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CR-fuzzy sets and their applications

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Abstract

Pythagorean fuzzy set is one of the successful extensions of the fuzzy set for handling uncertainties in information. Under this environment, in this paper, we introduce a new type of generalized fuzzy sets is called CR-fuzzy sets and compare CR-fuzzy sets with Pythagorean fuzzy sets and Fermatean fuzzy sets. The set operations, score function and accuracy function of CR-fuzzy sets will study along with their several properties.

Keywords: CR-fuzzy sets, operations, score function, accuracy function.

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1. Introduction

In 1965, Zadeh [15] introduced fuzzy sets. After the introduction of the concept of fuzzy sets, several researches were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets have been discussed in [1, 4, 5]. The concept of intuitionistic fuzzy set has been introduced by Atanassov [3] as a generalization concept of fuzzy sets. The intuitionistic fuzzy set theory is useful in various application areas, such as algebraic structures, control systems and various engineering fields. Many researchers have explored various applications of intuitionistic fuzzy set such as medical application, real life situations, education and networking [7–9]. Recently, Yager [14] launched a nonstandard fuzzy set referred to as Pythagorean fuzzy set which is the generalization of intuitionistic fuzzy sets. The construct of Pythagorean fuzzy sets can be used to characterize uncertain information more sufficiently and accurately than intuitionistic fuzzy set. Garg [6] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets. Ibrahim et al. [10] defined a new generalized Pythagorean fuzzy set is called (3, 2)-Fuzzy sets. In 2020, Fermatean fuzzy sets proposed by Senapati and Yager [13], can handle uncertain information more easily in the process of decision making. They also defined basic operations over the Fermatean fuzzy sets. The main advantage of Fermatean fuzzy sets is that it can describe more uncertainties than Pythagorean fuzzy sets, which can be applied in many decision-making problems. The relevant research can be referred to [11, 12]. Al-shami [2] introduced a new extensions of fuzzy sets called square-root fuzzy sets (briefly, SR-Fuzzy sets).

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2. CR-fuzzy sets

In this section, we study the notion of CR-fuzzy sets in details, and introduce some new operations on CR-fuzzy sets along with their several properties. For computations, we use only six decimal places in whole paper.

Definition 2.1. Let Y be a universal set. Then, the CR-fuzzy set (briefly, CR-FS) H which is a set of ordered pairs over Y is defined as follows:

$$H = \{\langle y, \lambda_H(y), \mu_H(y)\rangle : y \in Y\},$$

where $\lambda_H(y) : Y \to [0,1]$ is the degree of membership and $\mu_H(y) : Y \to [0,1]$ is the degree of non-membership of $y \in Y$ to H, such that

$$0 \leqslant (\lambda_H(y))^3 + \sqrt[3]{\mu_H(y)} \leqslant 1,$$

then, there is a degree of indeterminacy of $y \in Y$ to H defined by

$$\pi_H(y) = 1 - [(\lambda_H(y))^3 + \sqrt[3]{\mu_H(y)}].$$

It is clear that $(\lambda_H(y))^3 + \sqrt[3]{\mu_H(y)} + \pi_H(y) = 1$. Otherwise, $\pi_H(y) = 0$ whenever $(\lambda_H(y))^3 + \sqrt[3]{\mu_H(y)} = 1$.

In the interest of simplicity, we shall mention the symbol $H = (\lambda_H, \mu_H)$ for the CR-FS $H = \{\langle y, \lambda_H(y), \mu_H(y)\rangle : y \in Y\}$.

The space of CR-fuzzy membership grades is given in Figure 1.

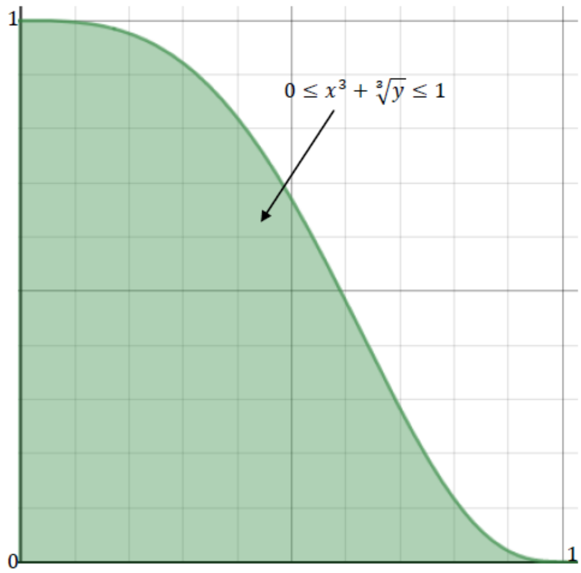

Figure 1: Grades space of CS-FSs.

