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Existence and uniqueness results of mild solutions for integro-differential Volterra-Fredholm equations

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Abstract

In this paper, we demonstrate the existence and uniqueness of mild and classical solutions to an integro-differential non-local Volterra-Fredholm quasilinear delay. The findings are derived by applying the fixed point theorems of $\mathfrak{R}_0$-Semigroup and the Banach.

Keywords: Volterra-Fredholm integro-differential equation, nonlocal condition, Banach fixed point theorem.

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1. Introduction

In recent years, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, we can remember the works [1, 14, 17–23, 30]. In Banach space, Byszewski [8] studied a solution for the development of non-local equations. In that article, he demonstrated the existence and uniqueness of the non-local Cauchy issue, which is so mild, powerful and classical

$$\vartheta'(\epsilon) = -\varpi\vartheta(\epsilon) + \Omega(\epsilon, \vartheta(\epsilon)),\ \epsilon \in (0, ŧ], \tag{1.1}$$

$$\vartheta(\epsilon_0) + \Omega(\epsilon_1, \epsilon_2, \cdots, \epsilon_p, \vartheta(\epsilon_1), \vartheta(\epsilon_2), \cdots, \vartheta(\epsilon_p)) = \vartheta_0, \tag{1.2}$$

where $0 < \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_p \leqslant a$, $-\varpi$ is a $\mathfrak{R}_0$-semigroup generator infinitesimal in a Banach space ξ, $\vartheta_0 \in \xi$ and $\Omega : [0, ŧ] \times \xi \longrightarrow \zeta$, $\Omega : [0, ŧ]^p \times \xi^p \longrightarrow \xi$ are given functions. The $\Omega(\epsilon_1, \cdots, \epsilon_p, \vartheta(.))$ is used in the sense that in the place of "." only parts of the collection can be replaced $(\epsilon_1, \cdots, \epsilon_p)$, e.g.

$$\Omega(\epsilon_1, \cdots, \epsilon_p, \vartheta(.)) = \mathfrak{R}_1\vartheta(\epsilon_1) + \mathfrak{R}_2\vartheta(\epsilon_2) + \cdots + \mathfrak{R}_p\vartheta(\epsilon_p),$$

when a constant of $\mathfrak{R}_i (i = 1, 2, \cdots, p)$ is specified. The study then extended to different nonlinear equations of evolution [4, 5, 9, 10]. This was followed by numerous writers. In Banach space [2, 6, 7, 13–16, 18–20, 22, 23, 29, 30] several writers have investigated solution of abstract quasilinear evolution. The

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existence of solutions to integro-differences quasilinear in Banach space was discussed by Oka [26], Oka and Tanaka [27] and Bahuguna [3]. The non-homogeneous evolutionary equations were investigated by Kato [24], and the non-linear integro-differential equation was demonstrated by Chandrasekaran [11]. Dhakne and Pachpatte [12] found in Banach spaces an unparalleled strong, abstract, functional and integral solution for a quasilineary abstract equation. A nonlinear conversational law with memory provides an equation of this type

$$\vartheta'(\epsilon,\rho)+\Psi(\vartheta(\epsilon,\rho))_\rho=\int_0^\epsilon b(\epsilon-\upsilon)\Psi(\vartheta(\epsilon,\rho))_\rho d\upsilon+f(\epsilon,\rho),\ \epsilon\in[0,\text{ŧ}], \tag{1.3}$$

$$\vartheta(0,\rho)=\phi(\rho),\quad \rho\in\mathbb{R}.$$

It is apparent that if (1.2) is introduced to (1.3), then $\vartheta(0,\rho)=\phi(\rho)$, the nonlocal condition (1.3) also has better impact. Therefore we want a range of integro-differential equations in Banach spaced classes for the findings of (1.1)–(1.2).

