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Exact solution for commensurate and incommensurate linear systems of fractional differential equations

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Abstract

In this paper, we introduce exact solutions for the initial value problems of two classes of a linear system of fractional ordinary differential equations with constant coefficients. This article concerns a linear system of fractional order, where the orders are equal or different rational numbers between zero and one. The conformable fractional derivative presented by [R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, J. Comput. Appl. Math., **264** (2014), 65–70] is considered. Two different approaches are adopted to give analytic solutions for fractional order systems. The presented methods are illustrated by analyzing some numerical examples that show the effectiveness of the implemented methods.

Keywords: Conformable fractional derivative, fractional Laplace transform, commensurate and incommensurate fractional order systems, asymptotically stable.

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1. Introduction

Fractional derivatives are significant tools for describing memory and genetic properties of a wide range of materials and phenomena [8, 29, 31]. This is the most significant benefit of fractional derivatives over classical integer-order derivatives.

Many articles have been published on the differential equations of fractional order in several fields of science and engineering [8, 11, 14, 20–23, 27]. The analytic and approximate solutions of linear and nonlinear systems of ordinary differential equations of fractional order have been discussed by several authors, see [4, 13, 19, 25, 30]. In particular, Odibat [26] investigated the existence, uniqueness, and stability of the exact solution for linear Caputo fractional differential equations systems.

In 2014, the authors Khalil et al. defined a new fractional derivative in [18]. It is based on the definition of the basic limit of the derivative. The new simple fractional derivative is called the conformable fractional derivative. Since then the conformable fractional derivatives have been the focus of many studies. In [1], the author appointed the basic concepts in fractional calculus based on the conformable fractional derivative. In particular, the fractional chain rule, the fractional power series, and the fractional version of the Laplace transform have been presented. A general solution of the fractional Cauchy Euler equation

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has been investigated in [5]. Moreover, in [2], the form of the Wronskian for conformable fractional linear differential equations with variable coefficient and a formulation of Abel's formula have been discussed. Fractional Fourier series has been introduced to solve the fractional partial differential equations in [3]. For more detailed information, refer to references [6, 7, 9, 10, 12, 15, 17, 24].

In this paper, we provide an exact solution for the nonhomogeneous linear system of fractional ordinary differential equations with constant coefficients:

$$D^{\alpha_1}x_1(t)=\sum_{j=1}^{n}a_{1j}x_j+f_1(t),\ \ D^{\alpha_2}x_2(t)=\sum_{j=1}^{n}a_{2j}x_j+f_2(t),\ \ \dots\ \ D^{\alpha_n}x_n(t)=\sum_{j=1}^{n}a_{nj}x_j+f_n(t), \tag{1.1}$$

where $D^{\alpha_i}=\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the conformable fractional derivative of order $\alpha_i\in(0,1]$, for $i=1,2,\dots,n$ and the coefficients a_{ij} are constants. The system is subject to the initial conditions

$$x_1(0)=c_1,\ x_2(0)=c_2,\dots,\ x_n(0)=c_n.$$

If $\alpha=\alpha_1=\alpha_2=\cdots=\alpha_n$, then the system is called a commensurate linear order system, otherwise system (1.1) denotes an incommensurate linear order system.

The article is organized as follows. In Section 2, we present some basic definitions and notations adopted throughout the article. Section 3 is devoted to solve commensurate linear system of fractional order, where two methods have been implemented. Finding an exact solution of incommensurate linear system has been discussed in Section 4. Finally, conclusions are presented in Section 5.

2. Preliminaries and notations

In this section, we present the main definitions and properties of the conformable fractional derivative and the fractional Laplace transform.

Definition 2.1. The conformable fractional derivative of order α, $0<\alpha\leqslant 1$ of $f:(0,\infty)\to\mathbb{R}$ is defined by

$$T_\alpha(f)(x):=\lim_{\epsilon\to 0}\frac{f(x+\epsilon x^{1-\alpha})-f(x)}{\epsilon}$$

for all $x>0$. If the limit exists, we say that f is α-differentiable at x. Moreover, if f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{x\to 0^+}T_\alpha(f)(x)$ exists, then define

$$T_\alpha(f)(0):=\lim_{x\to 0^+}T_\alpha(f)(x).$$

In contrast with other common fractional derivatives, f can be α-differentiable at a point but not necessarily differentiable at that point. For example, the function $f(x)=\sqrt{x}$ is not differentiable at 0 but $T_{1/2}(f)(0)=\lim_{x\to 0^+}T_{1/2}(f)(x)=1/2$.

The conformable fractional derivative satisfies all main properties of the usual derivative, e.g., linearity, multiplication, quotient rules, besides the chain rule.

The relation between the conformable fractional and usual first derivative is given by the following remark.

