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$(3,4)$-fuzzy sets and their topological spaces

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Abstract

The aim of this paper is to introduce the concept of $(3,4)$-fuzzy sets. We compare $(3,4)$-fuzzy sets with intuitionistic fuzzy sets, Pythagorean fuzzy sets, and Fermatean fuzzy sets. We focus on the complement of $(3,4)$-fuzzy sets. We construct some of the fundamental set of operations of the $(3,4)$-fuzzy sets. Due to their larger range of describing membership grades, $(3,4)$-fuzzy sets can deal with more uncertain situations than other types of fuzzy sets. For ranking $(3,4)$-fuzzy sets, we define a score function and an accuracy function. In addition, we introduce the concept of $(3,4)$-fuzzy topological space. Ultimately, we define $(3,4)$-fuzzy continuity of a map defined between $(3,4)$-fuzzy topological spaces and we characterize this concept.

Keywords: $(3,4)$-fuzzy sets, operations, $(3,4)$-fuzzy topology, $(3,4)$-fuzzy continuous.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [9]. After that several researchers developed the idea of fuzzy set theory. The concept of intuitionistic fuzzy sets was published by Atanassov [1], this idea was useful in real-life situations such as medical diagnosis and considered one of the extensions of fuzzy sets with enable for the presentation of a bigger body of nonstandard membership grades than fuzzy membership grades. The Pythagorean fuzzy set was offered by Yager [7] as a new fuzzy set. Senapati et al. [8] introduced Fermatean fuzzy sets and constructed some fundamental operations over Fermatean fuzzy sets. Recently, (3, 2)-fuzzy sets were released by Ibrahim et al. [5].

The concept of fuzzy topological space was published by Chang [2]. He defined some basic concepts of topology like the open set, closed set, continuity, and compactness via fuzzy topological spaces. Moreover, the concept of intuitionistic fuzzy topological spaces was introduced by Coker [3]. He also defined some fundamental notions of classical topology such as continuity and compactness. Besides, Pythagorean fuzzy topological spaces were presented by Olgun et al. [6], and Fermatean fuzzy topological spaces were defined by Ibrahim [4].

In this paper, we define a $(3,4)$-fuzzy set, which is a new type of fuzzy set extension and introduce their relationship with other kinds of fuzzy sets. We describe some of the basic set operations on $(3,4)$-fuzzy sets. Furthermore, we investigate the notion of topology for $(3,4)$-fuzzy sets. Finally, we study $(3,4)$-fuzzy continuous maps in details.

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2. Preliminaries

Definition 2.1 ([1]). The intuitionistic fuzzy sets (IFSs) are defined on a non-empty set X as objects having the form $I = \{\langle x, \lambda_I(x), \omega_I(x)\rangle : x \in X\}$, where $\lambda_I(x) : X \to [0,1]$ and $\omega_I(x) : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set I, respectively, and $0 \leqslant \lambda_I(x) + \omega_I(x) \leqslant 1$, for all $x \in X$.

Definition 2.2 ([7]). The Pythagorean fuzzy sets (PFSs) are defined on a non-empty set X as objects having the form $P = \{\langle x, \lambda_P(x), \omega_P(x)\rangle : x \in X\}$, where $\lambda_P(x) : X \to [0,1]$ and $\omega_P(x) : X \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set P, respectively, and $0 \leqslant (\lambda_P(x))^2 + (\omega_P(x))^2 \leqslant 1$, for all $x \in X$.

Definition 2.3 ([8]). Let X be a universe of discourse. A Fermatean fuzzy set (FFS) F in X is an object having the form $F = \{\langle x, \lambda_F(x), \omega_F(x)\rangle : x \in X\}$, where $\lambda_F(x) : X \to [0,1]$ and $\omega_F(x) : X \to [0,1]$, including the condition $0 \leqslant (\lambda_F(x))^3 + (\omega_F(x))^3 \leqslant 1$, for all $x \in X$. The numbers $\lambda_F(x)$ and $\omega_F(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set F.
For any FFS F and $x \in X$, $\pi_F(x) = \sqrt[3]{1 - (\lambda_F(x))^3 + (\omega_F(x))^3}$ is identified as the degree of indeterminacy of x to F.

Definition 2.4 ([5]). Let X be a universe of discourse. A (3, 2)-Fuzzy set ((3, 2)-FS) D in X is an object having the form $D = \{\langle x, \lambda_D(x), \omega_D(x)\rangle : x \in X\}$, where $\lambda_D(x) : X \to [0,1]$ and $\omega_D(x) : X \to [0,1]$, including the condition $0 \leqslant (\lambda_D(x))^3 + (\omega_D(x))^2 \leqslant 1$, for all $x \in X$. The numbers $\lambda_D(x)$ and $\omega_D(x)$ denote, respectively, the degree of membership and the degree of non-membership of the element x in the set D.
For any (3, 2)-FS D and $x \in X$, $\pi_D(x) = \sqrt[5]{1 - (\lambda_D(x))^3 + (\omega_D(x))^2}$ is identified as the degree of indeterminacy of x to D.

3. (3,4)-fuzzy Sets

In this section, we initiate the notion of (3,4)-fuzzy sets and study their relationship with other kinds of fuzzy sets. Then, we furnish some operations to (3,4)-fuzzy sets.

Definition 3.1. Let E be a discourse universe. Then, the (3,4)-fuzzy set (briefly, (3,4)-FS) S in E is an object having the form:

$$S = \{\langle e, \lambda_S(e), \omega_S(e)\rangle : e \in E\},$$

where $\lambda_S(e) : E \to [0,1]$ and $\omega_S(e) : E \to [0,1]$, with condition

$$0 \leqslant (\lambda_S(e))^3 + (\omega_S(e))^4 \leqslant 1,$$

for all $e \in E$, the functions $\lambda_S(e)$ and $\omega_S(e)$ denote the degree of membership and the degree of non-membership, respectively, of the element $e \in E$ in the set S. For any (3,4)-FS and $e \in E$,

$$\pi_S(e) = \sqrt[7]{1 - [(\lambda_S(e))^3 + (\omega_S(e))^4]}$$

is identified as the degree of indeterminacy of e in the set S and $\pi_S(e) \in [0,1]$. In what follows, $(\lambda_S(e))^3 + (\omega_S(e))^4 + (\pi_S(e))^7 = 1$. Otherwise, $\pi_S(e) = 0$ whenever $(\lambda_S(e))^3 + (\omega_S(e))^4 = 1$. For simplicity, we shall mention the symbol $S = (\lambda_S, \omega_S)$ for the (3,4)-FS $S = \{\langle e, \lambda_S(e), \omega_S(e)\rangle : e \in E\}$.

To understand the importance of (3,4)-FS to extend the grades of membership and non-membership degrees, assume that $\lambda_S(e) = 0.9$ and $\omega_S(e) = 0.7$ for $X = \{x\}$. We can get $0.9 + 0.7 = 1.6 > 1$, $(0.9)^2 + (0.7)^2 = 1.3 > 1$, $(0.9)^3 + (0.7)^2 = 1.219 > 1$ and $(0.9)^3 + (0.7)^3 = 1.072 > 1$ which does not obey the condition of IFS, PFS, (3, 2)-FS and FFS. However, we can get $(0.9)^3 + (0.7)^4 = 0.9691 < 1$, which means we can apply the (3,4)-FS to control it. That is $S = \{\langle e, 0.9, 0.7\rangle : e \in E\}$ is a (3,4)-FS.