This study aims to prove that there is a mild and traditional solutions to the integro-differential Volterra-Fredholm equation between quasilinear delays and non-local form conditions

$$\begin{aligned}\vartheta'(\epsilon)+\varpi(\epsilon,\vartheta)\vartheta(\epsilon)&=f(\epsilon,\vartheta(\epsilon),\vartheta(\alpha(\epsilon)))+\int_0^\epsilon \Xi(\epsilon,\upsilon,\vartheta(\upsilon),\vartheta(\beta(\upsilon)))d\upsilon\\&\quad+\int_0^\sigma \Xi_1(\epsilon,\upsilon,\vartheta(\upsilon),\vartheta(\beta(\upsilon)))d\upsilon,\end{aligned} \tag{1.4}$$

$$\vartheta(0)+\eth(\vartheta)=\vartheta_0, \tag{1.5}$$

where $\epsilon\in[0,\text{ŧ}]$, $\varpi(\epsilon,\vartheta)$ is the infinitesimal generator of a $\mathfrak{R}_0$ semigroup in a Banach space ζ, $\vartheta_0\in\zeta$, $f:\top\times\zeta\times\zeta\longrightarrow\zeta$, $\Xi,\Xi_1:\Delta\times\zeta\times\zeta\longrightarrow\zeta$, $\eth:C(\top,\zeta)\longrightarrow\zeta$, $\alpha,\beta:\top\longrightarrow\top$ are known functions. Here $\top=[0,\text{ŧ}]$ and $\Delta=\{(\upsilon,\epsilon):0\leqslant\upsilon\leqslant\epsilon\leqslant\text{ŧ}\}$. The findings obtained here are generalizations of results provided by [6, 7, 25] and [28].

2. Auxiliary results

Let ζ and ξ are Banach spaces, ξ be tightly integrated with ζ continually. The standard of approximately Z is $|.|$ or $\|.\|_z$ for any Banach space. The amount of $\mathfrak{B}(\zeta,\xi)$ and $\mathfrak{B}(\zeta,\zeta)$ is expressed as $\mathfrak{B}(\zeta)$ in all the linear operators that are bound from ζ to ξ. We remember some of the Pazy [28] definitions and facts.

Definition 2.1. Let φ is an operator of linear in ζ, ξ is a subspace of ζ. Then $\tilde{\varphi}$ defined as

$$D(\tilde{\varphi})=\{\sigma\in D(\varphi)\cap\xi:\varphi\sigma\in\xi\},$$

and $\tilde{\varphi}\sigma=\varphi\sigma$ for $\sigma\in D(\tilde{\varphi})$ is called the portion of φ in ξ.

Definition 2.2. Let $\mathfrak{B}$ is a subset of ζ and for all $0\leqslant\epsilon\leqslant\sigma$, $o\in\mathfrak{B}$, let $\varpi(\epsilon,o)$ be the infinitesimal generator of a $\mathfrak{R}_0$ semigroup $\varphi_{\epsilon,o}(\upsilon),\upsilon\geqslant 0$, on ζ. The family of operators $\{\varpi(\epsilon,o)\},(\epsilon,o)\in\top\times\mathfrak{B}$, is stable if there exist ω and $\Delta\geqslant 1$ pleasing

$$\rho(\varpi(\epsilon,o))\supset(\omega,\infty),\quad (\epsilon,o)\in\top\times\mathfrak{B},$$

$$\|\prod_{l=1}^{n}E(N:\varpi(\epsilon_l,o_l))\|\leqslant\Delta(N-\omega)^{-n},$$

every finite sequences $0\leqslant\epsilon_1\leqslant\epsilon_2\leqslant\cdots\leqslant\epsilon_n\leqslant\sigma$, $o_l\in\mathfrak{B}$, $1\leqslant l\leqslant n$, for $\lambda>\omega$. The stability of $\{\varpi(\epsilon,o)\},(\epsilon,o)\in\top\times\mathfrak{B}$, then

$$\|\prod_{j=1}^{k}\varphi_{\epsilon_j,o_j}(\upsilon_j)\|\leqslant\Delta\exp\{\omega\sum_{j=1}^{k}s_j\},\ s_j\geqslant 0,$$

and any finite sequences $0\leqslant\epsilon_1\leqslant\epsilon_2\leqslant\cdots\leqslant\epsilon_k\leqslant\sigma,o_j\in\mathfrak{B},1\leqslant j\leqslant k$, $k=1,2,\cdots$.