Remark 2.2. Let $\alpha\in(0,1]$ and f be differentiable and α-differentiable for all $x>0$. Then

$$T_\alpha(f)(x)=x^{1-\alpha}\frac{df}{dx}(x).$$

The fractional exponential function, denoted by $e^{\frac{1}{\alpha}x^\alpha}$, is defined by

$$e^{\frac{1}{\alpha}x^\alpha}=\sum_{j=0}^{\infty}\frac{x^{\alpha j}}{\alpha^j j!}.$$

Remark 2.3. The conformable fractional derivative of common functions are

- $T_\alpha(c) = 0$, for any constant c;
- $T_\alpha(x^r) = rx^{r-\alpha}, r \in \mathbb{R}$;
- $T_\alpha(\sin \frac{1}{\alpha}x^\alpha) = \cos \frac{1}{\alpha}x^\alpha$;
- $T_\alpha(\cos \frac{1}{\alpha}x^\alpha) = -\sin \frac{1}{\alpha}x^\alpha$;
- $T_\alpha(\sinh \frac{1}{\alpha}x^\alpha) = \cosh \frac{1}{\alpha}x^\alpha$;
- $T_\alpha(\cosh \frac{1}{\alpha}x^\alpha) = \sinh \frac{1}{\alpha}x^\alpha$.
- $T_\alpha(e^{\frac{1}{\alpha}x^\alpha}) = e^{\frac{1}{\alpha}x^\alpha}$.

Next, we introduce the fractional integral of order α.

Definition 2.4. Let $\alpha \in (0,1]$ and $x \in [a, \infty)$, $a \geqslant 0$. The conformable fractional integral of order α is given by

$$(I_\alpha^a f)(x) := \int_a^x f(t)d\alpha(t,a) = \int_a^x (t-a)^{\alpha-1} f(t)dt.$$

When $a = 0$, we use $d\alpha(t)$.

Theorem 2.5. *Let* $f : [a, \infty) \to \mathbb{R}$ *be any continuous function, then for all* $x > 0$ *and* $\alpha \in (0,1]$,

$$T_\alpha I_\alpha^a(f)(x) = f(x).$$

Now we define a fractional version of Laplace transform which will be very useful in solving a linear system of fractional differential equations.

Definition 2.6. The fractional Laplace transform of order $\alpha \in (0,1]$ is defined by

$$\mathcal{L}_\alpha\{f(t)\} = F_\alpha(s) := \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\ d\alpha(t) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\ t^{\alpha-1}dt.$$

The following theorem gives the fractional Laplace transform of the derivative of a function.

Theorem 2.7. *Consider* $f : [0,\infty) \to \mathbb{R}$ *be differentiable real valued function. Then*

$$\mathcal{L}_\alpha\{f^{(\alpha)}(t)\} = sF_\alpha(s) - f(0),\ s > 0.$$

Proof. By the definition, using Remark 2.2, we obtain

$$\mathcal{L}_\alpha\{f^{(\alpha)}(t)\} = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f^{(\alpha)}(t)\ t^{\alpha-1}dt = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f'(t)\ dt.$$

Using integration by parts with $u = e^{-s\frac{t^\alpha}{\alpha}}$ and $dv = f'(t)$, we obtain

$$\mathcal{L}_\alpha\{f^{(\alpha)}(t)\} = \lim_{k\to\infty}(e^{-s\frac{t^\alpha}{\alpha}} f(t)\Big|_0^k) + s\int_0^\infty f(t)t^{\alpha-1}e^{-s\frac{t^\alpha}{\alpha}}dt = -f(0) + sF_\alpha(s).$$

□

Remark 2.8. The following are fractional Laplace transform of common functions

- $\mathcal{L}_\alpha\{1\}(s) = \frac{1}{s}, \quad s > 0$;

- $\mathscr{L}_\alpha\{t\}(s) = \alpha^{1/\alpha}\frac{\Gamma(1+1/\alpha)}{s^{1+1/\alpha}}, \quad s > 0;$
- $\mathscr{L}_\alpha\{t^r\}(s) = \alpha^{r/\alpha}\frac{\Gamma(1+r/\alpha)}{s^{1+r/\alpha}}, \quad s > 0;$
- $\mathscr{L}_\alpha\{e^{\lambda\frac{t^\alpha}{\alpha}}\}(s) = \frac{1}{s-\lambda}, \quad s > \lambda;$
- $\mathscr{L}_\alpha\{\sin\omega\frac{t^\alpha}{\alpha}\}(s) = \frac{\omega}{s^2+\omega^2}, \quad s > 0;$
- $\mathscr{L}_\alpha\{\cos\omega\frac{t^\alpha}{\alpha}\}(s) = \frac{s}{s^2+\omega^2}, \quad s > 0;$
- $\mathscr{L}_\alpha\{\sinh\omega\frac{t^\alpha}{\alpha}\}(s) = \frac{\omega}{s^2-\omega^2}, \quad s > \omega;$
- $\mathscr{L}_\alpha\{\cosh\omega\frac{t^\alpha}{\alpha}\}(s) = \frac{s}{s^2-\omega^2}, \quad s > \omega;$
- $\mathscr{L}_\alpha\{e^{-c\frac{t^\alpha}{\alpha}}\sin\frac{t^\alpha}{\alpha}\}(s) = \frac{1}{(s+c)^2+1}, \quad s > -c.$

If f and g are functions that are equal to zero for $t < 0$, then the convolution of f and g, denoted by $f * g$ is defined by

$$(f * g)(t) = \int_0^t f(t-x)g(x)dx.$$

Remark 2.9. If $\mathscr{L}_\alpha(f)(t) = F_\alpha(s)$ and $\mathscr{L}_\alpha(g)(t) = G_\alpha(s)$, then

$$\mathscr{L}_\alpha\{(f * g)(t)\} = F_\alpha(s).G_\alpha(s).$$

The sequential conformable fractional derivative of order n is defined by

$$T_{n\alpha}f(x) = \underbrace{T_\alpha T_\alpha \dots T_\alpha}_{n \text{ times}} f(x).$$

Another useful property of the fractional Laplace transform which is used in solving incommensurate system is given in the following theorem. For proof refer to [16].

