(\acute{c}) \subseteq H_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$. Therefore $\mathfrak{H}_1 \widetilde{\Subset} \mathfrak{H}_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. It follows that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\circledcirc} \mathfrak{G}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} (\mathfrak{F}\widetilde{\circledcirc}\mathfrak{G})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ as desired. □

Proposition 4.6. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be an approximation space type III. If* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *and* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *are hypersoft sets over* V, *then* $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\circledcirc} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} (\mathfrak{F}\widetilde{\circledcirc}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proof. Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are hypersoft sets over V. Let $\mathfrak{H}_1 := (H_1, \prod_{i\in\mathbb{N}} C_{1_i}) = \mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\circledcirc} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_1(c) = (F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))(G\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_{1_i}$. Let $\mathfrak{H}_2 := (H_2, \prod_{i\in\mathbb{N}} C_{2_i}) = \mathfrak{F}\widetilde{\circledcirc}\mathfrak{G}$. Then $\prod_{i\in\mathbb{N}} C_{2_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_2(c) = (F(c))(G(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_{2_i}$. We shall show that $\mathfrak{H}_1 \widetilde{\Subset} \mathfrak{H}_2\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Clearly $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} C_{2_i}$. Let $\acute{c} \in \prod_{i\in\mathbb{N}} C_{1_i}$, and let $\nu_1 \in H_1(\acute{c})$. Then, we have $\nu_1 \in (F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}))(G\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}))$. Thus, there exist $\nu_2 \in F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ and $\nu_3 \in G\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ such that $\nu_1 = \nu_2\nu_3$. Thus, we obtain that $[\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(\acute{c})$ and $[\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq G(\acute{c})$. Now

$$[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} = [\nu_2\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)} = ([\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)})([\nu_3]^{cs}_{\mathfrak{R},(\varphi,\chi)}) \subseteq (F(\acute{c}))(G(\acute{c})) = H_2(\acute{c}).$$

Thus $\nu_1 \in H_2\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$. Hence $H_1(\acute{c}) \subseteq H_2\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$. It follows that $\mathfrak{H}_1 \widetilde{\Subset} \mathfrak{H}_2\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$. Therefore $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\circledcirc} \mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} (\mathfrak{F}\widetilde{\circledcirc}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ as required. □

Under the restricted product of two hypersoft sets as introduced above, if we put $A = \prod_{i\in\mathbb{N}} A_i$, then $(F, \prod_{i\in\mathbb{N}} A_i)$ in items (i)-(iv) of Definitions 2.26 and 2.28 is called a hypersoft left ideal (resp., a hypersoft right ideal, a hypersoft ideal, a hypersoft quasi-ideal and a hypersoft semigroup) over V. Furthermore, it is easy to see that $\mathfrak{F}$ is a hypersoft left ideal (resp., a hypersoft right ideal, a hypersoft ideal, and a hypersoft quasi-ideal) if and only if $F(a)$ is either empty or a left ideal (resp., a right ideal, an ideal and a quasi-ideal) of V for all $a \in \prod_{i\in\mathbb{N}} A_i$ due to Proposition 2.27.

We now come to the main results.

Theorem 4.7. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be an approximation space type II. If* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over* V, *then* $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over* V.

Proof. Suppose that $\mathfrak{F}$ is a hypersoft left ideal over V. Then $\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} \widetilde{\circledcirc} \mathfrak{F} \widetilde{\Subset} \mathfrak{F}$. Using Propositions 3.16 (i), 3.18, and 4.5, we have

$$\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} \widetilde{\circledcirc} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\circledcirc} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} (\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} \widetilde{\circledcirc} \mathfrak{F})]^{cs}_{\mathfrak{R},(\varphi,\chi)} \widetilde{\Subset} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

Hence $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal over V. Similarly, we can prove that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft right ideal over V. It follows that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft ideal over V as desired. □

Proposition 4.8. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be a given approximation space type II, where* V *is regular. If* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *is a hypersoft right ideal and* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *is a hypersoft left ideal over* V, *then the following items are identical:*

(i) $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Cap}\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iii) $(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{G})\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iv) $(\mathfrak{F}\widetilde{\Cap}\mathfrak{G})\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proof. Suppose $\mathfrak{F}$ is a hypersoft right ideal and $\mathfrak{G}$ is a hypersoft left ideal over V. Then, by Theorem 4.7, it follows that $\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft right ideal and $\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal over V. Thus $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is either empty or a right ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$, and also have $G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(b)$ is either empty or a left ideal of V for all $b \in \prod_{i\in\mathbb{N}} B_i$. Let $\mathfrak{H}_1 := (H_1, \prod_{i\in\mathbb{N}} C_{1_i}) = \mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Cap}\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_1(c) = F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(c) \cap G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(c)$ for all $c \in \prod_{i\in\mathbb{N}} C_{1_i}$. Let $\mathfrak{H}_2 := (H_2, \prod_{i\in\mathbb{N}} C_{2_i}) = \mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{G}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $\prod_{i\in\mathbb{N}} C_{2_i} = \prod_{i\in\mathbb{N}} A_i \cap \prod_{i\in\mathbb{N}} B_i$ and $H_2(c) = (F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))(G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(c))$ for all $c \in \prod_{i\in\mathbb{N}} C_{2_i}$. Note that $\prod_{i\in\mathbb{N}} C_{1_i} = \prod_{i\in\mathbb{N}} C_{2_i}$. Let $\acute{c} \in \prod_{i\in\mathbb{N}} C_{1_i}$ be given. Obviously, $H_1(\acute{c}) = H_2(\acute{c})$ if we consider several empty set cases. Suppose $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ and $G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})$ are non-empty. Thus, by Proposition 2.5, we get that

$$H_1(\acute{c}) = F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}) \cap G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}) = (F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c}))(G\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{c})) = H_2(\acute{c}).$$

Therefore $\mathfrak{H}_1 = \mathfrak{H}_2$. It follows that (i) and (ii) are identical. Using Proposition 2.5, once again, it is easy to prove that (iii) and (iv) are identical. From Proposition 3.17 (i), we obtain that (iv) is a hypersoft subset of (ii). By Proposition 4.5, we get that (i) is a hypersoft subset of (iii). It follows that the statement is true as required. □

Theorem 4.9. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type III. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over V, then $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over V.*

Proof. Suppose that $\mathfrak{F}$ is a hypersoft left ideal over V. Then $\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}\widetilde{\Subset}\mathfrak{F}$. Using Propositions 3.16 (i), 3.18, and 4.6, we have

$$\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Subset}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

Hence $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal over V. Similarly, we can verify that $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft right ideal over V. This implies that $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft ideal over V. □

Proposition 4.10. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type III, where V is regular. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft right ideal and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ is a hypersoft left ideal over V, then the following statements are identical:*

(i) $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(ii) $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Cap}\mathfrak{G}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iii) $(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$;