Definition 2.3. Let $\varphi_{\epsilon,o}(\upsilon), \upsilon \geqslant 0$, be the $\mathfrak{R}_0$-semigroup generatated by $\varpi(\epsilon, o), (\epsilon, o) \in \top \times \mathfrak{B}$. Let ξ be a subspace of ζ is called $\varpi(\epsilon, o)$-admissible if ξ is invariant subspace of $\varphi_{\epsilon,o}(\upsilon)$ and the restriction of $\varphi_{\epsilon,o}(\upsilon)$ to ξ is a $\mathfrak{R}_0$-semigroup in ξ.

Let $\mathfrak{B} \subset \zeta$ such that for all $(\epsilon, o) \in \top \times \mathfrak{B}$, $\varpi(\epsilon, o)$, is the infinitesimal generator of a $\mathfrak{R}_0$-semigroup $\varphi_{\epsilon,o}(\upsilon), \upsilon \geqslant 0$, on ζ. Our assumptions are as follows:

(A1) The family $\{\varpi(\epsilon, o)\}, (\epsilon, o) \in \top \times \mathfrak{B}$, is stable.

(A2) ξ is $\varpi(\epsilon, o)$-admissible for $(\epsilon, o) \in \top \times \mathfrak{B}$, and the family $\{\bar{A}(\epsilon, o)\}, (\epsilon, o) \in \top \times \mathfrak{B}$, of parts $\bar{A}(\epsilon, o)$ of $\varpi(\epsilon, o)$ in ξ, is stable in ξ.

(A3) For $(\epsilon, o) \in \top \times \mathfrak{B}, D(\varpi(\epsilon, o)) \supset \xi, \varpi(\epsilon, o)$ is a bounded from ξ to ζ and $\epsilon \longrightarrow \varpi(\epsilon, o)$ is continuous in the $\mathfrak{B}(\xi, \zeta)$ norm $\|.\|$, for all $o \in \mathfrak{B}$.

(A4) There exists $L > 0$ such that

$$\|\varpi(\epsilon, o_1) - \varpi(\epsilon, o_2)\|_{\xi \longrightarrow x} \leqslant L\|o_1 - o_2\|_{\zeta},$$

holds for all $o_1, o_2 \in \mathfrak{B}$ and $0 \leqslant \epsilon \leqslant a$.

Let $\mathfrak{B} \subset \zeta$ and $\{\varpi(\epsilon, o)\}, (\epsilon, o) \in \top \times \mathfrak{B}$, are operators family pleasing (A1)–(A4). When $\vartheta \in C(\top, \zeta)$ in $\mathfrak{B}$ then there is the unique evolution system $\eta(\epsilon, \upsilon, \vartheta)$, $0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma$, in ζ pleasing [28]:

(i) $\|\eta(\epsilon, \upsilon, \vartheta)\| \leqslant \Delta e^{\omega(\epsilon-\upsilon)}$, for $0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma$, where Δ, ω are stability constants.

(ii) $\frac{\partial^+}{\partial \epsilon}\eta(\epsilon, \upsilon, \vartheta)\omega = \varpi(\upsilon, \vartheta(\upsilon))\eta(\epsilon, \upsilon, \vartheta)\omega$, for $\omega \in \xi$, for $0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma$.

(ii) $\frac{\partial}{\partial \epsilon}\eta(\epsilon, \upsilon, \vartheta)\omega = -\eta(\epsilon, \upsilon, \vartheta)\varpi(\upsilon, \vartheta(\upsilon))\omega$, for $\omega \in \xi$ for $0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma$.

We also take it for granted:

(A5) For every $\vartheta \in C(\top, \zeta)$ pleasing $\vartheta(\epsilon) \in \mathfrak{B}$ for $0 \leqslant \epsilon \leqslant \sigma$, we have

$$\eta(\epsilon, \upsilon, \vartheta)\xi \subset \xi, \quad 0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma,$$

and $\eta(\epsilon, \upsilon, \vartheta)$ is strongly continuous in ξ for $0 \leqslant \upsilon \leqslant \epsilon \leqslant \sigma$.