Theorem 3.2. *The set of* $(3,4)$*-fuzzy membership grades are larger than the set of intuitionistic membership grades, Pythagorean membership grades, and Fermatean membership grades.*

Proof. It is well known that for any two numbers $e_1, e_2 \in [0,1]$, we have

$$e_1^3 \leqslant e_1^2 \leqslant e_1 \quad \text{and} \quad e_2^4 \leqslant e_2^3 \leqslant e_2^2 \leqslant e_2.$$

Then, we have

$$e_1 + e_2 \leqslant 1 \;\Rightarrow\; e_1^2 + e_2^2 \leqslant 1 \;\Rightarrow\; e_1^3 + e_2^3 \leqslant 1 \;\Rightarrow\; e_1^3 + e_2^4 \leqslant 1.$$

Hence, the space of $(3,4)$-fuzzy membership grades is larger than the space of intuitionistic membership grades, Pythagorean membership grades, and Fermatean membership grades. □

This development can be significantly noticed in Figure 1.

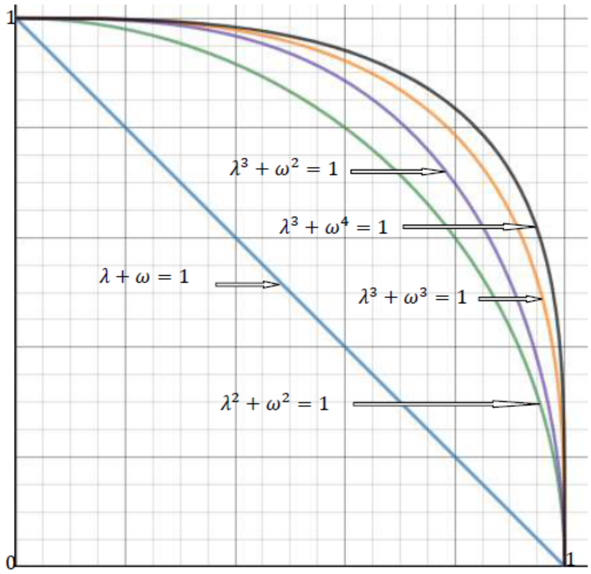

Figure 1: Comparison of grade space of IFSs, PFSs, FFSs, (3, 2)-FSs, and $(3,4)$-FSs.

Remark 3.3. From Figure 1, we notice that the set of $(3,4)$-fuzzy membership grades is larger than the set of (3, 2)-fuzzy membership grades.

Lemma 3.4. *Let* $E = \{e_i\}$ *be a universal set, for* $i = 1, \dots, n$ *and* S *be a* $(3,4)$*-FS. If* $\pi_S(e) = 0$*, then*

1. $|\lambda_S(e_i)| = \sqrt[3]{|(\omega_S(e_i)^2 - 1)(\omega_S(e_i)^2 + 1)|}$;
2. $|\omega_S(e_i)| = \sqrt[4]{|(\lambda_S(e_i) - 1)(\lambda_S(e_i)^2 + \lambda_S(e_i) + 1)|}$.

Proof. Assume that S is a $(3,4)$-FS and $\pi_S(e_i) = 0$ for $e_i \in E$, then

1.

$$\begin{aligned}
(\lambda_S(e_i))^3 + (\omega_S(e_i))^4 = 1 \;&\Rightarrow\; -(\lambda_S(e_i))^3 = (\omega_S(e_i))^4 - 1 \\
&\Rightarrow\; -(\lambda_S(e))^3 = (\omega_S(e_i)^2 - 1)(\omega_S(e_i)^2 + 1) \\
&\Rightarrow\; |-(\lambda_S(e_i))^3| = |(\omega_S(e_i)^2 - 1)(\omega_S(e_i)^2 + 1)| \\
&\Rightarrow\; |(\lambda_S(e_i))|^3 = |(\omega_S(e_i)^2 - 1)(\omega_S(e_i)^2 + 1)| \\
&\Rightarrow\; |\lambda_S(e_i)| = \sqrt[3]{|(\omega_S(e_i)^2 - 1)(\omega_S(e_i)^2 + 1)|}.
\end{aligned}$$

2.

$$\begin{aligned}(\lambda_S(e_i))^3+(\omega_S(e_i))^4=1 &\Rightarrow -(\omega_S(e_i))^4=(\lambda_S(e_i))^3-1\\ &\Rightarrow -(\omega_S(e_i))^4=(\lambda_S(e_i)-1)(\lambda_S(e_i)^2+\lambda_S(e_i)+1)\\ &\Rightarrow |-(\omega_S(e_i))^4|=|(\lambda_S(e_i)-1)(\lambda_S(e_i)^2+\lambda_S(e_i)+1)|\\ &\Rightarrow w|(\omega_S(e_i))|^4=|(\lambda_S(e_i)-1)(\lambda_S(e_i)^2+\lambda_S(e_i)+1)|\\ &\Rightarrow |\omega_S(e_i)|=\sqrt[4]{|(\lambda_S(e_i)-1)(\lambda_S(e_i)^2+\lambda_S(e_i)+1)|}.\end{aligned}$$

□

Example 3.5. Assume that S is a $(3,4)$-FS and $e\in E$ such that $\pi_S(e)=0$ and $\omega_S(e)=0.7$, then

$$|\lambda_S(e)|=\sqrt[3]{|(\omega_S(e)^2-1)(\omega_S(e)^2+1)|},\quad |\lambda_S(e)|=\sqrt[3]{|(0.7^2-1)(0.7^2+1)|},\quad |\lambda_S(e)|=\sqrt[3]{0.7599}.$$

Definition 3.6. Let $S=(\lambda_S,\omega_S)$, $S_1=(\lambda_{S_1},\omega_{S_1})$, and $S_2=(\lambda_{S_2},\omega_{S_2})$ be three $(3,4)$-FSs, then their operations are defined as follows:

1. $S_1\cap S_2=(\min\{\lambda_{S_1},\lambda_{S_2}\},\max\{\omega_{S_1},\omega_{S_2}\})$;
2. $S_1\cup S_2=(\max\{\lambda_{S_1},\lambda_{S_2}\},\min\{\omega_{S_1},\omega_{S_2}\})$;
3. $S^c=(\omega_S^{\frac{4}{3}},\lambda_S^{\frac{3}{4}})$.

We will use supremum "sup" (resp. infimum "inf") instead of maximum "max" (resp. minimum "min") if the union and the intersection are infinite.

Example 3.7. Let $S_1=(0.3,0.6)$, and $S_2=(0.7,0.9)$ be two $(3,4)$-FSs, then

1. $S_1\cap S_2=(\min\{\lambda_{S_1},\lambda_{S_2}\},\max\{\omega_{S_1},\omega_{S_2}\})=(\min\{0.3,0.7\},\max\{0.6,0.9\})=(0.3,0.9)$;
2. $S_1\cup S_2=(\max\{\lambda_{S_1},\lambda_{S_2}\},\min\{\omega_{S_1},\omega_{S_2}\})=(\max\{0.3,0.7\},\min\{0.6,0.9\})=(0.7,0.6)$;
3. $S_1^c=(\omega_{S_1},\lambda_{S_1})=((0.6)^{\frac{4}{3}},(0.3)^{\frac{3}{4}})$.