(iv) $(\mathfrak{F}\widetilde{\Cap}\mathfrak{G})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Proof. According to Propositions 2.5, 3.17 (ii), 4.6, and Theorem 4.9, we can prove that the statement holds. □

Theorem 4.11. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type II, where V is regular. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V, then $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V.*

Proof. Suppose $\mathfrak{F}$ is a hypersoft quasi-ideal over V. Then $(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})\widetilde{\Subset}\mathfrak{F}$. Notice that $\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}$ and $\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}$ can be viewed as a hypersoft right ideal and a hypersoft left ideal over V, respectively. By Theorem 4.7, we obtain that $(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ are a hypersoft right ideal and a hypersoft left ideal over V, respectively. By Propositions 3.16 (i), 3.18, 4.5, and 4.8, we observe that

$$
\begin{aligned}
&(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})\\
&\quad= (\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}]^{cs}_{\mathfrak{R},(\varphi,\chi)})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})\\
&\quad\widetilde{\Subset}(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})]^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad= (\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})]^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad\widetilde{\Subset}((\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\circledcirc}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}))]^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad= ((\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}))]^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad\widetilde{\Subset}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}.
\end{aligned}
$$

This implies that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V. □

Theorem 4.12. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type III. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V, then $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V.*

Proof. Assume $\mathfrak{F}$ is a hypersoft quasi-ideal over V. Then $(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})\widetilde{\Subset}\mathfrak{F}$. Using Propositions 3.16 (i), 3.17 (ii), 3.18, and 4.6, we see that

$$
\begin{aligned}
&(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})\\
&\quad= (\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})\\
&\quad\widetilde{\Subset}(\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F})\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad= ((\mathfrak{F}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}}\widetilde{\circledcirc}\mathfrak{F}))\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}\\
&\quad\widetilde{\Subset}\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}.
\end{aligned}
$$

It follows that $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal, and the proof is complete. □

We next provide the characterization of fuzzy quasi-ideals of semigroups in terms of hypersoft sets.

Proposition 4.13. *Let f be a fuzzy subset of V. Then, we have the following statements:*

(i) *f is a fuzzy ideal of V if and only if $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft ideal over V for all $\iota \in +I$;*

(ii) *f is a fuzzy quasi-ideal of V if and only if $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V for all $\iota \in +I$.*

Proof.

(i) Assume f is a fuzzy ideal of V. Let $\iota \in +I$. We shall prove that $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is either empty or an ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Now, let $a \in \prod_{i\in\mathbb{N}} A_i$, and assume $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \neq \emptyset$. Then, by Proposition 2.11, we get that $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is an ideal of V. Therefore $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft ideal over V. On the other hand, suppose $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft ideal over V for all $\iota \in +I$. We see that for all $a \in \prod_{i\in\mathbb{N}} A_i, \iota \in +I$, if $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \neq \emptyset$, then $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is an ideal of V. From Proposition 2.11, once again, it follows that f is a fuzzy ideal of V.

(ii) Suppose f is a fuzzy quasi-ideal of V. Let $\iota \in +I$ be given. We shall prove that $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is either empty or a quasi-ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Now, we let $a \in \prod_{i\in\mathbb{N}} A_i$ be given. Assume that $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \neq \emptyset$. Then, by Proposition 2.12, we get that $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is a quasi-ideal of V. Therefore $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V. On the other hand, assume that $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V for all $\iota \in +I$. We see that for all $a \in \prod_{i\in\mathbb{N}} A_i, \iota \in +I$, if $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a) \neq \emptyset$, then $V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)$ is a quasi-ideal of V. Using Proposition 2.12, once again, we obtain that f is a fuzzy quasi-ideal of V. □

Theorem 4.14. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type II. If f is a fuzzy ideal of V, then $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy ideal of V.*

Proof. Suppose f is a fuzzy ideal of V. Then, by Proposition 4.13 (i), we have $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft ideal over V for all $\iota \in +I$. As showed in the proof of Theorem 4.7, it holds that $\ulcorner(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft ideal over V for all $\iota \in +I$. By Proposition 3.31 (i), we get that $(V^{(\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft ideal over V for all $\iota \in +I$. From Proposition 4.13 (i), once again, we obtain that $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy ideal of V. □

Theorem 4.15. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type III. If f is a fuzzy ideal of V, then $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy ideal of V.*

Proof. We can verify that the statement is true by using Propositions 3.31 (ii), 4.13 (i), and Theorem 4.9. □

Theorem 4.16. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type II, where V is regular. If f is a fuzzy quasi-ideal of V, then $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy quasi-ideal of V.*

Proof. Assume f is a fuzzy quasi-ideal of V. Then, by Proposition 4.13 (ii), we have $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V for all $\iota \in +I$. Thus, by Theorem 4.11, it follows that $\ulcorner(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V for all $\iota \in +I$. By Proposition 3.31 (i), we obtain that $(V^{(\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft quasi-ideal over V for all $\iota \in +I$. Using Proposition 4.13 (ii), once again, it follows that $\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy quasi-ideal of V. □

Theorem 4.17. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type III. If f is a fuzzy quasi-ideal of V, then $\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy quasi-ideal of V.*

Proof. Based on Propositions 3.31 (ii), 4.13 (ii), and Theorem 4.12, we can show that the statement holds. □

Based on Definition 2.29, if we put $A = \prod_{i\in\mathbb{N}} A_i$ and $B = \prod_{i\in\mathbb{N}} B_i$ such that $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ and $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ are hypersoft semigroups over V and W, respectively, then we call $(\Gamma, \Lambda)_h$ a hypersoft homomorphism from $\mathfrak{F}$ to $\mathfrak{G}$.

Given a hypersoft set $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ over V with respect to $\prod_{i\in\mathbb{N}} A_i$, a hypersupport of $\mathfrak{F}$ is denoted by $\mathrm{Hsupp}(\mathfrak{F})$, where

$$\mathrm{Hsupp}(\mathfrak{F}) := \{a \in \prod_{i\in\mathbb{N}} A_i : F(a) \neq \emptyset\}.$$

As introduced above, these definitions lead to the following proposition under hypersoft homomorphism problems.