Theorem 2.10. *If* $f : [a, \infty) \to \mathbb{R}$ *be any function in* C^n*, then*

$$\mathscr{L}_\alpha\{T_{n\alpha}(f)(t)\} = s^n F_\alpha(s) - s^{n-1}f(0), \quad 0 < \alpha \leqslant 1/n.$$

3. Commensurate fractional order linear system

In this section, we derive the exact solution for the linear system of ordinary differential equations of fractional order Eq. (1.1), where all the fractional derivatives are in the same order, i.e., $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$. The system can be expressed by

$$\frac{d^\alpha}{dt^\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad t \in (0, a], \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^\mathbf{n}$, $A = [a_{ij}] \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathbf{f}(t) = [f_1(t), f_2(t), \dots, f_n(t)]^T$ and $\frac{d^\alpha}{dt^\alpha}$ is the α-conformable fractional derivative. The corresponding homogeneous system of (3.1) is given by

$$\frac{d^\alpha}{dt^\alpha}\mathbf{x}(t) = A\mathbf{x}(t), \quad t \in (0, a]. \tag{3.2}$$

Definition 3.1. The solution $\mathbf{x}(t)$ of system (3.2) is called stable if, for any initial condition $\mathbf{x}_0$, there exists $\epsilon > 0$ such that $\|\mathbf{x}(t)\| \leqslant \epsilon$ for all $t > 0$. The solution is called asymptotically stable if it is stable and $\|\mathbf{x}(\mathbf{t})\| \to \mathbf{0}$ as $t \to \infty$.

Stability of conformable fractional systems has been investigated in [28] where some stability conditions have been derived. In [28], the following results are presented

Theorem 3.2. *The solution of system* (3.2) *is given by*

$$\mathbf{x}(t) = \mathbf{x}_0 e^{\frac{1}{\alpha}At^{\alpha}},$$

whenever the solution is differentiable on $(0, \infty)$.

Theorem 3.3. *The system* (3.2) *is asymptotically stable if and only if the eigenvalues of* A *have strictly negative real parts, i.e.,* $\lambda_i + \overline{\lambda_i} < 0$ *for all* $\lambda_i \in \sigma(A)$.

In this paper, we investigate two methods to solve the system (3.1). The first approach is based on finding the eigenvalues and the corresponding eigenvectors of the matrix A, while the second approach is based on using the fractional Laplace transform.

3.1. The eigenvalue-eigenvector approach

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of A and their corresponding eigenvectors are $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n)}$, respectively, i.e., satisfy $(A - \lambda_i \mathbf{I})\mathbf{v}^{(i)} = 0$, for all $i = 1, 2, \dots, n$. Therefore the general solution of the system (3.2) is

$$\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} e^{\lambda_1 \frac{t^{\alpha}}{\alpha}} + c_2 \mathbf{v}^{(2)} e^{\lambda_2 \frac{t^{\alpha}}{\alpha}} + \cdots + c_n \mathbf{v}^{(n)} e^{\lambda_n \frac{t^{\alpha}}{\alpha}},$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants. If $[x_1(t), x_2(t), \dots, x_n(t)]^T$ is the solution of the system (3.2) and the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, then the initial value problem

$$\frac{d^{\alpha}}{dt^{\alpha}}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad t \in (0, a], \ \mathbf{x}(0) = \mathbf{x}_0,$$

has the solution ([26])

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} \int_0^t x_1(\tau - t) f_1(\tau) d\tau \\ \int_0^t x_2(\tau - t) f_2(\tau) d\tau \\ \vdots \\ \int_0^t x_n(\tau - t) f_n(\tau) d\tau \end{pmatrix}.$$

3.2. The fractional Laplace transform approach

Applying the fractional Laplace transform $\mathscr{L}_{\alpha}$ on both sides of the system (3.1), using Theorem 2.7, we obtain

$$\begin{pmatrix} sX_1(s) - x_1(0) \\ sX_2(s) - x_2(0) \\ \vdots \\ sX_n(s) - x_n(0) \end{pmatrix} = A \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} + \begin{pmatrix} F_1(s) \\ F_2(s) \\ \vdots \\ F_n(s) \end{pmatrix},$$

where $X_i(s) = \mathscr{L}_{\alpha}\{x_i(t)\}$ and $F_i(s) = \mathscr{L}_{\alpha}\{f_i(t)\}$, for $i = 1, 2, \dots, n$. By reordering the terms, we have

$$\begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} = \begin{pmatrix} \frac{1}{s - a_{11}}\left(x_1(0) + F_1(s) + \sum_{\substack{j=1, \\ j \neq 1}}^{n} a_{1j} X_j(s)\right) \\ \frac{1}{s - a_{22}}\left(x_2(0) + F_2(s) + \sum_{\substack{j=1, \\ j \neq 2}}^{n} a_{2j} X_j(s)\right) \\ \vdots \\ \frac{1}{s - a_{nn}}\left(x_n(0) + F_n(s) + \sum_{\substack{j=1, \\ j \neq n}}^{n} a_{nj} X_j(s)\right) \end{pmatrix}. \tag{3.3}$$

Solving system (3.3) and applying the inverse Laplace transform, we obtain the exact solution of the system (3.1). For the homogeneous system (3.2), one can define

$$M(s) = \begin{pmatrix} a_{11}-s & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-s & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{2n} & \dots & a_{nn}-s \end{pmatrix}.$$

It follows from Cramer's rule that the solution can be obtained by

$$X_k(s) = \frac{\det(M_k(s))}{\det(M(s))},$$

for $k = 1, 2, \dots, n$, where $M_k(s)$ is the matrix reshaped by replacing the kth column of $M(s)$ by

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}.$$

Example 3.4. Consider the homogeneous commensurate linear system of fractional ordinary differential equations

$$\begin{pmatrix} \frac{d^\alpha x_1}{dt^\alpha} \\ \frac{d^\alpha x_2}{dt^\alpha} \\ \frac{d^\alpha x_3}{dt^\alpha} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.4}$$

where $0 < \alpha \leqslant 1$.