Definition 3.8. Let $S=(\lambda_S,\omega_S)$, $S_1=(\lambda_{S_1},\omega_{S_1})$, and $S_2=(\lambda_{S_2},\omega_{S_2})$ be three $(3,4)$-FSs and $n>0$, then their operations are defined as follows:

1. $S_1\oplus S_2=\left(\sqrt[3]{\lambda_{S_1}^3+\lambda_{S_2}^3-\lambda_{S_1}^3\lambda_{S_2}^3},\omega_{S_1}\omega_{S_2}\right)$;
2. $S_1\otimes S_2=\left(\lambda_{S_1}\lambda_{S_2},\sqrt[4]{\omega_{S_1}^4+\omega_{S_2}^4-\omega_{S_1}^4\omega_{S_2}^4}\right)$;
3. $nS=\left(\sqrt[3]{1-(1-\lambda_S^3)^n},\omega_S^n\right)$;
4. $S^n=\left(\lambda_S^n,\sqrt[4]{1-(1-\omega_S^4)^n}\right)$.

Theorem 3.9. *Let $S=(\lambda_S,\omega_S)$, $S_1=(\lambda_{S_1},\omega_{S_1})$, and $S_2=(\lambda_{S_2},\omega_{S_2})$ be three $(3,4)$-FSs and $n,n_1,n_2>0$, then*

1. $S_1\oplus S_2=S_2\oplus S_1$;
2. $S_1\otimes S_2=S_2\otimes S_1$;
3. $n(S_1\oplus S_2)=nS_2\oplus nS_1$;
4. $(n_1+n_2)S=n_1S\oplus n_2S$;
5. $(S_1\otimes S_2)^n=S_1^n\otimes S_2^n$;
6. $S^{n_1}\otimes S^{n_2}=S^{n_1+n_2}$.

Proof. Assume that S, S_1 and S_2 are three $(3,4)$-FSs and $n,n_1,n_2>0$, then

1.

$$S_1\oplus S_2=\left(\sqrt[3]{\lambda_{S_1}^3+\lambda_{S_2}^3-\lambda_{S_1}^3\lambda_{S_2}^3},\omega_{S_1}\omega_{S_2}\right)=\left(\sqrt[3]{\lambda_{S_2}^3+\lambda_{S_1}^3-\lambda_{S_2}^3\lambda_{S_1}^3},\omega_{S_2}\omega_{S_1}\right)=S_2\oplus S_1;$$

2.
$$S_1 \otimes S_2 = \left(\lambda_{S_1}\lambda_{S_2}, \sqrt[4]{\omega_{S_1}^4 + \omega_{S_2}^4 - \omega_{S_1}^4\omega_{S_2}^4}\right) = \left(\lambda_{S_2}\lambda_{S_1}, \sqrt[4]{\omega_{S_2}^4 + \omega_{S_1}^4 - \omega_{S_2}^4\omega_{S_1}^4}\right) = S_2 \otimes S_1;$$

3.
$$\begin{aligned} n(S_1 \oplus S_2) = n\left(\sqrt[3]{\lambda_{S_1}^3 + \lambda_{S_2}^3 - \lambda_{S_1}^3\lambda_{S_2}^3}, \omega_{S_1}\omega_{S_2}\right) &= \left(\sqrt[3]{1-(1-(\lambda_{S_1}^3 + \lambda_{S_2}^3 - \lambda_{S_1}^3\lambda_{S_2}^3))^n}, (\omega_{S_1}\omega_{S_2})^n\right) \\ &= \left(\sqrt[3]{1-(1-\lambda_{S_1}^3)^n(1-\lambda_{S_2}^3)^n}, (\omega_{S_1})^n(\omega_{S_2})^n\right), \end{aligned}$$

and

$$\begin{aligned} nS_1 \oplus nS_2 &= \left(\sqrt[3]{1-(1-\lambda_{S_1}^3)^n}, (\omega_{S_1})^n\right) \oplus \left(\sqrt[3]{1-(1-\lambda_{S_2}^3)^n}, (\omega_{S_2})^n\right) \\ &= \left(\sqrt[3]{1-(1-\lambda_{S_1}^3)^n(1-\lambda_{S_2}^3)^n}, (\omega_{S_1})^n(\omega_{S_2})^n\right) = n(S_1 \oplus S_2); \end{aligned}$$

4.
$$\begin{aligned} (n_1+n_2)S = \left(\sqrt[3]{1-(1-\lambda_S^3)^{n_1+n_2}}, (\omega_S)^{n_1+n_2}\right) &= \left(\sqrt[3]{1-(1-\lambda_S^3)^{n_1}(1-\lambda_S^3)^{n_2}}, (\omega_S)^{n_1+n_2}\right) \\ &= \left(\sqrt[3]{1-(1-\lambda_S^3)^{n_1}}, \omega_S^{n_1}\right) \oplus \left(\sqrt[3]{1-(1-\lambda_S^3)^{n_2}}, \omega_S^{n_2}\right) \\ &= n_1 S \oplus n_2 S; \end{aligned}$$

5.
$$\begin{aligned} (S_1 \otimes S_2)^n &= \left(\lambda_{S_1}\lambda_{S_2}, \sqrt[4]{\omega_{S_1}^4 + \omega_{S_2}^4 - \omega_{S_1}^4\omega_{S_2}^4}\right)^n \\ &= \left((\lambda_{S_1}\lambda_{S_2})^n, \sqrt[4]{1-(1-\omega_{S_1}^4 - \omega_{S_2}^4 + \omega_{S_1}^4\omega_{S_2}^4)^n}\right) \\ &= \left(\lambda_{S_1}^n\lambda_{S_2}^n, \sqrt[4]{1-(1-\omega_{S_1}^4)^n(1-\omega_{S_2}^4)^n}\right) \\ &= \left(\lambda_{S_1}^n, \sqrt[4]{1-(1-\omega_{S_1}^4)^n}\right) \otimes \left(\lambda_{S_2}^n, \sqrt[4]{1-(1-\omega_{S_2}^4)^n}\right) \\ &= S_1^n \otimes S_2^n; \end{aligned}$$

6.
$$\begin{aligned} S^{n_1} \otimes S^{n_2} &= \left(\lambda_S^{n_1}, \sqrt[4]{1-(1-\omega_S^4)^{n_1}}\right) \otimes \left(\lambda_S^{n_2}, \sqrt[4]{1-(1-\omega_S^4)^{n_2}}\right) \\ &= \left(\lambda_S^{n_1+n_2}, \sqrt[4]{1-(1-\omega_S^4)^{n_1+n_2}}\right) = S^{(n_1+n_2)}. \end{aligned}$$

□

Theorem 3.10. *Let* $S_1 = (\lambda_{S_1}, \omega_{S_1})$ *and* $S_2 = (\lambda_{S_2}, \omega_{S_2})$ *be two* $(3,4)$*-FSs, then the following properties are valid:*

1. $S_1 \cap S_2 = S_2 \cap S_1$;
2. $S_1 \cup S_2 = S_2 \cup S_1$;
3. $(S_1 \cap S_2) \cup S_2 = S_2$;
4. $(S_1 \cup S_2) \cap S_2 = S_2$.