(A6) ξ is reflexive.

(A7) $\forall\ (\epsilon, o_1, o_2) \in \top \times \mathfrak{B} \times \mathfrak{B}, f(\epsilon, o_1, o_2) \in \xi$.

(A8) $\eth : C(\top, \mathfrak{B}) \longrightarrow \xi$ is bounded in ξ, that is, there exist $\gamma > 0$, $\gamma_1 > 0$ such that

$$\|\eth(\vartheta)\|_{\xi} \leqslant \gamma,$$

$$\|\eth(\vartheta) - \eth(\nu)\|_{\xi} \leqslant \gamma_1 \max_{\epsilon \in \top} \|\vartheta(\epsilon) - \nu(\epsilon)\|_{\zeta}.$$

For (A9) and (A10) let Z be considered as two ζ and ξ:

(A9) $k : \Delta \times Z \longrightarrow Z$ is continuous and there exist $\Lambda_1, \Lambda_1 > 0$ and $\Lambda_2, \Lambda_2 > 0$ such that

$$\int_0^{\epsilon} \|\Xi(\epsilon, \upsilon, \vartheta_1, \nu_1) - \Xi(\epsilon, \upsilon, \vartheta_2, \nu_2)\|_Z d\upsilon \leqslant \Lambda_1(\|\vartheta_1 - \vartheta_2\|_Z + \|\nu_1 - \nu_2\|_Z),$$

$$\int_0^{\epsilon} \|\Xi_1(\epsilon, \upsilon, \vartheta_1, \nu_1) - \Xi_1(\epsilon, \upsilon, \vartheta_2, \nu_2)\|_Z d\upsilon \leqslant \Lambda_1(\|\vartheta_1 - \vartheta_2\|_Z + \|\nu_1 - \nu_2\|_Z),$$

$$\Lambda_2 = \max\{\int_0^{\epsilon} \|\Xi(\epsilon, \upsilon, 0, 0))\|_Z d\upsilon : (\epsilon, \upsilon) \in \Delta\},$$

$$\Lambda_2 = \max\{\int_0^{\epsilon} \|\Xi_1(\epsilon, \upsilon, 0, 0))\|_Z d\upsilon : (\epsilon, \upsilon) \in \Delta\}.$$

(A10) $f : \top \times Z \longrightarrow Z$ is continuous and there exists $\Lambda_3, \Lambda_4 > 0$. That's how it is

$$\|f(\epsilon, \vartheta_1, M_1) - f(\epsilon, \vartheta_2, M_2)\|_Z \leqslant \Lambda_3(\|\vartheta_1 - \vartheta_2\|_Z + \|M_1 - M_2\|_Z),$$
$$\max_{\epsilon \in \top} \|f(\epsilon, 0, 0))\|_Z = \Lambda_4.$$

Consider $\kappa_0 = \max\{\|\eta(\epsilon, \upsilon, \vartheta)\|_{\mathfrak{B}(Z)}, \vartheta \in \mathfrak{B}, 0 \leqslant s \leqslant \epsilon \leqslant \sigma\}$.

(A11) $\alpha, \beta : \top \longrightarrow \top$ be continuous absolutely and there exist $o > 0$ and $\mho > 0$ such that $\alpha'(\epsilon) \geqslant o$ and $\beta'(\epsilon) \geqslant \mho$ respectively for $\epsilon \in \top$.