Proposition 4.18. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I, and let* $(\Gamma, \Lambda)_h$ *be a hypersoft homomorphism from a hypersoft semigroup* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *over* W, *where*

$$(R(k))^-(\acute{v}, \grave{v}) = (S(k))^-(\Gamma(\acute{v}), \Gamma(\grave{v})) \tag{4.1}$$

and

$$(R(k))^+(\acute{v}, \grave{v}) = (S(k))^+(\Gamma(\acute{v}), \Gamma(\grave{v})) \tag{4.2}$$

for all $k \in \prod_{i\in\mathbb{N}} K_i, \acute{v}, \grave{v} \in V$. *Then, we have following statements:*

(i) *for all* $\acute{v}, \grave{v} \in V, \acute{v} \in [\grave{v}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *and* $\Gamma(\acute{v}) \in [\Gamma(\grave{v})]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *are equivalent;*

(ii) $\Gamma(F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) = G\rceil^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$ *for all* $a \in \prod_{i\in\mathbb{N}} A_i$;

(iii) $\Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) \subseteq G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$ *for all* $a \in \prod_{i\in\mathbb{N}} A_i$;

(iv) *if* Γ *is injective, then* $\Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$ *is equal to* $G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$ *for all* $a \in \prod_{i\in\mathbb{N}} A_i$;

(v) $\Lambda(\mathrm{Hsupp}(\mathfrak{F}\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)})) = \mathrm{Hsupp}(\mathfrak{G}\rceil^{cs}_{\mathfrak{S},(\varphi,\chi)})$;

(vi) $\Lambda(\mathrm{Hsupp}(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)})) \subseteq \mathrm{Hsupp}(\mathfrak{G}\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)})$;

(vii) *if* Γ *is injective, then* $\Lambda(\mathrm{Hsupp}(\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}))$ *is equal to* $\mathrm{Hsupp}(\mathfrak{G}\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)})$;

(viii) $\mathfrak{R}$ *is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft symmetric relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft compatible relation if and only if* $\mathfrak{S}$ *is a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft symmetric relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft compatible relation, respectively.*

(ix) *if* $\mathfrak{R}$ *is a bipolar fuzzy perfect antisymmetric relation and a bipolar fuzzy complete relation, then* $\mathfrak{S}$ *is a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation and a bipolar fuzzy complete relation, respectively.*

(x) *if* Γ *is injective, then* $\mathfrak{R}$ *is a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation and a bipolar fuzzy complete relation if and only if* $\mathfrak{S}$ *is a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation and a bipolar fuzzy complete relation, respectively.*

Proof. In this proposition, we shall check (i)-(v). The proofs of remaining items (vi)-(x) are straightforward, so we omit it.

(i) In order to prove the argument, we let v_1 and v_2 be given in V. Suppose that $v_1 \in [v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then $[v_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Note that $\Gamma(v_1), \Gamma(v_2) \in W$. Now, we let $w_1 \in [\Gamma(v_1)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Since Γ is surjective, there exists $v_3 \in V$ such that $\Gamma(v_3) = w_1$. Observe that

$$(R(k))^-(v_1, v_3) = (S(k))^-(\Gamma(v_1), \Gamma(v_3)) \subseteq \varphi$$

and

$$(R(k))^+(v_1, v_3) = (S(k))^+(\Gamma(v_1), \Gamma(v_3)) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Thus $v_3 \in [v_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. It follows that $v_3 \in [v_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Now

$$(S(k))^-(\Gamma(v_2), \Gamma(v_3)) = (R(k))^-(v_2, v_3) \subseteq \varphi$$

and

$$(S(k))^{+}(\Gamma(\nu_2),\Gamma(\nu_3)) = (R(k))^{+}(\nu_2,\nu_3) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Whence $\Gamma(\nu_3) \in [\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Thus, we get $[\Gamma(\nu_1)]^{s}_{\mathfrak{S},(\varphi,\chi)} \subseteq [\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Conversely, we can prove that $[\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)} \subseteq [\Gamma(\nu_1)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Hence $[\Gamma(\nu_1)]^{s}_{\mathfrak{S},(\varphi,\chi)} = [\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. It follows that $\Gamma(\nu_1) \in [\Gamma(\nu_2)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. On the other hand, suppose that $\Gamma(\nu_1) \in [\Gamma(\nu_2)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. Whence $[\Gamma(\nu_1)]^{s}_{\mathfrak{S},(\varphi,\chi)} = [\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Now, we let $\nu_4 \in [\nu_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Then

$$(S(k))^{-}(\Gamma(\nu_1),\Gamma(\nu_4)) = (R(k))^{-}(\nu_1,\nu_4) \subseteq \varphi$$

and

$$(S(k))^{+}(\Gamma(\nu_1),\Gamma(\nu_4)) = (R(k))^{+}(\nu_1,\nu_4) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Thus $\Gamma(\nu_4) \in [\Gamma(\nu_1)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Hence $\Gamma(\nu_4) \in [\Gamma(\nu_2)]^{s}_{\mathfrak{S},(\varphi,\chi)}$. Now

$$(R(k))^{-}(\nu_2,\nu_4) = (S(k))^{-}(\Gamma(\nu_2),\Gamma(\nu_4)) \subseteq \varphi$$

and

$$(R(k))^{+}(\nu_2,\nu_4) = (S(k))^{+}(\Gamma(\nu_2),\Gamma(\nu_4)) \supseteq \chi$$

for all $k \in \prod_{i\in\mathbb{N}} K_i$. Thus, we get that $\nu_4 \in [\nu_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. It follows that $[\nu_1]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [\nu_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Conversely, we can show that $[\nu_2]^{s}_{\mathfrak{R},(\varphi,\chi)} \subseteq [\nu_1]^{s}_{\mathfrak{R},(\varphi,\chi)}$, which yields $[\nu_1]^{s}_{\mathfrak{R},(\varphi,\chi)} = [\nu_2]^{s}_{\mathfrak{R},(\varphi,\chi)}$. Consequently $\nu_1 \in [\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. The proof is complete.

(ii) Let $a \in \prod_{i\in\mathbb{N}} A_i$, and let $w_1 \in \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. Then, there exists $\nu_1 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ such that $\Gamma(\nu_1) = w_1$. Observe that $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset$. There exists $\nu_2 \in V$ such that $\nu_2 \in [\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $\nu_2 \in F(a)$. By item (i), we have $\Gamma(\nu_2) \in [\Gamma(\nu_1)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ and $\Gamma(\nu_2) \in \Gamma(F(a))$. Since $\Gamma(F(a)) = G(\Lambda(a))$, we have $\Gamma(\nu_2) \in G(\Lambda(a))$. Now

$$[w_1]^{cs}_{\mathfrak{S},(\varphi,\chi)} \cap G(\Lambda(a)) = [\Gamma(\nu_1)]^{cs}_{\mathfrak{S},(\varphi,\chi)} \cap G(\Lambda(a)) \neq \emptyset.$$

Thus, we get $w_1 \in G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$. It follows that $\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) \subseteq G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$. Conversely, we let $w_2 \in G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$. Then $[w_2]^{cs}_{\mathfrak{S},(\varphi,\chi)} \cap G(\Lambda(a)) \neq \emptyset$. Thus, there exists $w_3 \in W$ such that $w_3 \in [w_2]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ and $w_3 \in G(\Lambda(a))$. Since $\Gamma(F(a)) = G(\Lambda(a))$, we have $w_3 \in \Gamma(F(a))$. There exists $\nu_3 \in F(a)$ such that $\Gamma(\nu_3) = w_3$. Since Γ is surjective, there exists $\nu_4 \in V$ such that $\Gamma(\nu_4) = w_2$. We see that $\Gamma(\nu_3) \in [\Gamma(\nu_4)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. By item (i), we get $\nu_3 \in [\nu_4]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Hence $[\nu_4]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(a) \neq \emptyset$. Thus $\nu_4 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Whence, we obtain that $w_2 = \Gamma(\nu_4) \in \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. It follows that $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a)) \subseteq \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. This implies that $\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$ as required.