Two approaches are used for solving the initial value problem consisting of the system (3.4) and the initial condition $\mathbf{x}(0) = \mathbf{x_0}$. The first is based on finding the eigenvalues and the corresponding eigenvectors of the coefficients matrix A, while the second is based on using the fractional Laplace transform.

First method: The eigenvalues of the matrix A are $\lambda_1 = -2$, $\lambda_2 = 0$, $\lambda_3 = 1$ and their corresponding eigenvectors are $\mathbf{v}^{(1)} = [1, -2, 1]^T$, $\mathbf{v}^{(2)} = [1, 0, -1]^T$, $\mathbf{v}^{(3)} = [1, 1, 1]^T$, respectively. Therefore, the general solution of (3.4) is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-\frac{2}{\alpha}t^\alpha} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{\frac{1}{\alpha}t^\alpha},$$

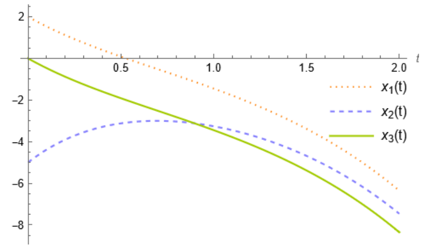
where c_1, c_2 and c_3 are arbitrary constants. In particular, the initial value problem consisting of the system (3.4) and the initial condition

$$\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} \tag{3.5}$$

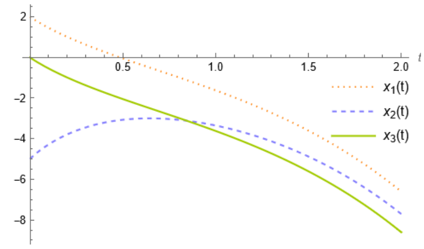
has the unique solution

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{\frac{-2}{\alpha}t^\alpha} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{\frac{1}{\alpha}t^\alpha}. \tag{3.6}$$

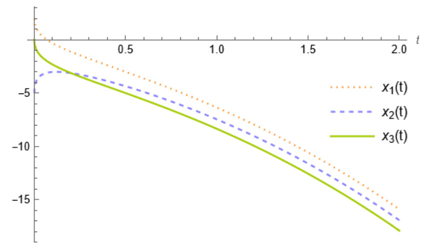
The exact solution (3.6) is shown in Figure 1 when $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.5$ and $\alpha = 0.1$. The solution components decay towards $-\infty$ as t increases due to the existence of the positive eigenvalue λ_3, the negative coefficient c_3, and the nonzero components of the corresponding eigenvector. As shown in Figure 1, the system (3.4) is not stable.



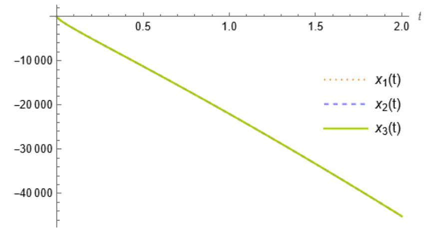
(a) $\alpha = 1$.



(b) $\alpha = 0.95$.



(c) $\alpha = 0.5$.



(d) $\alpha = 0.1$.

Figure 1: Plots of $x_1(t)$, $x_2(t)$ and $x_3(t)$ versus t for Example 3.4.

Second method: Applying fractional Laplace transform $\mathscr{L}_\alpha$ to both sides of the system (3.4), we obtain

$$sX_1(s) - X_2(s) = x_1(0), \qquad -X_1(s) + (s+1)X_2(s) - X_3(s) = x_2(0), \qquad -X_2(s) + sX_3(s) = x_3(0), \tag{3.7}$$

where $X_1(s) = \mathscr{L}_\alpha\{x_1(t)\}$, $X_2(s) = \mathscr{L}_\alpha\{x_2(t)\}$ and $X_3(s) = \mathscr{L}_\alpha\{x_3(t)\}$. Solving the linear system (3.7), we obtain

$$\begin{aligned} X_1(s) &= \frac{1}{s}\left[x_1(0) + \frac{x_1(0) + x_3(0)) + sx_2(0}{s^2+s-2}\right], \\ X_2(s) &= \frac{x_1(0) + x_3(0) + sx_2(0)}{s^2+s-2}, \\ X_3(s) &= \frac{1}{s}\left[x_3(0) + \frac{x_1(0) + x_3(0) + sx_2(0)}{s^2+s-2}\right]. \end{aligned}$$