(A12)

$$\kappa_0\Big[\|\vartheta_0\|_\xi + \gamma + r[\Lambda_3\sigma(1 + 1/o) + (\Lambda_1 + \Lambda_1)\sigma(1 + 1/\mho)] + \sigma(\Lambda_4 + \Lambda_2 + \Lambda_2)\Big] \leqslant r,$$
$$q = \Big[\delta\sigma\|\vartheta_0\|_\xi + \gamma\delta\sigma + \kappa_0\gamma_1 + \kappa_0[\Lambda_3\sigma(1 + 1/o) + \Lambda_1\sigma(1 + 1/\mho)] + \delta\sigma[r(\Lambda_3\sigma(1 + 1/o) + (\Lambda_1 + \Lambda_1)\sigma(1 + 1/\mho))] + \sigma(\Lambda_4 + \Lambda_2 + \Lambda_2)\Big] < 1.$$

We establish next that there are traditional local quasilinear solutions of (1.4)–(1.5). For a mild solution of (1.4)–(1.5) let a function $\vartheta \in C(\top, \zeta)$, $\vartheta_0 \in \zeta$ in $\mathfrak{B}$ satisfying

$$\vartheta(\epsilon) = \eta(\epsilon, 0, \vartheta)\vartheta_0 - \eta(\epsilon, 0, \vartheta)\eth(\vartheta) + \int_0^\epsilon \eta(\epsilon, \upsilon, \vartheta)[f(\upsilon, \vartheta(\upsilon), \vartheta(\alpha(\upsilon))) + \int_0^s \Xi(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau]d\upsilon.$$

A function $\vartheta \in (\top, \zeta)$ such that $\vartheta(\epsilon) \in D(\varpi(\epsilon, \vartheta(\epsilon)))$ for $\epsilon \in (0, ɨ]$, $\vartheta \in C^1((0, ɨ], \zeta)$ and satisfies (1.4)–(1.5) in ζ is called a classical solution of (1.4)–(1.5) on $\top$. Further, there exists $\delta > 0$ such that for all $\vartheta, \nu \in C(\top, \zeta)$ with values in $\mathfrak{B}$ and for all $w \in \xi$ we have

$$\|\eta(\epsilon, \upsilon, \vartheta)\omega - \eta(\epsilon, \upsilon, \vartheta)\omega\| \leqslant \delta\|\omega\|_\xi \int_0^\epsilon \|\vartheta(\tau) - \nu(\tau)\|d\tau.$$

3. Main results

Theorem 3.1. *Let $\vartheta_0 \in \xi$ and let $\mathfrak{B} = \{\vartheta \in \zeta : \|\vartheta\|_\xi \leqslant r\}, r > 0$. If (A1)–(A12) are satisfied, then (1.4)–(1.5) has a unique solution $\vartheta \in C([0, ɨ], \xi) \cap C^1((0, ɨ], \zeta)$.*

Proof. Allow S to be a closed nonempty subset of $C([0, ɨ], \zeta)$ defined by

$$S = \{\vartheta : \vartheta \in C([0, ɨ], \zeta),\ \|\vartheta(\epsilon)\|_\xi \leqslant r,\ \text{for } 0 \leqslant \epsilon \leqslant \sigma\}.$$

Let a mapping P on S defined by

$$(P\vartheta)(\epsilon) = \eta(\epsilon, 0, \vartheta)\vartheta_0 - \eta(\epsilon, 0, \vartheta)\eth(\vartheta) + \int_0^\epsilon \eta(\epsilon, \upsilon; \vartheta)[f(\upsilon, \vartheta(\upsilon), \vartheta(\alpha(\upsilon))) + \int_0^s \Xi(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau]d\upsilon.$$

We say P maps S to S. we are saying. We've got for $\vartheta \in S$

$$\|P\vartheta(\epsilon)\|_\xi = \|\eta(\epsilon, 0, \vartheta)\vartheta_0 - \eta(\epsilon, 0, \vartheta)\eth(\vartheta) + \int_0^\epsilon \eta(\epsilon, \upsilon, \vartheta)[f(\upsilon, \vartheta(\upsilon), \vartheta(\alpha(\upsilon))) + \int_0^\epsilon \Xi(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon, \tau, \vartheta(\tau), \vartheta(\beta(\tau)))d\tau]d\upsilon\|$$