(iii) Let $a \in \prod_{i\in\mathbb{N}} A_i$, and let $w_1 \in \Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. Then, there exists $\nu_1 \in F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ such that $\Gamma(\nu_1) = w_1$. We observe that $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)$. Now, we let $w_2 \in [w_1]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. Then, there exists $\nu_2 \in V$ such that $\Gamma(\nu_2) = w_2$. Thus $\Gamma(\nu_2) \in [\Gamma(\nu_1)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. By item (i), we get $\nu_2 \in [\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, and so $\nu_2 \in F(a)$. Thus $\Gamma(\nu_2) \in \Gamma(F(a))$. Since $\Gamma(F(a)) = G(\Lambda(a))$, we have $w_2 = \Gamma(\nu_2) \in G(\Lambda(a))$. Whence $[w_1]^{cs}_{\mathfrak{S},(\varphi,\chi)} \subseteq G(\Lambda(a))$, which yields $w_1 \in G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$. Therefore, it follows that $\Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) \subseteq G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$.

(iv) Let $a \in \prod_{i\in\mathbb{N}} A_i$, and let $w_1 \in G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$. Then $[w_1]^{cs}_{\mathfrak{S},(\varphi,\chi)} \subseteq G(\Lambda(a))$. Since $\Gamma(F(a)) = G(\Lambda(a))$, we have $[w_1]^{cs}_{\mathfrak{S},(\varphi,\chi)} \subseteq \Gamma(F(a))$. Since Γ is surjective, there exists $\nu_1 \in V$ such that $\Gamma(\nu_1) = w_1$. Thus, we get $[\Gamma(\nu_1)]^{cs}_{\mathfrak{S},(\varphi,\chi)} \subseteq \Gamma(F(a))$. Now, we shall prove that $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)$. Suppose $\nu_2 \in [\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Then, by item (i), we obtain $\Gamma(\nu_2) \in [\Gamma(\nu_1)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. Thus $\Gamma(\nu_2) \in \Gamma(F(a))$. There exists $\nu_3 \in F(a)$ such that $\Gamma(\nu_2) = \Gamma(\nu_3)$. Since Γ is injective, we have $\nu_2 = \nu_3$. Observe that $\nu_2 \in F(a)$. It follows that $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \subseteq F(a)$. Therefore $\nu_1 \in F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$. Thus, we see that $w_1 = \Gamma(\nu_1) \in \Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. Thus $G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a)) \subseteq \Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$. As item (iii), we get $\Gamma(F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) = G\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(a))$ as required.

(v) Suppose that $\acute{b} \in \Lambda(\mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}))$. Then, there exists $\acute{a} \in \mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ such that $\acute{b} = \Lambda(\acute{a})$. Observe that $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a}) \neq \emptyset$. There exists $v_1 \in V$ such that $v_1 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})$. By item (ii), observe that

$$\Gamma(v_1) \in \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}).$$

Thus $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}) \neq \emptyset$. Therefore $\acute{b} \in \mathrm{Hsupp}(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)})$. Hence $\Lambda(\mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})) \subseteq \mathrm{Hsupp}(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)})$. Conversely, let $\grave{b} \in \mathrm{Hsupp}(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)})$. Then $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\grave{b}) \neq \emptyset$. Thus, there exists $w \in W$ such that $w \in G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\grave{b})$. Since Λ is surjective, there exists $\grave{a} \in \prod_{i\in\mathbb{N}} A_i$ such that $\Lambda(\grave{a}) = \grave{b}$. Using item (ii), we get that

$$w \in G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\grave{b}) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a})) = \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})).$$

Then, there exists $v_2 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$ such that $\Gamma(v_2) = w$. We observe that $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}) \neq \emptyset$. Then $\grave{a} \in \mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})$. It follows that $\grave{b} \in \Lambda(\mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}))$. Hence $\mathrm{Hsupp}(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}) \subseteq \Lambda(\mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}))$. This substantiates that $\Lambda(\mathrm{Hsupp}(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})) = \mathrm{Hsupp}(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)})$. □

Theorem 4.19. *Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ and $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be approximation spaces type I. Let $(\Gamma, \Lambda)_h$ be a given hypersoft homomorphism from a hypersoft semigroup $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ over V to a hypersoft semigroup $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ over W satisfying equations (4.1) and (4.2). Then $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over V if and only if $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over W.*

Proof. We only prove the case of a hypersoft left ideal, the other arguments are similar.

Suppose that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal over V. Then $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is either empty or a left ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Note that $K \cap B = B$. Now, we let $\acute{b} \in \prod_{i\in\mathbb{N}} B_i$. Then, there exists $\acute{a} \in \prod_{i\in\mathbb{N}} A_i$ such that $\Lambda(\acute{a}) = \acute{b}$. We consider the following two cases.

Case 1. Assume $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is empty for all $a \in \prod_{i\in\mathbb{N}} A_i$. Then $\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$ is also empty for all $a \in \prod_{i\in\mathbb{N}} A_i$. Thus, by Proposition 4.18 (ii), we obtain $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b})$ is empty. Observe that $(W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b}))(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b})) = (G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))$. Therefore $\mathfrak{W}_{W_{\prod_{i\in\mathbb{N}} B_i}} \widetilde{\circledcirc} \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)} = \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$.

Case 2. Suppose $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is a left ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Then, we have $V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) \subseteq F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. By Proposition 4.18 (ii), we observe that

$$\begin{aligned}(W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b}))(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b})) &= W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})))\\ &= (\Gamma(V))(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})))\\ &= \Gamma(V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a}))) \subseteq \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}).\end{aligned}$$

Whence $\mathfrak{W}_{W_{\prod_{i\in\mathbb{N}} B_i}} \widetilde{\circledcirc} \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)} \widetilde{\Subset} \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. This implies that $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a hypersoft left ideal.

Conversely, assume that $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a hypersoft left ideal over W. Then $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is either empty or a left ideal of W for all $b \in \prod_{i\in\mathbb{N}} B_i$. Note that $\prod_{i\in\mathbb{N}} K_i \cap \prod_{i\in\mathbb{N}} A_i = \prod_{i\in\mathbb{N}} A_i$. Let $\grave{a} \in \prod_{i\in\mathbb{N}} A_i$. Then, we consider the following two cases.