Substituting the initial condition (3.5), using the following decompositions

$$\begin{aligned} \frac{1}{(s+2)(s-1)} &= -\frac{1}{3(s+2)} + \frac{1}{3(s-1)}, \\ \frac{s}{(s+2)(s-1)} &= \frac{2}{3(s+2)} + \frac{1}{3(s-1)}, \\ \frac{1}{s(s+2)(s-1)} &= -\frac{1}{2s} + \frac{1}{6(s+2)} + \frac{1}{3(s-1)}, \end{aligned}$$

we obtain

$$X_1(s) = \frac{1}{s} + \frac{2}{(s+2)} - \frac{1}{(s-1)}, \qquad X_2(s) = \frac{-4}{s+2} - \frac{1}{s-1}, \qquad X_3(s) = \frac{-1}{s} + \frac{2}{(s+2)} - \frac{1}{(s-1)}. \tag{3.8}$$

Now, applying the inverse fractional Laplace transform to the system (3.8), and using Remark 2.8, we get the solution

$$x_1(t) = 1 + 2e^{\frac{-2}{\alpha}t^\alpha} - e^{\frac{1}{\alpha}t^\alpha}, \qquad x_2(t) = -4e^{\frac{-2}{\alpha}t^\alpha} - e^{\frac{1}{\alpha}t^\alpha}, \qquad x_3(t) = -1 + 2e^{\frac{-2}{\alpha}t^\alpha} - e^{\frac{1}{\alpha}t^\alpha},$$

which agrees with the solution given by the first method.

Example 3.5. Consider the nonhomogeneous commensurate linear system

$$\begin{pmatrix} \frac{d^{1/2}x_1}{dt^{1/2}} \\ \frac{d^{1/2}x_2}{dt^{1/2}} \\ \frac{d^{1/2}x_3}{dt^{1/2}} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ t \\ t^{1/2} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.9}$$

subject to the initial condition $[x_1(0),\ x_2(0),\ x_3(0)] = [1,\ -1,\ 0]$. Applying fractional Laplace transform $\mathscr{L}_{1/2}$ to the system (3.9), using the initial condition, and partial fractions decomposition, we obtain

$$\begin{aligned} X_1(s) &= -\frac{15}{48s} + \frac{1}{8s^2} - \frac{3}{4s^3} + \frac{5}{6(s-1)} + \frac{23}{48(s+2)}, \\ X_2(s) &= -\frac{7}{8s} - \frac{3}{4s^2} + \frac{5}{6(s-1)} - \frac{23}{24(s+2)}, \\ X_3(s) &= -\frac{63}{48s} - \frac{7}{8s^2} + \frac{1}{4s^3} + \frac{5}{6(s-1)} + \frac{23}{48(s+2)}. \end{aligned} \tag{3.10}$$

Applying the inverse Laplace transform to Eq. (3.10) and using Remark 2.8, we get the following exact solution

$$\begin{aligned} x_1(t) &= -\frac{15}{48} + \frac{1}{8}t^{1/2} - \frac{3}{2}t + \frac{5}{6}e^{2t^{1/2}} + \frac{23}{48}e^{-4t^{1/2}}, \\ x_2(t) &= -\frac{7}{8} - \frac{3}{4}t^{1/2} + \frac{5}{6}e^{2t^{1/2}} - \frac{23}{24}e^{-4t^{1/2}}, \\ x_3(t) &= -\frac{21}{16} - \frac{7}{8}t^{1/2} + \frac{1}{2}t + \frac{5}{6}e^{2t^{1/2}} + \frac{23}{48}e^{-4t^{1/2}}. \end{aligned}$$

The exact solution of the initial value problem is shown in Figure 2.

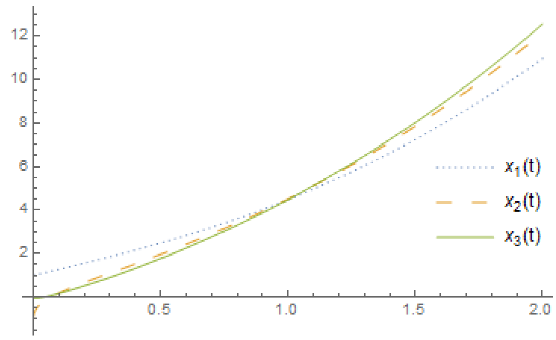


Figure 2: Plots of $x_1(t)$, $x_2(t)$, and $x_3(t)$ versus t for Example 3.5.

Example 3.6. Consider the initial value problem

$$\begin{pmatrix} \frac{d^{1/4}x}{dt^{1/4}} \\ \frac{d^{1/4}y}{dt^{1/4}} \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.11}$$

According to our method, the exact solution is given by

$$x(t) = \frac{4}{5}e^{-24t^{1/4}} + \frac{1}{5}e^{-4t^{1/4}}, \qquad y(t) = -\frac{4}{5}e^{-24t^{1/4}} + \frac{4}{5}e^{-4t^{1/4}}.$$

All eigenvalues of A are negative real numbers. Therefore, the system (3.11) is asymptotically stable, see Figure 3.

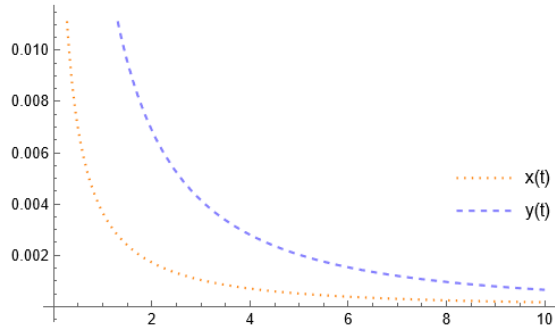


Figure 3: Plots of $x(t)$ and $y(t)$ versus t for Example 3.6.