Proof. Assume that S_1 and S_2 are two $(3,4)$-FSs. Then,

1.
$$S_1 \cap S_2 = (\min\{\lambda_{S_1}, \lambda_{S_2}\}, \max\{\omega_{S_1}, \omega_{S_2}\}) = (\min\{\lambda_{S_2}, \lambda_{S_1}\}, \max\{\omega_{S_2}, \omega_{S_1}\}) = S_2 \cap S_1.$$

2. The proof is similar to (1).
3.

$$(S_1 \cap S_2) \cup S_2 = (\min\{\lambda_{S_1}, \lambda_{S_2}\}, \max\{\omega_{S_1}, \omega_{S_2}\}) \cap (\lambda_{S_2}, \omega_{S_2})$$
$$= (\max\{\min\{\lambda_{S_1}, \lambda_{S_2}\}, \lambda_{S_2}\}, \min\{\max\{\omega_{S_1}, \omega_{S_2}\}, \omega_{S_2}\}) = (\lambda_{S_2}, \omega_{S_2}) = S_2.$$

4. The proof is similar to (3).

□

Theorem 3.11. *Let $S_1 = (\lambda_{S_1}, \omega_{S_1})$, $S_2 = (\lambda_{S_2}, \omega_{S_2})$ and $S_3 = (\lambda_{S_3}, \omega_{S_3})$ be three $(3, 4)$-FSs and $n > 0$, then the following properties are valid:*

1. $S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$;
2. $S_1 \cup (S_2 \cup S_3) = (S_1 \cup S_2) \cup S_3$;
3. $n(S_1 \cup S_2) = nS_1 \cup nS_2$;
4. $(S_1 \cup S_2)^n = S_1^n \cup S_2^n$.

Proof. Assume that S_1, S_2 and S_3 are three $(3, 4)$-FSs and $n > 0$. Then,

1.

$$S_1 \cap (S_2 \cap S_3) = (\lambda_{S_1}, \omega_{S_1}) \cap (\min\{\lambda_{S_2}, \lambda_{S_3}\}, \max\{\omega_{S_2}, \omega_{S_3}\})$$
$$= (\min\{\lambda_{S_1}, \min\{\lambda_{S_2}, \lambda_{S_3}\}\}, \max\{\omega_{S_1}, \max\{\omega_{S_2}, \omega_{S_3}\}\})$$
$$= (\min\{\min\{\lambda_{S_1}, \lambda_{S_2}\}, \lambda_{S_3}\}, \max\{\max\{\omega_{S_1}, \omega_{S_2}\}, \omega_{S_3}\})$$
$$= (\min\{\lambda_{S_1}, \lambda_{S_2}\}, \max\{\omega_{S_1}, \omega_{S_2}\}) \cap (\lambda_{S_3}, \omega_{S_3}) = (S_1 \cap S_2) \cap S_3.$$

2. The proof is similar to (1).
3.

$$n(S_1 \cup S_2) = n(\max\{\lambda_{S_1}, \lambda_{S_2}\}, \min\{\omega_{S_1}, \omega_{S_2}\}) = (\sqrt[3]{1 - (1 - (\max\{\lambda^3_{S_1}, \lambda^3_{S_2}\})^n}, \min\{\omega^n_{S_1}, \omega^n_{S_2}\}).$$

And,

$$nS_1 \cup nS_2 = (\sqrt[3]{1 - (1 - \lambda^3_{S_1})^n}, \omega^n_{S_1}) \cup (\sqrt[3]{1 - (1 - \lambda^3_{S_2})^n}, \omega^n_{S_2})$$
$$= (\max\{\sqrt[3]{1 - (1 - \lambda^3_{S_1})^n}, \sqrt[3]{1 - (1 - \lambda^3_{S_2})^n}\}, \min\{\omega^n_{S_1}, \omega^n_{S_1}\})$$
$$= (\sqrt[3]{1 - (1 - \max\{\lambda^3_{S_1}, \lambda^3_{S_2}\})^n}, \min\{\omega^n_{S_1}, \omega^n_{S_2}\}) = n(S_1 \cup S_2).$$

4. The proof is similar to (3).

□

Theorem 3.12. *Let $S_1 = (\lambda_{S_1}, \omega_{S_1})$ and $S_2 = (\lambda_{S_2}, \omega_{S_2})$ be two $(3, 4)$-FSs, then*

1. $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$;
2. $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$;
3. $(S_1^c)^c = S_1$.

Proof. Let S_1 and S_2 be two $(3, 4)$-FSs. Then,

1.

$$(S_1 \cap S_2)^c = (\min\{\lambda_{S_1}, \lambda_{S_2}\}, \max\{\omega_{S_1}, \omega_{S_2}\})^c$$
$$= (\max\{\omega_{S_1}, \omega_{S_2}\}^{\frac{4}{3}}, \min\{\lambda_{S_1}, \lambda_{S_2}\}^{\frac{3}{4}})$$
$$= (\max\{\omega_{S_1}{}^{\frac{4}{3}}, \omega_{S_2}{}^{\frac{4}{3}}\}, \min\{\lambda_{S_1}{}^{\frac{3}{4}}, \lambda_{S_2}{}^{\frac{3}{4}}\}) = (\omega^{\frac{4}{3}}_{S_1}, \lambda^{\frac{3}{4}}_{S_1}) \cup (\omega^{\frac{4}{3}}_{S_2}, \lambda^{\frac{3}{4}}_{S_2}) = S_1^c \cup S_2^c.$$

2. The proof is similar to (1).
3. $(S_1^c)^c = (\omega_{S_1}^{\frac{4}{3}}, \lambda_{S_1}^{\frac{3}{4}})^c = ([\lambda_{S_1}^{\frac{3}{4}}]^{\frac{4}{3}}, [\omega_{S_1}^{\frac{4}{3}}]^{\frac{3}{4}}) = S_1$.

□

Definition 3.13. Let $S_1 = (\lambda_{S_1}, \omega_{S_1})$ and $S_2 = (\lambda_{S_2}, \omega_{S_2})$ be two $(3,4)$-FSs, then

1. $S_1 = S_2$ if and only if $\lambda_{S_1} = \lambda_{S_2}$ and $\omega_{S_1} = \omega_{S_2}$;
2. $S_1 \geqslant S_2$ if and only if $\lambda_{S_1} \geqslant \lambda_{S_2}$ and $\omega_{S_1} \leqslant \omega_{S_2}$;
3. $S_1 \supset S_2$ if $S_1 \geqslant S_2$.

Example 3.14.

1. If $S_1 = (0.9, 0.7)$ and $S_2 = (0.9, 0.7)$ for $E = e$, then $S_1 = S_2$;
2. If $S_1 = (0.9, 0.4)$ and $S_2 = (0.6, 0.7)$ for $E = e$, then $S_1 \geqslant S_2$ and $S_1 \supset S_2$.

Definition 3.15. Let $S = (\lambda_S, \omega_S)$ be a $(3,4)$-fuzzy set, then the score function of S can be defined as:

$$s(S) = \lambda_S^3 - \omega_S^4.$$

Proposition 3.16. *For any $(3,4)$-fuzzy set $S = (\lambda_S, \omega_S)$, the score function $s(S) \in [-1, 1]$.*

Proof. We know that for $(3,4)$-FS, $\lambda_S^3 + \omega_S^4 \leqslant 1$. Then $\lambda_S^3 - \omega_S^4 \leqslant \lambda_S^3 \leqslant 1$ and $\lambda_S^3 - \omega_S^4 \geqslant -\omega_S^4 \geqslant -1$. Therefore, $-1 \leqslant \lambda_S^3 - \omega_S^4 \leqslant 1$, namely $s(S) \in [-1, 1]$. In particular if $S = (0, 1)$, then $s(S) = -1$, and if $S = (1, 0)$, then $s(S) = 1$. □

Definition 3.17. Let $S = (\lambda_S, \omega_S)$ be a $(3,4)$-fuzzy set, then the accuracy function of S can be defined as

$$a(S) = \lambda_S^3 + \omega_S^4.$$

Remark 3.18. For any $(3,4)$-fuzzy set $S = (\lambda_S, \omega_S)$, the suggested accuracy function $a(S) \in [0, 1]$.