Case 1. Suppose $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is empty for all $b \in \prod_{i\in\mathbb{N}} B_i$. Then, by Proposition 4.18 (ii), we get that

$$\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a})) = \emptyset.$$

If $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}) \neq \emptyset$, then there exists $v \in V$ such that $v \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Thus $\Gamma(v) \in \emptyset$, a contradiction. Hence $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}) = \emptyset$. It follows that $(V_{\prod_{i\in\mathbb{N}} A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})) = (F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$. Hence $\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} \widetilde{\circledcirc} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} = \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Case 2. Suppose that $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is a left ideal of W for all $b \in \prod_{i\in\mathbb{N}} B_i$. Then $W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)) \subseteq G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ for all $b \in \prod_{i\in\mathbb{N}} B_i$. Now, assume that $v_1 \in (V_{\prod_{i\in\mathbb{N}} A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$. Thus, by Proposition 4.18 (ii), we see that

$$\begin{aligned}\Gamma(v_1) &\in \Gamma((V_{\prod_{i\in\mathbb{N}} A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))\\ &= \Gamma(V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))\\ &= (\Gamma(V))(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))) = W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a}))) \subseteq G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a})) = \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})).\end{aligned}$$

There is $v_2 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$ such that $\Gamma(v_1) = \Gamma(v_2)$. Using Proposition 3.4, we have $\Gamma(v_1) \in [\Gamma(v_2)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. Thus, by Proposition 4.18 (i), we obtain $v_1 \in [v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. From Proposition 3.5, we get $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} = [v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Observe that $[v_2]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(\grave{a}) \neq \emptyset$. Hence $[v_1]^{cs}_{\mathfrak{R},(\varphi,\chi)} \cap F(\grave{a}) \neq \emptyset$, which yields $v_1 \in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Thus $(V_{\prod_{i\in\mathbb{N}} A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})) \subseteq F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Therefore $\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}} A_i}} \tilde{\circledcirc} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)} \tilde{\Subset} \mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. This shows that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft left ideal over V. □

Theorem 4.20. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* $(\Gamma, \Lambda)_h$ *be a given hypersoft homomorphism from a hypersoft semigroup* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *over* W *satisfying equations* (4.1) *and* (4.2)*. If* Γ *is injective, then* $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over* V *if and only if* $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a hypersoft left ideal (resp., a hypersoft right ideal and a hypersoft ideal) over* W*.*

Proof. Based on Proposition 4.18 (iv), we can show that the statement is true. □

Theorem 4.21. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* $(\Gamma, \Lambda)_h$ *be a given hypersoft homomorphism from a hypersoft semigroup* $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $\mathfrak{G} := (G, \prod_{i\in\mathbb{N}} B_i)$ *over* W *satisfying equations* (4.1) *and* (4.2)*. If* Γ *is injective, then* $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a hypersoft quasi-ideal over* V *if and only if* $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a hypersoft quasi-ideal over* W*.*

Proof. In order to verify this statement, we shall assume that Γ is an injective function. We observe that it is easy to prove that $\Gamma((F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))V \cap V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))) = (\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)))(\Gamma(V)) \cap (\Gamma(V))(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)))$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. Suppose $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V. Then $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is either empty or a quasi-ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Note that $(\prod_{i\in\mathbb{N}} B_i \cap \prod_{i\in\mathbb{N}} K_i) \cap (\prod_{i\in\mathbb{N}} K_i \cap \prod_{i\in\mathbb{N}} B_i) = \prod_{i\in\mathbb{N}} B_i$. Now, we let $\acute{b} \in \prod_{i\in\mathbb{N}} B_i$. Then, there exists $\acute{a} \in \prod_{i\in\mathbb{N}} A_i$ such that $\Lambda(\acute{a}) = \acute{b}$. We consider the following two cases.

Case 1. Suppose $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is empty for all $a \in \prod_{i\in\mathbb{N}} A_i$. Then $\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$ is also empty for all $a \in \prod_{i\in\mathbb{N}} A_i$. From Proposition 4.18 (ii), we get $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b})$ is empty. Thus $(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))(W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b})) \cap (W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b}))(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))$ is equal to $(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))$. Therefore

$$(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)} \tilde{\circledcirc} \mathfrak{W}_{W_{\prod_{i\in\mathbb{N}} B_i}}) \tilde{\Cap} (\mathfrak{W}_{W_{\prod_{i\in\mathbb{N}} B_i}} \tilde{\circledcirc} \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}) = \mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}.$$

Case 2. Suppose that $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ is a quasi-ideal of V for all $a \in \prod_{i\in\mathbb{N}} A_i$. Then $(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))V \cap V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) \subseteq F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. Using Proposition 4.18 (ii), we observe that

$$\begin{aligned}&(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))(W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b})) \cap (W_{\prod_{i\in\mathbb{N}} B_i}(\acute{b}))(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}))\\ &= (G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})))W \cap W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})))\\ &= (\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})))(\Gamma(V)) \cap (\Gamma(V))(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})))\\ &= \Gamma((F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a}))V \cap V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})))\\ &\subseteq \Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\acute{a})) = G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\acute{b}).\end{aligned}$$

Wherefore $(\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{W_{\prod_{i\in\mathbb{N}}B_i}})\widetilde{\Cap}(\mathfrak{W}_{W_{\prod_{i\in\mathbb{N}}B_i}}\widetilde{\circledcirc}\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)})\widetilde{\Subset}\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. It follows that $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a hypersoft quasi-ideal. On the other hand, suppose $\mathfrak{G}]^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a hypersoft quasi-ideal over W. Then $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is either empty or a quasi-ideal of W for all $b\in\prod_{i\in\mathbb{N}}B_i$. Note that $(\prod_{i\in\mathbb{N}}A_i\cap\prod_{i\in\mathbb{N}}K_i)\cap(\prod_{i\in\mathbb{N}}K_i\cap\prod_{i\in\mathbb{N}}A_i)=\prod_{i\in\mathbb{N}}A_i$. Let $\grave{a}\in\prod_{i\in\mathbb{N}}A_i$. We consider the following two cases.

Case 1. Suppose $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is empty for all $b\in\prod_{i\in\mathbb{N}}B_i$. From Proposition 4.18 (ii), we get that

$$\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))=G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a}))=\emptyset.$$

Assume by contradiction that $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})\neq\emptyset$. Then, there exists $\nu\in V$ such that $\nu\in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Thus $\Gamma(\nu)\in\emptyset$, a contradiction. Hence $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})=\emptyset$. It holds that $(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))\cap(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$ is equal to $(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$. This implies that

$$(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}}A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}}A_i}}\widetilde{\circledcirc}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})=\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

Case 2. Suppose $G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ is a quasi-ideal of W for all $b\in\prod_{i\in\mathbb{N}}B_i$. Then $(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b))W\cap W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b))\subseteq G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(b)$ for all $b\in\prod_{i\in\mathbb{N}}B_i$. Assume that $\nu_1\in(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))\cap(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$. Thus, by Proposition 4.18 (ii), we see that

$$\begin{aligned}\Gamma(\nu_1)&\in\Gamma((F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))\cap(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))\\&=\Gamma((F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))V\cap V(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))\\&=(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))(\Gamma(V))\cap(\Gamma(V))(\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})))\\&=(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a})))W\cap W(G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a})))\\&\subseteq G]^{cs}_{\mathfrak{S},(\varphi,\chi)}(\Lambda(\grave{a}))\\&=\Gamma(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})).\end{aligned}$$