$$\begin{aligned}
&\leqslant \|\eta(\epsilon,0,\vartheta)\vartheta_0\| + \|\eta(\epsilon,0,\vartheta)\eth(\vartheta)\| \\
&\quad + \int_0^\epsilon \|\eta(\epsilon,\upsilon,\vartheta)\| \Big[\|f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon))) - f(\upsilon,0,0)\| + \|f(\upsilon,0,0)\| \\
&\quad + \|\int_0^s [\Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau))) - \Xi(\upsilon,\tau,0,0)]d\tau\| + \|\int_0^s \Xi(\upsilon,\tau,0,0)d\tau\| \\
&\quad + \|\int_0^\sigma [\Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau))) - \Xi_1(\upsilon,\tau,0,0)]d\tau\| + \|\int_0^\sigma \Xi_1(\upsilon,\tau,0,0)d\tau\|\Big] d\upsilon.
\end{aligned}$$

From $(A8)$–$(A11)$, we get

$$\begin{aligned}
\|P\vartheta(\epsilon)\|_\xi = \kappa_0\|\vartheta_0\|_\xi + \kappa_0\gamma + \int_0^\epsilon \kappa_0\Big[&\Lambda_3(\|\vartheta(\upsilon)\| + \|\vartheta(\alpha(\upsilon))\|) + \Lambda_4 \\
&+ \int_0^s \Lambda_1(\|\vartheta(\upsilon)\| + \|\vartheta(\beta(\epsilon))\|)d\tau \\
&+ \int_0^\sigma \Lambda_1(\|\vartheta(\upsilon)\| + \|\vartheta(\beta(\epsilon))\|)d\tau + \int_0^s \Lambda_2 d\tau + \int_0^\sigma \Lambda_2 d\tau\Big] d\upsilon \\
\leqslant \kappa_0\|\vartheta_0\|_\xi + \kappa_0\gamma + \kappa_0\Big[&\Lambda_3\sigma r + \Lambda_3\int_0^\epsilon (\|\vartheta(\alpha(\upsilon))\|\vartheta(\alpha'(\upsilon)/o))d\upsilon \\
&+ \Lambda_4\sigma + (\Lambda_1+\Lambda_1)\sigma r + (\Lambda_1+\Lambda_1)\int_0^\epsilon (\|\vartheta(\beta(\upsilon))\|(\beta'(\upsilon)/\mho)d\upsilon + (\Lambda_2+\Lambda_2)\sigma\Big] \\
\leqslant \kappa_0\|\vartheta_0\|_\xi + \kappa_0\gamma + \kappa_0\Big[&\Lambda_3\sigma r + (\Lambda_3/o)\int_{\alpha(0)}^{\alpha(\epsilon)} (\|\vartheta(\upsilon)\|d\upsilon + \Lambda_4\sigma \\
&+ (\Lambda_1+\Lambda_1)\sigma r + ((\Lambda_1+\Lambda_1)/\mho)\int_{\beta(0)}^{\beta(\epsilon)} (\|\vartheta(\upsilon))\|d\upsilon + (\Lambda_2+\Lambda_2)\sigma\Big] \\
\leqslant \kappa_0\Big[&\|\vartheta_0\|_\xi + \gamma + r[\Lambda_3\sigma(1+1/o) + (\Lambda_1+\Lambda_1)\sigma(1+1/\mho)] + \sigma(\Lambda_4+\Lambda_2+\Lambda_2)\Big].
\end{aligned}$$

From $(A12)$, we get $\|P\vartheta(\epsilon)\|_\xi \leqslant r$. Then $P : S \longrightarrow S$. For $\vartheta, \nu \in S$,