Example 3.7. Consider the system

$$\begin{pmatrix} \frac{d^\alpha x_1}{dt^\alpha} \\ \frac{d^\alpha x_2}{dt^\alpha} \\ \frac{d^\alpha x_3}{dt^\alpha} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} 5 & -4 & -6 \\ 2 & 1 & -2 \\ 2 & -3 & -3 \end{pmatrix}, \tag{3.12}$$

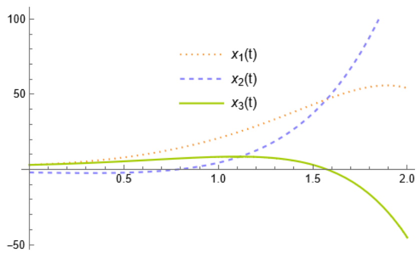
where $0 < \alpha \leqslant 1$. The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = 2+i$, $\lambda_3 = 2-i$ and their corresponding eigenvectors are $\mathbf{v}^{(1)} = [1,0,1]^T$, $\mathbf{v}^{(2)} = [1-i,-1-i,1]^T$, $\mathbf{v}^{(3)} = [1+i,-1+i,1]^T$, respectively. Therefore, the general solution of the system (3.12) is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-\frac{1}{\alpha}t^\alpha} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{\frac{2+i}{\alpha}t^\alpha} + c_3 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{\frac{2-i}{\alpha}t^\alpha} + \frac{c_2}{i} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{\frac{2+i}{\alpha}t^\alpha} - \frac{c_3}{i} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{\frac{2-i}{\alpha}t^\alpha}, \tag{3.13}$$

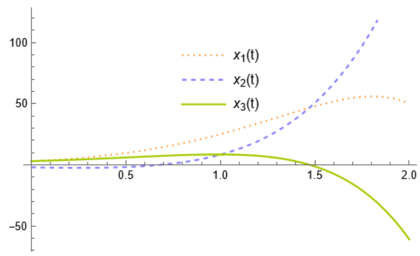
where c_1, c_2, and c_3 are arbitrary constants. If we consider $\alpha = 1$, then the general solution (3.13) can be written as

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 e^{2t} \left[\cos t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right] + c_3 e^{2t} \left[\cos t \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right].$$

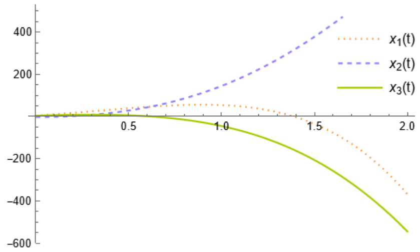
By Figure 4, we can detect that the system (3.12) is not asymptotically stable for the given values of α.



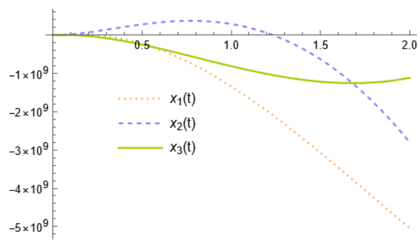
(a) $\alpha = 1$.



(b) $\alpha = 0.9$.



(c) $\alpha = 0.5$.



(d) $\alpha = 0.1$.

Figure 4: Plots of $x_1(t)$, $x_2(t)$ and $x_3(t)$ versus t for Example 3.7.

Example 3.8. Consider the system

$$\begin{pmatrix} \frac{d^\alpha x_1}{dt^\alpha} \\ \frac{d^\alpha x_2}{dt^\alpha} \\ \frac{d^\alpha x_3}{dt^\alpha} \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{3.14}$$

where $0 < \alpha \leqslant 1$. The eigenvalues of the matrix A are $\lambda_1 = -1$, $\lambda_2 = -2$ with multiplicity of 2, and their corresponding eigenvectors are $\mathbf{v}^{(1)} = [0, 1, 0]^T$, $\mathbf{v}^{(2)} = [-3, 1, 0]^T$, respectively. To find the third eigenvector, we solve

$$(A + 2I)\mathbf{v}^{(3)} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

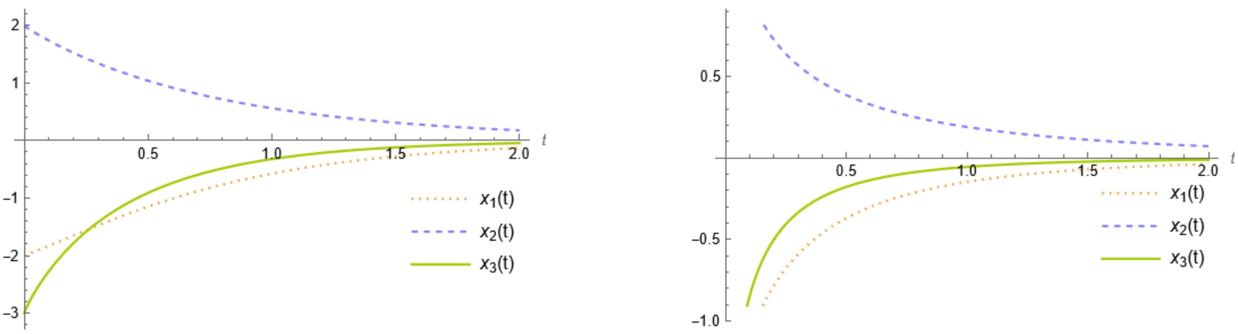
to get

$$\mathbf{v}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}.$$

Therefore, the general solution of (3.14) is

$$\mathbf{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-\frac{1}{\alpha}t^\alpha} + c_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} e^{-\frac{2}{\alpha}t^\alpha} + c_3 \left(\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \frac{t^\alpha}{\alpha} e^{-2\frac{1}{\alpha}t^\alpha} + \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} e^{-2\frac{1}{\alpha}t^\alpha} \right).$$

As shown in Figure 5, the system (3.14) is asymptotically stable.