There exists $\nu_2\in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$ such that $\Gamma(\nu_1)=\Gamma(\nu_2)$. Using Proposition 3.4, we see that $\Gamma(\nu_1)\in[\Gamma(\nu_2)]^{cs}_{\mathfrak{S},(\varphi,\chi)}$. By Proposition 4.18 (i), we obtain that $\nu_1\in[\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. From Proposition 3.5, we get that $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}=[\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. Observe that $[\nu_2]^{cs}_{\mathfrak{R},(\varphi,\chi)}\cap F(\grave{a})\neq\emptyset$. Hence $[\nu_1]^{cs}_{\mathfrak{R},(\varphi,\chi)}\cap F(\grave{a})\neq\emptyset$. Whence $\nu_1\in F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Thus, we get $(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))\cap(V_{\prod_{i\in\mathbb{N}}A_i}(\grave{a}))(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a}))$ is a subset of $F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(\grave{a})$. Hence $(\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}\widetilde{\circledcirc}\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}}A_i}})\widetilde{\Cap}(\mathfrak{W}_{V_{\prod_{i\in\mathbb{N}}A_i}}\widetilde{\circledcirc}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)})\widetilde{\Subset}\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$. We conclude that $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a hypersoft quasi-ideal over V. □

Theorem 4.22. *Let* $(V,V,[V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}}K_i),(\varphi,\chi)})$ *and* $(W,W,[W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}}K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* $(\Gamma,\Lambda)_h$ *be a given hypersoft homomorphism from a hypersoft semigroup* $\mathfrak{F}:=(F,\prod_{i\in\mathbb{N}}A_i)$ *over* V *to a hypersoft semigroup* $\mathfrak{G}:=(G,\prod_{i\in\mathbb{N}}B_i)$ *over* W *satisfying equations (4.1) and (4.2). If* Γ *is injective, then* $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a hypersoft quasi-ideal over* V *if and only if* $\mathfrak{G}\rfloor^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a hypersoft quasi-ideal over* W.

Proof. If we use Proposition 4.18 (iv), then we can verify that the statement holds. □

Theorem 4.23. *Let* $(V,V,[V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}}K_i),(\varphi,\chi)})$ *and* $(W,W,[W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}}K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* f *and* g *be fuzzy subsets of* V *and* W*, respectively, and let* $\iota,\kappa\in+I$. *Let* $(\Gamma,\Lambda)_h$ *be a hypersoft homomorphism from a hypersoft semigroup* $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}}A_i},\prod_{i\in\mathbb{N}}A_i)$ *over* V *to a hypersoft semigroup* $(W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}}B_i},\prod_{i\in\mathbb{N}}B_i)$ *over* W *satisfying equations (4.1) and (4.2). Then* $\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a fuzzy ideal of* V *if and only if* $\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a fuzzy ideal of* W.

Proof. From Propositions 3.31 (i), 4.13 (i), and Theorem 4.19, we observe that

$$\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}\text{ is a fuzzy ideal of }V\Leftrightarrow(V^{(\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}}A_i},\prod_{i\in\mathbb{N}}A_i)\text{ is a hypersoft ideal over }V\text{ for all }\iota\in+I$$

$$\Leftrightarrow (V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)} \text{ is a hypersoft ideal over } V \text{ for all } \iota \in +I$$
$$\Leftrightarrow (W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i)\rceil^{cs}_{\mathfrak{S},(\varphi,\chi)} \text{ is a hypersoft ideal over } W \text{ for all } \kappa \in +I$$
$$\Leftrightarrow (W^{(\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)},\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i) \text{ is a hypersoft ideal over } W \text{ for all } \kappa \in +I$$
$$\Leftrightarrow \ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)} \text{ is a fuzzy ideal of } W.$$

Thus $\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy ideal of V if and only if $\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a fuzzy ideal of W. □

Theorem 4.24. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* f *and* g *be fuzzy subsets of* V *and* W*, respectively, and let* $\iota, \kappa \in +I$*. Let* $(\Gamma, \Lambda)_h$ *be a given hypersoft homomorphism from a hypersoft semigroup* $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $(W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i)$ *over* W *satisfying equations* (4.1) *and* (4.2)*. If* Γ *is injective, then* $\llcorner f\lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a fuzzy ideal of* V *if and only if* $\llcorner g\lrcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a fuzzy ideal of* W*.*

Proof. Applying Propositions 3.31 (ii), 4.13 (i), and Theorem 4.20, this statement is easily provided. □

Theorem 4.25. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* f *and* g *be fuzzy subsets of* V *and* W*, respectively, and let* $\iota, \kappa \in +I$*. Let* $(\Gamma, \Lambda)_h$ *be a given hypersoft homomorphism from a hypersoft semigroup* $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $(W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i)$ *over* W *satisfying equations* (4.1) *and* (4.2)*. If* Γ *is injective, then* $\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a fuzzy quasi-ideal of* V *if and only if* $\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a fuzzy quasi-ideal of* W*.*

Proof. Suppose Γ is injective. Then, by Propositions 3.31 (i), 4.13 (ii), and Theorem 4.21, we observe that

$$\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)} \text{ is a fuzzy quasi-ideal of } V$$
$$\Leftrightarrow (V^{(\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i) \text{ is a hypersoft quasi-ideal over } V \text{ for all } \iota \in +I$$
$$\Leftrightarrow (V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)} \text{ is a hypersoft quasi-ideal over } V \text{ for all } \iota \in +I$$
$$\Leftrightarrow (W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i)\rceil^{cs}_{\mathfrak{S},(\varphi,\chi)} \text{ is a hypersoft quasi-ideal over } W \text{ for all } \kappa \in +I$$
$$\Leftrightarrow (W^{(\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)},\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i) \text{ is a hypersoft quasi-ideal over } W \text{ for all } \kappa \in +I$$
$$\Leftrightarrow \ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)} \text{ is a fuzzy quasi-ideal of } W.$$

Therefore $\ulcorner f\urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ is a fuzzy quasi-ideal of V if and only if $\ulcorner g\urcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ is a fuzzy quasi-ideal of W. □

Theorem 4.26. *Let* $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *and* $(W, W, [W]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ *be approximation spaces type I. Let* f *and* g *be fuzzy subsets of* V *and* W*, respectively, and let* $\iota, \kappa \in +I$*. Let* $(\Gamma, \Lambda)_h$ *be a hypersoft homomorphism from a hypersoft semigroup* $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ *over* V *to a hypersoft semigroup* $(W^{(g,\kappa,\geqslant)}_{\prod_{i\in\mathbb{N}} B_i}, \prod_{i\in\mathbb{N}} B_i)$ *over* W *satisfying equations* (4.1) *and* (4.2)*. If* Γ *is injective, then* $\llcorner f\lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)}$ *is a fuzzy quasi-ideal of* V *if and only if* $\llcorner g\lrcorner^{cs}_{\mathfrak{S},(\varphi,\chi)}$ *is a fuzzy quasi-ideal of* W*.*

Proof. We can verify that the statement is true by using Propositions 3.31 (ii), 4.13 (ii), and Theorem 4.22. □

5. Observations and conclusions

In this research article, the concept related to the hesitant bipolar-valued fuzzy soft set theory and hypersoft set theory was developed to hesitant bipolar-valued fuzzy hypersoft relations. We have adapted the methodologies of [15, 39, 43, 45] to extended approximation spaces and novel rough approximation models induced by hesitant bipolar-valued fuzzy hypersoft relations as the following.