Definition 3.19. For any $(3,4)$-FSs $S_i = (\lambda_{S_i}, \omega_{S_i})$ the comparison technique is supposed as,

1. if $s(S_1) < s(S_2)$, then $S_1 < S_2$;
2. if $s(S_1) > s(S_2)$, then $S_1 > S_2$;
3. if $s(S_1) = s(S_2)$, then
   (a) $a(S_1) < a(S_2)$, then $S_1 < S_2$;
   (b) $a(S_1) > a(S_2)$, then $S_1 > S_2$;
   (c) $a(S_1) = a(S_2)$, then $S_1 = S_2$.

4. Topology on $(3,4)$-fuzzy sets

In this section, we formulate the concept of $(3,4)$-fuzzy topology on the family of $(3,4)$-fuzzy sets and scrutinize main properties.

Definition 4.1. Suppose that τ is a class of $(3,4)$-fuzzy subsets of a non-empty set E. If

1. $1_E, 0_E \in \tau$, where $1_E = (1, 0)$ and $0_E = (0, 1)$;
2. $S_1 \cap S_2 \in \tau$, for any $S_1, S_2 \in E$;
3. $\cup_{i \in I} S_i \in \tau$, for any $\{S_i\}_{i \in I} \in E$,

then, τ is called a $(3,4)$-fuzzy topology on E and (E, τ) is a $(3,4)$-fuzzy topological space. Each member of τ is called an open $(3,4)$-fuzzy subset. The complement of an open $(3,4)$-fuzzy subset is called a closed $(3,4)$-fuzzy subset.

Remark 4.2. The class $\{1_E, 0_E\}$ is called indiscrete (3,4)-fuzzy topological space and the topology that contains all subsets is called discrete (3,4)-fuzzy topological space. In addition, a (3,4)-fuzzy topology τ_1 on a set is called coarser than a (3,4)-fuzzy topology τ_2 on the same set if $\tau_1 \subset \tau_2$.

Example 4.3. Suppose that $\tau = \{1_E, 0_E, S_1, S_2, S_3, S_4, S_5\}$ is a class of (3,4)-fuzzy subsets of $E = \{e_1, e_2\}$, where

$$S_1 = \{\langle e_1, 0.9, 0.62\rangle, \langle e_2, 0.91, 0.61\rangle\}, \qquad S_2 = \{\langle e_1, 0.93, 0.53\rangle, \langle e_2, 0.92, 0.62\rangle\},$$
$$S_3 = \{\langle e_1, 0.89, 0.63\rangle, \langle e_2, 0.90, 0.63\rangle\}, \qquad S_4 = \{\langle e_1, 0.93, 0.53\rangle, \langle e_2, 0.92, 0.61\rangle\},$$
$$S_5 = \{\langle e_1, 0.9, 0.62\rangle, \langle e_2, 0.91, 0.62\rangle\}.$$

Thus, (E, τ) is a (3,4)-fuzzy topological space.

Remark 4.4. We showed that every fuzzy set S is a (3,4)-fuzzy set having the form $S = \{\langle e, \lambda_S(e), 1-\lambda_S(e)\rangle : e \in E\}$. Then, every fuzzy topological space (E, τ_1) in the sense of Chang is obviously a (3,4)-fuzzy topological space in the form $\tau = \{S : \lambda_S \in \tau_1\}$ whenever we identify a fuzzy set in E whose membership function is λ_S with its counterpart $S = \{\langle e, \lambda_S(e), 1-\lambda_S(e)\rangle : e \in E\}$. In the same way, one can note that every intuitionistic fuzzy topology, Pythagorean fuzzy topology and Fermatean fuzzy topology are a (3,4)-fuzzy topology. The following examples illustrate this note.

Example 4.5. Suppose that $\tau = \{1_E, 0_E, S_1, S_2\}$ is a class of fuzzy subsets of $E = \{e\}$, where

$$\begin{aligned}
1_E &= \{\langle e, \lambda_{1_E}(e) = 1, 1-\lambda_{1_E}(e) = \omega_{1_E}(e) = 0\rangle\},\\
0_E &= \{\langle e, \lambda_{0_E}(e) = 0, 1-\lambda_{0_E}(e) = \omega_{0_E}(e) = 1\rangle\},\\
S_1 &= \{\langle e, \lambda_{S_1}(e) = 0.6, 1-\lambda_{S_1}(e) = \omega_{S_1}(e) = 0.4\rangle\},\\
S_2 &= \{\langle e, \lambda_{S_2}(e) = 0.1, 1-\lambda_{S_2}(e) = \omega_{S_2}(e) = 0.9\rangle\}.
\end{aligned}$$

Then, τ is a fuzzy topology on E, and thus it is (3,4)-fuzzy topology.

Example 4.6. Suppose that $\tau = \{1_E, 0_E, S_1, S_2\}$ is a class of fuzzy subsets of $E = \{e_1, e_2\}$, where $S_1 = \{\langle e_1, 0.9, 0.53\rangle, \langle e_2, 0.91, 0.62\rangle\}$ and $S_2 = \{\langle e_1, 0.93, 0.53\rangle, \langle e_2, 0.92, 0.62\rangle\}$. Then, τ is (3,4)-fuzzy topology but τ is not intuitionistic fuzzy topology, Pythagorean fuzzy topology and Fermatean fuzzy topology.

Definition 4.7. Suppose that (E, τ) is a (3,4)-fuzzy topological space and S is a (3,4)-FS in E. Then, the (3,4)-fuzzy interior and the (3,4)-fuzzy closure of S are, respectively, defined as:

1. $\mathrm{int}(S) = \cup\{B : B$ is an open (3,4)-FS in E and $B \subset S\}$;
2. $\mathrm{cl}(S) = \cap\{A : A$ is a closed (3,4)-FS in E and $S \subset A\}$.

Remark 4.8. Let (E, τ) be a (3,4)-fuzzy topological space and S be any (3,4)-FS in E. Then,

1. int(S) is an open (3,4)-FS;
2. cl(S) is a closed (3,4)-FS;
3. $\mathrm{int}(1_E) = \mathrm{cl}(1_E) = 1_E$ and $\mathrm{int}(0_E) = \mathrm{cl}(0_E) = 0_E$.

Example 4.9. Consider the (3,4)-fuzzy topological space in Example 4.3 if

$$S = \{\langle e_1, 0.65, 0.91\rangle, \langle e_2, 0.73, 0.71\rangle\},$$

then $\mathrm{int}(S) = 0_E$ and $\mathrm{cl}(S) = 1_E$.

Theorem 4.10. *Let (E, τ) be a (3,4)-fuzzy topological space. If S_1 and S_2 are two (3,4)-FSs in E, then the following axioms hold:*

1. $\mathrm{int}(S_1) \subset S_1$ *and* $S_1 \subset \mathrm{cl}(S_1)$;

2. *if* $S_1 \subset S_2$, *then* $\mathrm{int}(S_1) \subset \mathrm{int}(S_2)$ *and* $\mathrm{cl}(S_1) \subset \mathrm{cl}(S_2)$;
3. S_1 *is open if and only if* $S_1 \subset \mathrm{int}(S_1)$;
4. S_1 *is closed if and only if* $\mathrm{cl}(S_1) \subset S_1$.