(a) $\alpha = 0.9$. (b) $\alpha = 0.5$

Figure 5: Plots of $x_1(t)$, $x_2(t)$ and $x_3(t)$ versus t for Example 3.8.

4. Incommensurate fractional order linear system

This section concerns with the incommensurate linear system of differential equations of fractional order,

$$\frac{d^\alpha}{dt^\alpha}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t), \quad t \in (0, a], \tag{4.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^n \times \mathbb{R}^n$, $\mathbf{f}(t) = [f_1(t), f_2(t), \ldots, f_n(t)]^T$ and $\alpha = [\alpha_1, \ \alpha_2, \ \ldots, \ \alpha_n]$ denotes the fractional orders, $\frac{d^\alpha}{dt^\alpha} = \left[\frac{d^{\alpha_1}}{dt^{\alpha_1}}, \frac{d^{\alpha_2}}{dt^{\alpha_2}}, \ldots, \frac{d^{\alpha_n}}{dt^{\alpha_n}}\right]$ and $\frac{d^{\alpha_i}}{dt^{\alpha_i}}$ is the conformable fractional derivative of order $\alpha_i \in (0, 1]$. In fact, we can write $\alpha_i = r_i/m_i$ for some $r_i, \ m_i \in \mathbb{N}$ for $i = 1, \ 2, \ldots, \ n$. Let $\mu = 1/m$

where $m = \text{L.C.M.}\{m_1, m_2, \ldots, m_n\}$, then $\alpha_i = k_i r_i/m$ for some $k_i \in \mathbb{N}$. Applying the fractional Laplace transform $\mathscr{L}_\mu$ to the system (4.1), using Theorem 2.10, we obtain

$$\begin{pmatrix} s^{k_1}X_1(s) - s^{k_1-1}x_1(0) \\ s^{k_2}X_2(s) - s^{k_2-1}x_2(0) \\ \vdots \\ s^{k_n}X_n(s) - s^{k_n-1}x_n(0) \end{pmatrix} = A \begin{pmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{pmatrix} + \begin{pmatrix} F_1(s) \\ F_2(s) \\ \vdots \\ F_n(s) \end{pmatrix},$$

where $X_i(s) = \mathscr{L}_\mu\{x_i(t)\}$ and $F_i(s) = \mathscr{L}_\mu\{f_i(t)\}$ for all $i = 1, 2, \ldots, n$.

Example 4.1. Consider the initial value problem

$$\begin{pmatrix} \frac{d^{1/2}x}{dt^{1/2}} \\ \frac{d^{1/3}y}{dt^{1/3}} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.2}$$

where $a, c, d \in \mathbb{R}$. The initial value problem (4.2) can be rewritten as:

$$\begin{pmatrix} \frac{d^{3/6}x}{dt^{3/6}} \\ \frac{d^{2/6}y}{dt^{2/6}} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{4.3}$$

Applying Laplace transform $\mathscr{L}_{1/6}$ to (4.3), using Theorem 2.10, we obtain

$$X(s) = \frac{s^2}{s^3 - a}x_0, \qquad Y(s) = \frac{s^2}{(s^2-d)(s^3-a)}cx_0 + \frac{s}{s^2-d}y_0,$$

where $X(s) = \mathscr{L}_{1/6}\{x(t)\}$ and $Y(s) = \mathscr{L}_{1/6}\{y(t)\}$. In particular, in case $a = 1$, $c = 6$, and $d = -1$, using partial fractions decompositions, we obtain

$$X(s) = \frac{s^2}{s^3-1}x_0 = \left[\frac{1}{3(s-1)} + \frac{2s+1}{3(s^2+s+1)}\right]x_0,$$
$$Y(s) = \frac{6s^2}{(s^2+1)(s^3-1)}x_0 + \frac{s}{s^2+1}y_0 = x_0\left[\frac{1}{s-1} + \frac{2(s+1/2)-3}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{3(-s+1)}{s^2+1}\right] + y_0\frac{s}{s^2-1}.$$

Applying the inverse fractional Laplace transform, we get the exact solution

$$x(t) = \frac{x_0}{3}(e^{6t^{1/6}} + 2e^{-3t^{1/6}}\cos 3\sqrt{3}t^{1/6}),$$
$$y(t) = x_0(e^{6t^{1/6}} + 2e^{-3t^{1/6}}\cos 3\sqrt{3}t^{1/6} - 3e^{-3t^{1/6}}\sin 3\sqrt{3}t^{1/6} - 3\cos 6t^{1/6} + 3\sin 6t^{1/6}) + y_0\cos 6t^{1/6}.$$