- The basic element of the rough approximation of hypersoft sets constitutes upper and lower rough approximations, boundary regions, definable hypersoft sets, and rough hypersoft sets.
- The basic element of the rough approximation of fuzzy sets constitutes upper and lower rough approximations, definable fuzzy sets, and rough fuzzy sets.

Consequently, we obtained that a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation on a single universe generate both the definable hypersoft set and the definable fuzzy set.

As summarized above, we shall discuss to accuracy and roughness measures of hypersoft sets and fuzzy sets in terms of Pawlak's rough set theory [39]. In the study of accuracy and roughness measures, V and W are denoted as finite. Pawlak suggests two numerical measures for characterizing the imprecision in a Pawlak's approximation space (V, E) as follows.

Let X be a subset of V. An accuracy measure of X, denoted by $X|_E$, is defined by

$$X|_E := \frac{|\lfloor X \rfloor_E|}{|\lceil X \rceil_E|},$$

where $|\lceil X \rceil_E|$ and $|\lfloor X \rfloor_E|$ denote cardinalities of $\lceil X \rceil_E$ and $\lfloor X \rfloor_E$, respectively. We observe that $0 \leqslant X|_E \leqslant 1$. A roughness measure of X, denoted by $X\|_E$, is defined by

$$X\|_E := 1 - X|_E.$$

In the following, accuracy and roughness measures of hypersoft sets are considered in approximation spaces induced by hesitant bipolar-valued fuzzy hypersoft relations. We let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I. Let $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ be a hypersoft set over V. For $a \in \prod_{i\in\mathbb{N}} A_i$, an accuracy measure of $F(a)$ based on $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, denoted by $F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)}$, is defined by

$$F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} := \frac{|F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|},$$

where $|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|$ and $|F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|$ denote cardinalities of $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$ and $F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)$, respectively. Generally, observe that $F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} \in +I$ for all $a \in \prod_{i\in\mathbb{N}} A_i$. In what follows, for $a \in \prod_{i\in\mathbb{N}} A_i$, a roughness measure of $F(a)$ based on $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, denoted by $F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)}$, is defined by

$$F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)} := 1 - F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

In observation, the following arguments indeed hold.

- Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then we have $\mathfrak{F}$ is a definable hypersoft set within $(V, V, [V]^{cs}_{\mathfrak{R},(\varphi,\chi)})$ if and only if $F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} = 1$ or $F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)} = 0$ for all $a \in \mathrm{Hsupp}(\mathfrak{F})$.
- Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ and $(V, V, [V]^{cs}_{\mathfrak{S}:=(S,\prod_{i\in\mathbb{N}} K_i),(\gamma,\omega)})$ be approximation spaces type I with the property that the inclusion relation of the hesitant bipolar-valued fuzzy hypersoft reflexive relation $\mathfrak{R}$ and the hesitant bipolar-valued fuzzy hypersoft transitive relation $\mathfrak{S}$ is $\mathfrak{R} \subseteq_{ir} \mathfrak{S}$,

and $(\gamma,\omega) \subseteq_{sr} (\varphi,\chi)$. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then we have $F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} \geqslant F(a)|^{cs}_{\mathfrak{S},(\gamma,\omega)}$ for all $a \in \mathrm{Hsupp}(\mathfrak{F})$. In fact, we let $a \in \mathrm{Hsupp}(\mathfrak{F})$ be given. Then, by using Proposition 3.19, we see that $|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)| \leqslant |F\rceil^{cs}_{\mathfrak{S},(\gamma,\omega)}(a)|$ and $|F\rfloor^{cs}_{\mathfrak{S},(\gamma,\omega)}(a)| \leqslant |F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|$. Now

$$F(a)|^{cs}_{\mathfrak{S},(\gamma,\omega)} := \frac{|F\rfloor^{cs}_{\mathfrak{S},(\gamma,\omega)}(a)|}{|F\rceil^{cs}_{\mathfrak{S},(\gamma,\omega)}(a)|} \leqslant \frac{|F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{S},(\gamma,\omega)}(a)|} \leqslant \frac{|F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|} =: F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)}.$$

- Let $(V, V, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be a given approximation space type I with the property that $\mathfrak{R}$ is a hesitant bipolar-valued fuzzy hypersoft reflexive relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation over $V \times V$, and $(\varphi,\chi) \in \mathcal{P}(-I)\backslash\{-I\} \times \mathcal{P}(+I)\backslash\{\emptyset\}$. If $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then $F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} = 1$ and $F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)} = 0$ for all $a \in \mathrm{Hsupp}(\mathfrak{F})$ due to Proposition 3.20.
- We further study the fact under distance measurement concerning the classical concept of Marczewski and Steinhaus [28]. Let X and Y be subsets of V. Marczewski and Steinhaus propose the notion of distance measure of X and Y as follows.

  A symmetric difference between X and Y, denoted by $X \boxminus Y$, is defined by

  $$X \boxminus Y := (X \cup Y) - (X \cap Y).$$

  A distance measure of X and Y, denoted by $DM(X, Y)$, is defined by

  $$DM(X,Y) := \begin{cases} \frac{|X\boxminus Y|}{|X\cup Y|}, & \text{if } |X\cup Y| > 0, \\ 0, & \text{if } |X \cup Y| = 0, \end{cases}$$

  where $|X \cup Y|$ denotes the cardinality of $X \cup Y$, and $|X \boxminus Y|$ denotes the cardinality of the symmetric difference $X \boxminus Y$. Based on $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ type I, if $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ is a hypersoft set over V, then we obtain that $DM(F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a), F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a))$ is equal to $F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $a \in \mathrm{Hsupp}(\mathfrak{F})$. In fact, let $a \in \mathrm{Hsupp}(\mathfrak{F})$. Then $F(a) \neq \emptyset$. By Remark 3.14, we get that $F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \neq \emptyset$. We observe that $|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)| > 0$ and $|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cup F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)| > 0$. Now

$$\begin{aligned} DM(F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a), F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)) &:= \frac{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \boxminus F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cup F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|} \\ &= \frac{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cup F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cup F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|} - \frac{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cap F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a) \cup F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|} \\ &= 1 - \frac{|F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|}{|F\rceil^{cs}_{\mathfrak{R},(\varphi,\chi)}(a)|} = 1 - F(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)} =: F(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)}. \end{aligned}$$

In the fuzzy context, we further study to accuracy and roughness measures of fuzzy sets in approximation spaces induced by hesitant bipolar-valued fuzzy hypersoft relations as the following.