Proof. Obvious. □

Corollary 4.11. *Let* (E, τ) *be a* $(3, 4)$*-fuzzy topological space. If* S_1 *and* S_2 *are two* $(3, 4)$*-FSs in* E, *then*

1. $\mathrm{int}(S_1) \cup \mathrm{int}(S_2) \subset \mathrm{int}(S_1 \cup S_2)$;
2. $\mathrm{cl}(S_1 \cap S_2) \subset \mathrm{cl}(S_1) \cap (S_2)$;
3. $\mathrm{int}(S_1 \cap S_2) = \mathrm{int}(S_1) \cap \mathrm{int}(S_2)$;
4. $\mathrm{cl}(S_1) \cup \mathrm{cl}(S_2) = \mathrm{cl}(S_1 \cup S_2)$.

Proof. (1) and (2) are obvious by Theorem 4.10.

(3) Since $\mathrm{int}(S_1 \cap S_2) \subset \mathrm{int}(S_1)$ and $\mathrm{int}(S_1 \cap S_2) \subset \mathrm{int}(S_2)$ we get $\mathrm{int}(S_1 \cap S_2) \subset \mathrm{int}(S_1) \cap \mathrm{int}(S_2)$. On the contrary, from the fact $\mathrm{int}(S_1) \subset S_1$ and $\mathrm{int}(S_2) \subset S_2$ we get $\mathrm{int}(S_1) \cap \mathrm{int}(S_2) \subset S_1 \cap S_2$ and since $\mathrm{int}(S_1) \cap \mathrm{int}(S_2)$ is open, then we have $\mathrm{int}(S_1) \cap \mathrm{int}(S_2) \subset \mathrm{int}(S_1 \cap S_2)$ and hence $\mathrm{int}(S_1 \cap S_2) = \mathrm{int}(S_1) \cap \mathrm{int}(S_2)$.

(4) The proof is similar to (3). □

Theorem 4.12. *Let* (E, τ) *be a* $(3, 4)$*-fuzzy topological space and* S *be any* $(3, 4)$*-FS in* E. *Then, the following axioms hold:*

1. $\mathrm{cl}(S^c) = \mathrm{int}(S)^c$;
2. $\mathrm{int}(S^c) = \mathrm{cl}(S)^c$;
3. $\mathrm{cl}(S^c)^c = \mathrm{int}(S)$;
4. $\mathrm{int}(S^c)^c = \mathrm{cl}(S)$.

Proof. We will prove (1) and the others can be proved similarly. Let $S = \{\langle e, \lambda_S(e), \omega_S(e)\rangle : e \in E\}$ and assume that the class of open $(3, 4)$-fuzzy sets contained in S is indexed by class $A_i = \{\langle e, \lambda_{A_i}(e), \omega_{A_i}(e)\rangle : i \in I\}$. Then, $\mathrm{int}(S) = \{\langle \bigvee \lambda_{A_i}(e), \bigwedge \omega_{A_i}(e)\rangle\}$. Thus, $\mathrm{int}(S)^c = \{\left\langle \bigwedge [\omega_{A_i}(e)]^{\frac{4}{3}}, \bigvee [\lambda_{A_i}(e)]^{\frac{3}{4}} \right\rangle\}$. Now, $S^c = \{\left\langle [\omega_S(e)]^{\frac{4}{3}}, [\lambda_S(e)]^{\frac{3}{4}} \right\rangle\}$ such that $\lambda_{A_i} \leqslant \lambda_S$ and $\omega_{A_i} \geqslant \omega_S$ for each $i \in I$. This implies that $\{\left\langle [\omega_{A_i}(e)]^{\frac{4}{3}}, [\lambda_{A_i}(e)]^{\frac{3}{4}} \right\rangle i \in I\}$ is the class of closed $(3, 4)$-fuzzy sets containing S^c. That is, $\mathrm{cl}(S^c) = \{\left\langle \bigwedge [\omega_{A_i}(e)]^{\frac{4}{3}}, \bigvee [\lambda_{A_i}(e)]^{\frac{3}{4}} \right\rangle\}$. Hence, $\mathrm{cl}(S^c) = \mathrm{int}(S)^c$. □

5. $(3, 4)$-fuzzy continuous maps

In this section, we define $(3, 4)$-fuzzy continuous maps and give some characterizations.

Definition 5.1. Let $f : E \to T$ be a map with S and D are $(3, 4)$-fuzzy subsets of E and T, respectively. The functions of membership and non-membership of the image of S, denoted by $f[S]$, are respectively, defined by

$$\lambda_{f[S]}(t) = \begin{cases} \sup\limits_{z \in f^{-1}(t)} \lambda_S(z), & f^{-1}(t) \neq \phi, \\ 0, & \text{otherwise}, \end{cases} \qquad \omega_{f[S]}(t) = \begin{cases} \inf\limits_{z \in f^{-1}(t)} \omega_S(z), & f^{-1}(t) \neq \phi, \\ 1, & \text{otherwise}. \end{cases}$$

The functions of membership and non-membership of the inverse image of D, denoted by $f^{-1}[D]$, are respectively, defined by $\lambda_{f^{-1}[D]}(z) = \lambda_D(f(z))$ and $\omega_{f^{-1}[D]}(z) = \omega_D(f(z))$.

Remark 5.2. To prove f[S] and $f^{-1}[D]$ are (3,4)-fuzzy subsets, consider $(r_A(z))^7 = (\lambda_S(z))^3 + (\omega_S(z))^4$ and $t \in T$, then we obtain,

$$\begin{aligned}(\lambda_{f[S]}(t))^3 + (\omega_{f[S]}(t))^4 &= (\sup_{z\in f^{-1}(t)} \lambda_S(z))^3 + (\inf_{z\in f^{-1}(t)} \omega_S(z))^4 \\ &= \sup_{z\in f^{-1}(t)} (\lambda_S(z))^3 + \inf_{z\in f^{-1}(t)} (\omega_S(z))^4 \\ &= \sup_{z\in f^{-1}(t)} ((r_S(z))^7 - (\omega_S(z))^4) + \inf_{z\in f^{-1}(t)} (\omega_S(z))^4 \\ &\leqslant \sup_{z\in f^{-1}(t)} (1 - (\omega_S(z))^4) + \inf_{z\in f^{-1}(t)} (\omega_S(z))^4 = 1,\end{aligned}$$

whenever $f^{-1}(t) \neq \phi$. But, if $f^{-1}(t) = \phi$, then we have $(\lambda_S(z))^3 + (\omega_S(z))^4 = 1$. The proof is obvious for $f^{-1}[D]$.

Example 5.3. Let $E = \{e_1, e_2\}$, $T = \{t_1, t_2\}$ and $f : E \to T$ be defined as follows:

$$f(e) = \begin{cases} t_2, & e = e_1, \\ t_1, & e = e_2. \end{cases}$$

Let $S = \{\langle e_1, 0.6, 0.88\rangle, \langle e_2, 0.8, 0.6\rangle\}$, then $f[S] = \{\langle t_1, 0.8, 0.6\rangle, \langle t_2, 0.6, 0.88\rangle\}$.