Example 4.2. Consider the initial value problem

$$\begin{pmatrix} \frac{d^{1/2}x}{dt^{1/2}} \\ \frac{d^{1/3}y}{dt^{1/3}} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ t^{1/6} \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{4.4}$$

where $a, d \in \mathbb{R}$. Applying Laplace transform $\mathscr{L}_{1/6}$ to the initial value problem (4.4), we obtain

$$X(s) = \frac{s^2}{s^3-a}x_0 + \frac{1}{s(s^3-a)}, \qquad Y(s) = \frac{s}{s^2-d}y_0 + \frac{1}{6s^2(s^2-d)}.$$

In particular, in case $a = 1$, and $d = -1$ we have

$$X(s) = \frac{s^2}{s^3-1}x_0 + \frac{1}{s(s^3-1)}, \qquad Y(s) = \frac{s}{s^2+1}y_0 + \frac{1}{6s^2(s^2+1)}. \tag{4.5}$$

Applying the inverse fractional Laplace transform to Eq. (4.5), using the following decompositions

$$\begin{aligned}\frac{s^2}{s^3-1} &= \frac{1}{3(s-1)}+\frac{2}{3}\frac{s+1/2}{[(s+1/2)^2+(\sqrt{3}/2)^2]},\\ \frac{1}{s(s^3-1)} &= -\frac{1}{s}+\frac{1}{3(s-1)}+\frac{2}{3}\frac{s+1/2}{[(s+1/2)^2+(\sqrt{3}/2)^2]},\\ \frac{1}{s^2(s^2+1)} &= -\frac{1}{s^2}+\frac{1}{s^2+1},\end{aligned}$$

we obtain the following exact solution

$$\begin{aligned}x(t) &= \frac{x_0}{3}(e^{6t^{1/6}}+2e^{-3t^{1/6}}\cos 3\sqrt{3}t^{1/6})-1+\frac{1}{3}(e^{6t^{1/6}}+2e^{-3t^{1/6}}\cos 3\sqrt{3}t^{1/6}),\\ y(t) &= y_0\cos 6t^{1/6}-t^{1/6}+\frac{1}{6}\sin 6t^{1/6}.\end{aligned}$$

Example 4.3. Consider the initial value problem

$$\begin{pmatrix}\frac{dx}{dt}\\ \frac{d^{1/2}y}{dt^{1/2}}\\ \frac{d^{1/2}z}{dt^{1/2}}\end{pmatrix}=\begin{pmatrix}-2 & 3 & 1\\ 0 & -1 & -2\\ 0 & 4 & 5\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix},\quad \begin{pmatrix}x(0)\\ y(0)\\ z(0)\end{pmatrix}=\begin{pmatrix}x_0\\ y_0\\ z_0\end{pmatrix}. \tag{4.6}$$

In agreement with our approach, applying fractional Laplace transform $\mathscr{L}_{1/2}$ to the initial value problem (4.6), using Theorem 2.10, we achieve

$$\begin{aligned}X(s) &= \frac{x_0s^3-4x_0s^2+(3x_0+3y_0+z_0)s+(-11y_0-5z_0)}{(s^2+2)(s-1)(s-3)},\\ Y(s) &= \frac{y_0s-(5y_0+2z_0)}{(s-3)(s-1)},\\ Z(s) &= \frac{z_0s+(4y_0+z_0)}{(s-1)(s-3)}.\end{aligned} \tag{4.7}$$

Adopting the inverse Laplace transform to Eq. (4.7), using the following decompositions

$$\begin{aligned}\frac{1}{(s-3)(s-1)} &= \frac{1/2}{s-3}-\frac{1/2}{s-1},\\ \frac{s}{(s-3)(s-1)} &= \frac{3/2}{s-3}-\frac{1/2}{s-1},\\ \frac{1}{(s^2+2)(s-3)(s-1)} &= \frac{1/22}{s-3}-\frac{1/6}{s-1}+\frac{4s+1}{33(s^2+2)},\\ \frac{s}{(s^2+2)(s-3)(s-1)} &= \frac{3/22}{s-3}-\frac{1/6}{s-1}+\frac{s-8}{33(s^2+2)},\\ \frac{s^2}{(s^2+2)(s-3)(s-1)} &= \frac{9/22}{s-3}-\frac{1/6}{s-1}+\frac{-8s-2}{33(s^2+2)},\\ \frac{s^3}{(s^2+2)(s-3)(s-1)} &= \frac{27/22}{s-3}-\frac{1/6}{s-1}+\frac{-2s+16}{33(s^2+2)},\end{aligned}$$

and using Remark 2.8, we obtain the following exact solution

$$\begin{aligned} x(t) &= \frac{1}{3}(4y_0 + 2z_0)e^{2t^{1/2}} - \frac{1}{11}(y_0 + z_0)e^{6t^{1/2}} + \frac{1}{33}(33x_0 - 41y_0 - 19z_0)\cos 2\sqrt{2}t^{1/2} \\ &\quad - \frac{1}{33}(35y_0 + 13z_0)\sin 2\sqrt{2}t^{1/2}, \\ y(t) &= (2y_0 + z_0)e^{2t^{1/2}} - (y_0 + z_0)e^{6t^{1/2}}, \\ z(t) &= (-2y_0 - z_0)e^{2t^{1/2}} + 2(y_0 + z_0)e^{6t^{1/2}}. \end{aligned}$$

5. Conclusions

The main intent of this work has been to construct an analytic solution for commensurate and incommensurate classes of linear fractional order systems. The intent has been attained by using the eigenvalue-eigenvector method and the fractional Laplace transform method. These methods provide the solutions in terms of exponential functions. Numerical examples have been given to exhibit that the two methods are efficient.

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