Let $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$ be an approximation space type I, f a fuzzy subset of V and $\iota \in +I$. An ι-level accuracy measure of f based on $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, denoted by $(f,\iota,\geqslant)|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)}$, is defined by

$$(f,\iota,\geqslant)|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)} := \frac{|V^{(\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}|}{|V^{(\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}|},$$

where $|V^{(\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}|$ and $|V^{(\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}|$ denote cardinalities of finite sets $V^{(\ulcorner f \urcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}$ and $V^{(\llcorner f \lrcorner^{cs}_{\mathfrak{R},(\varphi,\chi)},\iota,\geqslant)}$, respectively. Now, observe that $f|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)} \in +I$. An ι-level roughness measure of f based on $[V]^{cs}_{\mathfrak{R},(\varphi,\chi)}$, denoted by $(f,\iota,\geqslant)\|^{cs}_{\mathfrak{R},(\varphi,\chi)}$, is defined by

$$(f,\iota,\geqslant)\|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)} := 1 - (f,\iota,\geqslant)|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)}.$$

Based on Proposition 3.31, if f is a fuzzy subset of V and $\iota \in +I$ such that $(V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}, \prod_{i\in\mathbb{N}} A_i)$ is a $(f, \iota, \geqslant)$-relative whole hypersoft set over V, then we observe that the following items are true.

- $(f, \iota, \geqslant)|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)} = V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)|^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.
- $(f, \iota, \geqslant)\|^{cs,\iota}_{\mathfrak{R},(\varphi,\chi)} = V^{(f,\iota,\geqslant)}_{\prod_{i\in\mathbb{N}} A_i}(a)\|^{cs}_{\mathfrak{R},(\varphi,\chi)}$ for all $a \in \prod_{i\in\mathbb{N}} A_i$.

This means that the concept of accuracy and roughness measures of fuzzy sets can be described in terms of hypersoft sets.

In order to obtain the optimal multi-parameter of a hypersoft set in general, we present the decision-making algorithm of associated rough hypersoft sets in approximation spaces induced by hesitant bipolar-valued fuzzy hypersoft relations as follows.

Step 1. Construct an information (maybe algebraic) system containing the approximation space $(V, W, [V]^{cs}_{\mathfrak{R}:=(R,\prod_{i\in\mathbb{N}} K_i),(\varphi,\chi)})$.

Step 2. Input a hypersoft set $\mathfrak{F} := (F, \prod_{i\in\mathbb{N}} A_i)$ over V.

Step 3. Compute $\mathfrak{F}]^{cs}_{\mathfrak{R},(\varphi,\chi)}$ and $\mathfrak{F}\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}$.

Step 4. If the value $\min_{1\leqslant i\leqslant n}\{m(F(a_i)) := \frac{DM(F]^{cs}_{\mathfrak{R},(\varphi,\chi)}(a_i), F\rfloor^{cs}_{\mathfrak{R},(\varphi,\chi)}(a_i))}{card(F(a_i))}\}$ is found, then the optimal decision is $F(a)$, where a is a multi-parameter generated the minimum value. Otherwise, the optimal decision does not exist. In this step, we call the multi-parameter a an optimal multi-parameter of $(F, \prod_{i\in\mathbb{N}} A_i)$.

We consider the corresponding example as follows. Based on Example 3.13, let $\prod_{i\in\mathbb{N}} A_i = \{a_i :$ i is a natural number with $1 \leqslant i \leqslant 4\}$ and let $(F, \prod_{i\in\mathbb{N}} A_i)$ be a hypersoft set over V defined by

$$\begin{aligned}
F(a_1) &= \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number with } i = 1\},\\
F(a_2) &= \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number with } 1 \leqslant i \leqslant 2\},\\
F(a_3) &= \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number with } 1 \leqslant i \leqslant 3\},\\
F(a_4) &= \{v_{3i} : i \text{ is a natural number}\}.
\end{aligned}$$

Using Definition 3.12, we obtain that

$$\begin{aligned}
F]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a) &= \{v_{3i} : i \text{ is a natural number}\} \cup \{v_{3i-2} : i \text{ is a natural number}\},\\
F\rfloor^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a) &= \{v_{3i} : i \text{ is a natural number}\}
\end{aligned}$$

for all $a \in \{a_1, a_2, a_3\}$. Moreover, we get that

$$F]^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a_4) = F\rfloor^{cs}_{\mathfrak{R},([-1,-0.7),[0.5,1])}(a_4).$$

By Remark 3.14, we see that

$$m(F(a_1)) > m(F(a_2)) > m(F(a_3)) > m(F(a_4)) = 0.$$

Therefore a_4 is the optimal multi-parameter of $(F, \prod_{i\in\mathbb{N}} A_i)$ such that $F(a_4)$ is the best choice. Observe that $F(a_4)$ is definable. Then, the definable-based approximation approach induces the optimal multi-parameter and the best alternative. Here, the notion of the set-valued distance measurement combined with a decision-making algorithm based on rough set theory generates the optimal multi-parameter as well as the best alternative of a hypersoft set. Furthermore, such an algorithm can be also applied to semigroup (or other algebraic structures) and several information systems under decision-making problems.

In the approach of semigroup theory, we used the novel models to study upper and lower rough approximations of hypersoft quasi-ideals over semigroups and fuzzy quasi-ideals of semigroups. Then, we demonstrated arguments like the following.

- In a regular semigroup, every upper rough approximation of a hypersoft quasi-ideal (resp., a fuzzy quasi-ideal) is a hypersoft quasi-ideal (resp., a fuzzy quasi-ideal) based on a hesitant bipolar-valued fuzzy hypersoft reflexive relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft compatible relation.
- Every lower rough approximation of a hypersoft quasi-ideal (resp., a fuzzy quasi-ideal) is a hypersoft quasi-ideal (resp., a fuzzy quasi-ideal) based on a bipolar fuzzy reflexive relation, a hesitant bipolar-valued fuzzy hypersoft transitive relation, and a hesitant bipolar-valued fuzzy hypersoft complete relation.

Moreover, we got that a hesitant bipolar-valued fuzzy hypersoft symmetric relation and a hesitant bipolar-valued fuzzy hypersoft antisymmetric relation are not sufficient conditions for all results. In the end, we used hypersoft homomorphisms to study upper and lower rough approximations of hypersoft quasi-ideals over semigroups and fuzzy quasi-ideals of semigroups. Then, we obtained necessary and sufficient conditions for upper and lower rough approximations of hypersoft quasi-ideals over semigroups and fuzzy quasi-ideals of semigroups.

Combined with other types of hypersoft sets and fuzzy sets, we shall verify the results of rough approximations for these and also consider other types of several algebraic structures in the future. Based on the interesting applicative concept in [30], we also further study the approximation of fuzzy hypersoft sets by hesitant bipolar-valued fuzzy hypersoft relation with image processing application in the next step.

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