Theorem 5.4. *Let* $f : E \to T$ *be a map with* S *and* D *are* (3,4)*-fuzzy subsets of* E *and* T*, respectively. Then, we have*

1. $f^{-1}[D^c] = (f^{-1}[D])^c$;
2. $f[S]^c \subset f[S^c]$;
3. *if* $D_1 \subset D_2$*, then* $f^{-1}[D_1] \subset f^{-1}[D_2]$*, where* D_1 *and* D_2 *are* (3,4)*-fuzzy subsets of* T;
4. *if* $S_1 \subset S_2$*, then* $f[S_1] \subset f[S_2]$*, where* S_1 *and* S_2 *are* (3,4)*-fuzzy subsets of* E;
5. $f[f^{-1}[D]] \subset D$;
6. $S \subset f^{-1}[f[S]]$.

Proof.

1. Let $e \in E$ and D be a (3,4)-fuzzy subsets of T. Then, $\lambda_{f^{-1}[D^c]}(e) = \lambda_{D^c}(f(e)) = (\omega_D(f(e)))^{\frac{3}{4}} = (\omega_{f^{-1}(D)}(e))^{\frac{3}{4}} = \lambda_{f^{-1}[D]^c}(e)$. Similarly, one can have $\omega_{f^{-1}[D^c]}(e) = \omega_{f^{-1}[D]^c}(e)$. Thus, $f^{-1}[D^c] = (f^{-1}[D])^c$.

2. For any $t \in T$ such that $f^{-1}(t) \neq \phi$ and for any (3,4)-fuzzy subsets S of E, we can write

$$\begin{aligned}(r_{f[S]}(t))^7 &= (\lambda_{f[S]}(t))^3 + (\omega_{f[S]}(t))^4 \\ &= \sup_{z\in f^{-1}(t)} \lambda_S(z)^3 + \inf_{z\in f^{-1}(t)} \omega_S(z)^4 \\ &= \sup_{z\in f^{-1}(t)} (r_S(z))^7 - (\omega_S(z))^4 + \inf_{z\in f^{-1}(t)} \omega_S(z)^4 \\ &\leqslant \sup_{z\in f^{-1}(t)} (r_S(z))^7 - \inf_{z\in f^{-1}(t)} (\omega_S(z))^4 + \inf_{z\in f^{-1}(t)} \omega_S(z)^4 = \sup_{z\in f^{-1}(t)} (r_S(z))^7.\end{aligned}$$

Now,

$$\begin{aligned}\lambda_{f[S^c]}(t) &= \sup_{z\in f^{-1}(t)} \lambda_{S^c}(z) \\ &= \sup_{z\in f^{-1}(t)} \sqrt[3]{(\omega_S(z))^4} \\ &= \sup_{z\in f^{-1}(t)} \sqrt[3]{r_S(z)^7 - (\lambda_S(z))^3} \\ &\geqslant \sqrt[3]{\sup_{z\in f^{-1}(t)} r_S(z)^7 - \sup_{z\in f^{-1}(t)} (\lambda_S(z))^3} \\ &\geqslant \sqrt[3]{r_{f[S]}(t) - (\lambda_{f[S]}(t))^3} = \sqrt[3]{(\omega_{f[S]}(t))^4} = \lambda_{f[S]^c}(t).\end{aligned}$$

The proof is obvious when $f^{-1}(t) = \phi$. Following a similar technique, we get

$$\omega_{f[S^c]}(t) \leqslant \omega_{f[S]^c}(t),$$

which means that

$$f[S]^c \subset f[S^c].$$

3. Suppose that $D_1 \subset D_2$. Then, for each $e \in E$,

$$\lambda_{f^{-1}[D_1]}(e) = \lambda_{D_1}(f(e)) \leqslant \lambda_{D_2}(f(e)) = \lambda_{f^{-1}[D_2]}(e).$$

Also, $\omega_{f^{-1}[D_1]}(e) \geqslant \omega_{f^{-1}[D_2]}(e)$. Thus, we get the desired result.

4. Suppose that $S_1 \subset S_2$ and $t \in T$. The proof is easy when $f(t) = \phi$, assume that $f(t) \neq \phi$. Then,

$$\lambda_{f[S_1]}(t) = \sup_{z \in f^{-1}(t)} \lambda_{S_1}(z) \leqslant \sup_{z \in f^{-1}(t)} \lambda_{S_2}(z) = \lambda_{f[S_2]}(t).$$

Similarly, we have $\omega_{f[S_1]}(t) \geqslant \omega_{f[S_2]}(t)$. Thus, we get the desired result.

5. For any $t \in T$ such that $f(t) \neq \phi$, we find that

$$\lambda_{f[f^{-1}[D]]}(t) = \sup_{z \in f^{-1}(t)} \lambda_{f^{-1}[D]}(z) = \sup_{z \in f^{-1}(t)} \lambda_D(f(z)) \leqslant \lambda_D(t).$$

On the other side, we have $\lambda_{f[f^{-1}[D]]}(t) = 0 \leqslant \lambda_D(t)$, when $f(t) = \phi$. Similarly, we have $\omega_{f[f^{-1}[D]]}(t) \geqslant \omega_D(t)$.

6. For any $e \in E$, we have

$$\lambda_{f^{-1}[f[S]]}(e) = \lambda_{f[S]}(f(e)) = \sup_{e \in f^{-1}(t)} \lambda_S(e) \geqslant \lambda_S(e).$$

Similarly, we have $\omega_{f^{-1}[f[S]]}(e) \leqslant \omega_S(e)$. □

The proof for the following result is straight, so it is omitted.

Theorem 5.5. *Let* $f : E \to T$ *be a map. Then,*

1. $f[\cup_{i \in I} S_i] = \cup_{i \in I} f[S_i]$, *for any* $(3,4)$*-fuzzy subsets* S_i *of* E;
2. $f^{-1}[\cup_{i \in I} D_i] = \cup_{i \in I} f^{-1}[D_i]$, *for any* $(3,4)$*-fuzzy subsets* D_i *of* T;
3. $f[S_1 \cap S_2] \subset f[S_1] \cap f[S_2]$, *for any two* $(3,4)$*-fuzzy subsets* S_1 *and* S_2 *of* E;
4. $f^{-1}[\cap_{i \in I} D_i] = \cap_{i \in I} f^{-1}[D_i]$, *for any* $(3,4)$*-fuzzy subsets* D_i *of* T.

Definition 5.6. Assume that S and U are two $(3,4)$-fuzzy subsets in a $(3,4)$-fuzzy topological space. Then, U is said to be a neighborhood of S, if there exists an open $(3,4)$-fuzzy subsets G such that $S \subset G \subset U$.

Theorem 5.7. *A* $(3,4)$*-fuzzy subset* S *is open if and only if it contains a neighborhood of each its subset.*

Proof. Let S be an open $(3,4)$-fuzzy set and A be a $(3,4)$-fuzzy set such that $A \subset S$. Since $A \subset S \subset S$ and S is an open $(3,4)$-fuzzy set, then S is a neighborhood of A.

Conversely, let S be a neighborhood of its each subset. For arbitrary $A \subset S$ there exists an open $(3,4)$-fuzzy set O_A such that $A \subset O_A \subset S$. Thus, we have $S \subset \bigcup_{A \subset S} O_A$ and since for all $A \subset S$ and $O_A \subset S$, we get $\bigcup_{A \subset S} O_A \subset S$. Consequently, we obtain $S = \bigcup_{A \subset S} O_A$ which implies S is an open $(3,4)$-fuzzy set. □

Definition 5.8. Let (E, τ_1) and (T, τ_2) be two $(3,4)$-fuzzy topological spaces and $f : E \to T$ be a map. Then, f is said to be $(3,4)$-fuzzy continuous if for any $(3,4)$-fuzzy subset S of E and for any neighborhood V of $f[S]$ there exists a neighborhood U of S such that $f[U] \subset V$.