Definition 2.2. Let Y be a universal set. Then, the intuitionistic fuzzy set (IFS) [3] (resp. Pythagorean fuzzy set (PFS) [14], Fermatean fuzzy set (FFS) [13] and SR-Fuzzy set (SR-FS) [2]) is defined by the following:

$$H = \{\langle y, \lambda_H(y), \mu_H(y)\rangle : y \in Y\},$$

including the condition $0 \leqslant \lambda_H(y) + \mu_H(y) \leqslant 1$ (resp. $0 \leqslant (\lambda_H(y))^2 + (\mu_H(y))^2 \leqslant 1$, $0 \leqslant (\lambda_H(y))^3 + (\mu_H(y))^3 \leqslant 1$ and $0 \leqslant (\lambda_H(y))^2 + \sqrt{\mu_H(y)} \leqslant 1$), where $\lambda_H(y) : Y \to [0,1]$ is the degree of membership and $\mu_H(y) : Y \to [0,1]$ is the degree of non-membership of every $y \in Y$ to H.

Remark 2.3. From Figure 2, we get that

1. the space of Fermatean membership grades is larger than the space of CR-fuzzy membership grades;
2. $H = (\lambda_H \approx 0.168678, \mu_H \approx 0.985671)$ is a point of intersection between CR-fuzzy and Pythagorean fuzzy;
3. for $\lambda_H \in (0, 0.168678)$ and $\mu_H \in (0.985671, 1)$ the space of CR-fuzzy membership grades start to be larger than the space of Pythagorean membership grades;
4. for $\lambda_H \in (0.168678, 1)$ and $\mu_H \in (0, 0.985671)$ the space of CR-fuzzy membership grades start to be smaller than the space of Pythagorean membership grades.

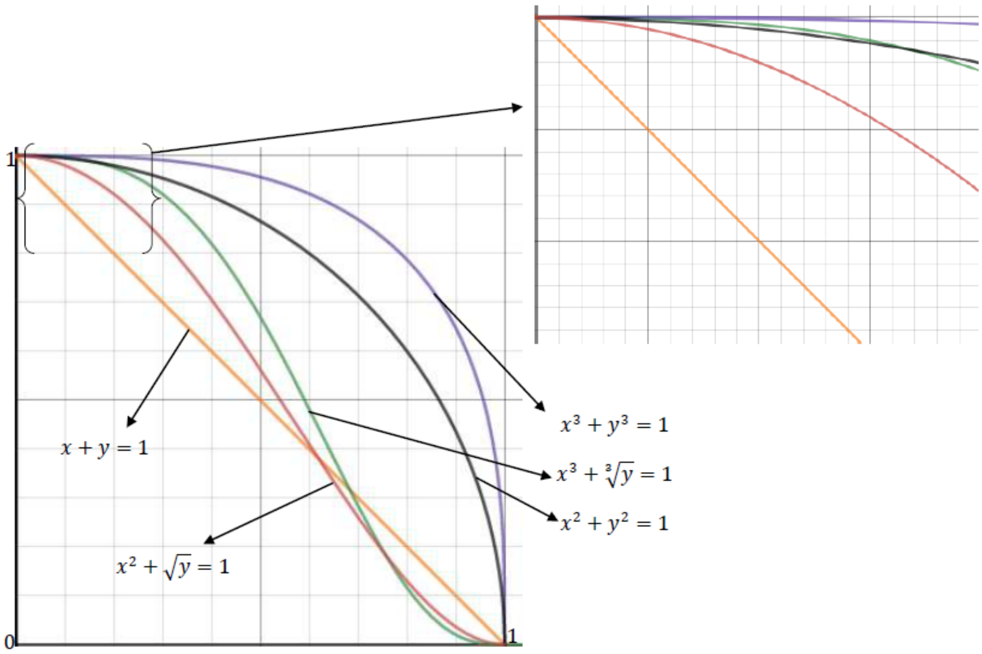

Figure 2: Comparison of grades space of IFSs, SR-FSs, CR-FSs, PFSs, and FFSs.