$$\begin{aligned}
\Big\|P\vartheta(\epsilon) - P\nu(\epsilon)\Big\| \leqslant &\Big\|\eta(\epsilon,0,\vartheta)\vartheta_0 - \eta(\epsilon,0,\nu)\vartheta_0\Big\| + \Big\|\eta(\epsilon,0,\vartheta)\eth(\vartheta) - \eta(\epsilon,0,\nu)\eth(\nu)\Big\| \\
&+ \int_0^\epsilon \Big\|\eta(\epsilon,\upsilon,\vartheta)\Big[f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon))) + \int_0^s \Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau \\
&+ \int_0^\sigma \Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau\Big] - \eta(\epsilon,\upsilon,\nu)\Big[f(\upsilon,\nu(\upsilon),\nu(\alpha(\upsilon))) \\
&+ \int_0^s \Xi(\upsilon,\tau,\nu(\tau),\nu(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon,\tau,\nu(\tau),\nu(\beta(\tau)))d\tau\Big]\Big\|d\upsilon \\
\leqslant &\Big\|\eta(\epsilon,0,\vartheta)\vartheta_0 - \eta(\epsilon,0,\nu)\vartheta_0\Big\| + \Big\|\eta(\epsilon,0,\vartheta)\eth(\vartheta) - \eta(\epsilon,0,\nu)\eth(\vartheta)\Big\| \\
&- \Big\|\eta(\epsilon,0,\nu)\eth(\vartheta) - \eta(\epsilon,0,\nu)\eth(\nu)\Big\| + \int_0^\epsilon \Big\{\Big\|\eta(\epsilon,\upsilon,\vartheta)\Big[f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon))) \\
&+ \int_0^s \Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau\Big] \\
&- \eta(\epsilon,\upsilon,\nu)\Big[f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon))) + \int_0^s \Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau \\
&+ \int_0^\sigma \Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau\Big] + \Big\|\eta(\epsilon,\upsilon,\nu)\Big[f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon))) \\
&+ \int_0^s \Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau + \int_0^\sigma \Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau\Big]\Big\|
\end{aligned}$$

$$-\eta(\epsilon,\upsilon,\nu)\Big[f(\upsilon,\nu(\upsilon),\nu(\alpha(\upsilon)))$$
$$+\int_0^s \Xi(\upsilon,\tau,\nu(\tau),\nu(\beta(\tau)))d\tau+\int_0^\sigma \Xi_1(\upsilon,\tau,\nu(\tau),\nu(\beta(\tau)))d\tau\Big]\Big\|\Big\}d\upsilon.$$

From (A8)–(A12), we get

$$\begin{aligned}
\|P\vartheta(\epsilon)-P\nu(\epsilon)\| &\leqslant \delta\sigma\|\vartheta_0\|_\xi \max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\|+\gamma\delta\sigma\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\| \\
&\quad+\kappa_0\gamma_1\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\| \\
&\quad+\delta\sigma\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\|\Big[\Lambda_3\int_0^\epsilon(\|\vartheta(\upsilon)\|+\Lambda_3\int_0^\epsilon\|\vartheta(\alpha(\upsilon))\|(\alpha'(\upsilon)/o))d\upsilon \\
&\quad+\Lambda_4\sigma+(\Lambda_1+\Lambda_1)\sigma r+(\Lambda_1+\Lambda_1)\int_0^\epsilon(\|\vartheta(\beta(\upsilon))\|(\beta'(\upsilon)/\mho)d\upsilon+(\Lambda_2+\Lambda_2)\sigma\Big] \\
&\quad+\kappa_0\Big[\Lambda_3\int_0^\epsilon(\|\vartheta(\upsilon)-\nu(\upsilon)\|+\Lambda_3\int_0^\epsilon\|\vartheta(\alpha(\upsilon))-\nu(\alpha(\upsilon))\|(\alpha'(\upsilon)/o))d\upsilon \\
&\quad+(\Lambda_1+\Lambda_1)\sigma\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\|+(\Lambda_1+\Lambda_1)\int_0^\epsilon(\|\vartheta(\beta(\upsilon))-\nu(\beta(\upsilon))\|(\beta'(\upsilon)/\mho)d\upsilon \\
&\leqslant\Big[\delta\sigma\|\vartheta_0\|_\xi+\kappa_0\gamma+\kappa_0[\Lambda_3\sigma(1+1/o)+(\Lambda_1+\Lambda_1)\sigma(1+1/\mho)] \\
&\quad+\delta\sigma[r[\Lambda_3\sigma(1+1/o)+(\Lambda_1+\Lambda_1)\sigma(1+1/\mho)]+\sigma(\Lambda_4+\Lambda_2+\Lambda_2)\Big]\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\| \\
&=q\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\|,
\end{aligned}$$