Theorem 5.9. *Let* $f : E \to T$ *be a map. Then, the following statements are equivalent:*

1. f *is* $(3,4)$*-fuzzy continuous;*
2. *for each* $(3,4)$*-fuzzy subset* S *of* E *and each neighborhood* V *of* $f[S]$ *there is a neighborhood* U *of* S *such that for each* $D \subset U$, *we have* $f[D] \subset V$;

3. *for each* $(3,4)$*-fuzzy subset* S *of* E *and each neighborhood* V *of* $f[S]$*, there is a neighborhood* U *of* S *such that* $U \subset f^{-1}[V]$*;*
4. *for each* $(3,4)$*-fuzzy subset* S *of* E *and each neighborhood* V *of* $f[S]$*,* $f^{-1}[V]$ *is a neighborhood of* S*.*

Proof.

(1) $\Rightarrow$ (2) Let f be $(3,4)$-fuzzy continuous and S be a $(3,4)$-fuzzy subset of E. Consider V as a neighborhood of $f[S]$. Then, there is a neighborhood U of S such that $f[U] \subset V$. Since $D \subset U$, we have $f[D] \subset f[U] \subset V$.

(2) $\Rightarrow$ (3) Let S be a $(3,4)$-fuzzy subset of E and V be a neighborhood of $f[S]$. From (2), there is a neighborhood U of S such that for each $D \subset U$, we have $f[D] \subset V$. Therefore, $D \subset f^{-1}[f[D]] \subset f^{-1}[V]$. Since D is an arbitrary subset of U, we obtain $U \subset f^{-1}[V]$.

(3) $\Rightarrow$ (4) Let S be a $(3,4)$-fuzzy subset of E and V be a neighborhood of $f[S]$. From (3), there is a neighborhood U of S such that $U \subset f^{-1}[V]$. Since U is a neighborhood of S there is an open $(3,4)$-fuzzy subset G of E such that $S \subset G \subset U$, and so $S \subset G \subset f^{-1}[V]$. Therefore, $f^{-1}[V]$ is a neighborhood of S.

(4) $\Rightarrow$ (1) Let S be a $(3,4)$-fuzzy subset of E and V be a neighborhood of $f[S]$. Then $f^{-1}[V]$ is a neighborhood S. Thus, there is an open $(3,4)$-fuzzy subset G of E such that $S \subset G \subset f^{-1}[V]$ which means $f[G] \subset f[f^{-1}[V]] \subset V$. Moreover, G is an open $(3,4)$-fuzzy subset, thus it is a neighborhood of S. Hence, f is $(3,4)$-fuzzy continuous. □

Theorem 5.10. *Let* (E,τ_1) *and* (T,τ_2) *be two (3, 4)-fuzzy topological spaces. A map* $f : E \to T$ *is (3, 4)-fuzzy continuous if and only if* $f^{-1}[D]$ *is an open (3, 4)-fuzzy subset of* E *for each open (3, 4)-fuzzy subset* D *of* T*.*

Proof. Suppose that f is (3, 4)-fuzzy continuous. Let D be any open (3, 4)-fuzzy subset of T and let $S \subset f^{-1}[D]$. Then, we have $f[S] \subset D$. By Theorem 5.7, there is a neighborhood V of $f[S]$ satisfying $V \subset D$. Since f is (3, 4)-fuzzy continuous, then by Theorem 5.9 we obtain that $f^{-1}[V]$ is a neighborhood of S. Therefore $f^{-1}[V] \subset f^{-1}[D]$, and so $f^{-1}[D]$ is a neighborhood of S. As S is an arbitrary subset of $f^{-1}[D]$, then $f^{-1}[D]$ is an open (3, 4)-fuzzy subset E.

Conversely, let S be a (3, 4)-fuzzy subset of E and V be a neighborhood of $f[S]$. Then, τ_2 contains a (3, 4)-fuzzy subset G of T such that $f[S] \subset G \subset V$ and so $S \subset f^{-1}[f[S]] \subset f^{-1}[G] \subset f^{-1}[V]$. Hence, $f^{-1}[V]$ is a neighborhood of S. This proves that f is (3, 4)-fuzzy continuous. □

The following two examples are constructed such that the first example shows a (3, 4)-fuzzy continuous map, while the second shows a fuzzy map that is not (3, 4)-fuzzy continuous.

Example 5.11. Consider $E = \{e_1, e_2\}$ with the (3, 4)-fuzzy topology $\tau_1 = \{1_E, 0_E, S\}$ and $T = \{t_1, t_2\}$ with the (3, 4)-fuzzy topology $\tau_2 = \{1_T, 0_T, D\}$, where

$$S = \{\langle e_1, 0.6, 0.88\rangle, \langle e_2, 0.8, 0.6\rangle\} \quad \text{and} \quad D = \{\langle t_1, 0.8, 0.6\rangle, \langle t_2, 0.6, 0.88\rangle\}.$$

Let $f : E \to T$ defined as follows:

$$f(e) = \begin{cases} t_2, & e = e_1, \\ t_1, & e = e_2. \end{cases}$$

Since $1_T, 0_T$ and D are open (3, 4)-fuzzy subsets of T, then

$$f^{-1}[1_T] = \{\langle e_1, 1, 0\rangle, \langle e_2, 1, 0\rangle\}, \quad f^{-1}[0_T] = \{\langle e_1, 0, 1\rangle, \langle e_2, 0, 1\rangle\}, \quad f^{-1}[D] = \{\langle e_1, 0.6, 0.88\rangle, \langle e_2, 0.8, 0.6\rangle\},$$

are open (3, 4)-fuzzy subsets of E. Hence, f is (3, 4)-fuzzy continuous.

Example 5.12. Consider $E = \{e_1, e_2\}$ with the (3, 4)-fuzzy topology $\tau_1 = \{1_E, 0_E\}$ and $T = \{t_1, t_2\}$ with the (3, 4)-fuzzy topology $\tau_2 = \{1_T, 0_T, D\}$, where

$$D = \{\langle t_1, 0.92, 0.52\rangle, \langle t_2, 0.62, 0.80\rangle\}.$$

Let $f : E \to T$ defined as follows:

$$f(e) = \begin{cases} t_1, & e = e_1, \\ t_2, & e = e_2. \end{cases}$$

Since D is an open (3, 4)-fuzzy subset of T, but $f^{-1}[D] = \{\langle e_1, 0.92, 0.52\rangle, \langle e_2, 0.62, 0.80\rangle\}$ is not an open (3, 4)-fuzzy subsets of E. Hence, f is not (3, 4)-fuzzy continuous.

6. Conclusions

In this paper, we constructed a new extension of intuitionistic fuzzy set called (3, 4)-fuzzy sets and compared with other classes of fuzzy sets such as intuitionistic fuzzy sets, Pythagorean fuzzy sets and Fermatean fuzzy sets. Further, some well-known operators have been proved over (3, 4)-fuzzy sets. The score function and accuracy function have been defined on (3, 4)-fuzzy sets. Moreover, the concept (3, 4)-fuzzy topology is given. Some fundamental concepts of classical topology are defined like open sets, closed sets, interior and closure. Finally, (3, 4)-fuzzy maps and (3, 4)-fuzzy continuity are presented.

In future works, we will try to present the notions of compactness and connectedness in (3,4)-fuzzy topology.

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