where $0<q<1$. Then for all $\epsilon\in\top$

$$\|P\vartheta(\epsilon)-P\nu(\epsilon)\|\leqslant q\max_{\tau\in\top}\|\vartheta(\tau)-\nu(\tau)\|,$$

then P is a contraction on S. It follows from the theory of P that $\vartheta\in S$ has a single point $\vartheta\in S$ which is the mild solution of (1.4)–(1.5) on $[0,ŧ]$. Note that $\vartheta(\epsilon)$ is in $C(\top,\xi)$ by (A6) see [28]. In fact, $\vartheta(\epsilon)$ be continuous weakly with ξ function. This implies that $\vartheta(\epsilon)$ is separably valued in ξ, then ϑ be measurable strongly. Hence $\|\vartheta(\epsilon)\|_\xi$ be measurable function and bounded. Then, $\vartheta(\epsilon)$ is Bochner integrable [31, Chap. V]. By relation $\vartheta(\epsilon)=P\vartheta(\epsilon)$, then $\vartheta(\epsilon)\in C(\top,\xi)$. Now consider

$$\nu'(\epsilon)+\mathfrak{B}\nu(\epsilon)=W(\epsilon),\ \ \epsilon\in(0,ŧ], \tag{3.1}$$

$$\nu(0)=\vartheta_0-\eth(\vartheta), \tag{3.2}$$

where $\mathfrak{B}(\epsilon)=\varpi(\epsilon,\vartheta(\epsilon))$ and

$$W(\epsilon)=f(\epsilon,\vartheta(\epsilon),\vartheta(\alpha(\epsilon)))+\int_0^\epsilon\Xi(\epsilon,\upsilon,\vartheta(\upsilon),\vartheta(\beta(\upsilon))d\upsilon+\int_0^\sigma\Xi_1(\epsilon,\upsilon,\vartheta(\upsilon),\vartheta(\beta(\upsilon))d\upsilon,\ \ \epsilon\in[0,ŧ],$$

and ϑ is the unique fixed point of P in S. We warrant that $\mathfrak{B}(\epsilon)$ satisfies (H1)–(H3) in [28] and $W\in C(\top,\xi)$. We have from [28] there exists a unique function $\nu\in C(\top,\xi)$ such that $\nu\in C^1((0,ŧ],\zeta)$ intimidating (3.1) and (3.2) in ζ and ν is

$$\nu(\epsilon)=\eta(\epsilon,0,\vartheta)\vartheta_0-\eta(\epsilon,0,\vartheta)\eth(\vartheta)+\int_0^\epsilon\eta(\epsilon,\upsilon,\vartheta)[f(\upsilon,\vartheta(\upsilon),\vartheta(\alpha(\upsilon)))$$
$$+\int_0^s\Xi(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau+\int_0^\sigma\Xi_1(\upsilon,\tau,\vartheta(\tau),\vartheta(\beta(\tau)))d\tau]d\upsilon,$$

where $\eta(\epsilon,\upsilon,\vartheta)$ is generated by $\{\varpi(\epsilon,\vartheta(\epsilon))\},\epsilon\in\top$ of the operators in ζ. The uniqueness of $\nu\implies\nu=\vartheta$ on $\top$ and hence ϑ is a unique solution of (1.4)–(1.5) and $\vartheta\in C([0,ŧ],\xi)\cap C^1((0,ŧ],\zeta)$. □

4. Conclusion

In this manuscript, we studied the existence and uniqueness of mild and traditional solutions for the nonlocal Volterra-Fredholm problem of nonlinear integro-differential equations in a Banach space. The theorem is proved by using some fixed point theorems based on $\mathfrak{R}_0$-Semigroup theory for condensing maps. By using same methodology and ideas as discussed in this paper, one can extended the results to Volterra-Fredholm integro-differential equations of the fractional derivative as Caputo, ψ-Caputo, Hadamard, Caputo-Fabrizio, Hilfer, etc.

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