Definition 2.4. Let $H = (\lambda_H, \mu_H)$, $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ be three CR-fuzzy sets (CR-FSs), then

1. $H_1 \cap H_2 = (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\})$;
2. $H_1 \cup H_2 = (\max\{\lambda_{H_1}, \lambda_{H_2}\}, \min\{\mu_{H_1}, \mu_{H_2}\})$;
3. $H^c = (\sqrt[9]{\mu_H}, (\lambda_H)^9)$.

Example 2.5. Suppose that $H_1 = (0.167, 0.986)$ and $H_2 = (0.8, 0.2)$ are both CR-FSs for $Y = \{y\}$. Then,

1. $H_1 \cap H_2 = (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) = (\min\{0.167, 0.8\}, \max\{0.986, 0.2\}) = (0.167, 0.986)$;
2. $H_1 \cup H_2 = (\max\{\lambda_{H_1}, \lambda_{H_2}\}, \min\{\mu_{H_1}, \mu_{H_2}\}) = (\max\{0.167, 0.8\}, \min\{0.986, 0.2\}) = (0.8, 0.2)$;
3. $H_1^c \approx (0.998435, 0.000000)$.

Theorem 2.6. *If* $H = (\lambda_H, \mu_H)$ *is a CR-FS, then* H^c *is also a CR-FS and* $(H^c)^c = H$.

Proof. Since $0 \leqslant \lambda_H^3 \leqslant 1$, $0 \leqslant \sqrt[3]{\mu_H} \leqslant 1$ and $0 \leqslant (\lambda_H)^3 + \sqrt[3]{\mu_H} \leqslant 1$, then

$$0 \leqslant (\sqrt[9]{\mu_H})^3 + \sqrt[3]{(\lambda_H)^9} = (\lambda_H)^3 + \sqrt[3]{\mu_H} \leqslant 1$$

and hence $0 \leqslant (\sqrt[9]{\mu_H})^3 + \sqrt[3]{(\lambda_H)^9} \leqslant 1$. Thus, H^c is a CR-FS and it is obvious that $(H^c)^c = (\sqrt[9]{\mu_H}, (\lambda_H)^9)^c = (\lambda_H, \mu_H)$. □

Theorem 2.7. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$ *and* $H_2 = (\lambda_{H_2}, \mu_{H_2})$ *be two CR-FSs, then the following properties hold:*

1. $H_1 \cap H_2 = H_2 \cap H_1$;
2. $H_1 \cup H_2 = H_2 \cup H_1$.

Proof. From Definition 2.4, we can obtain:

1. $H_1 \cap H_2 = (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) = (\min\{\lambda_{H_2}, \lambda_{H_1}\}, \max\{\mu_{H_2}, \mu_{H_1}\}) = H_2 \cap H_1$;
2. the proof is similar to 1. □

Theorem 2.8. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$ *and* $H_2 = (\lambda_{H_2}, \mu_{H_2})$ *be two CR-FSs, then*

1. $(H_1 \cap H_2) \cup H_2 = H_2$;
2. $(H_1 \cup H_2) \cap H_2 = H_2$.

Proof. From Definition 2.4, we can obtain:

1.

$$\begin{aligned}(H_1 \cap H_2) \cup H_2 &= (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) \cup (\lambda_{H_2}, \mu_{H_2}) \\ &= (\max\{\min\{\lambda_{H_1}, \lambda_{H_2}\}, \lambda_{H_2}\}, \min\{\max\{\mu_{H_1}, \mu_{H_2}\}, \mu_{H_2}\}) = (\lambda_{H_2}, \mu_{H_2}) = H_2;\end{aligned}$$

2. the proof is similar to 1. □

Theorem 2.9. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$, $H_2 = (\lambda_{H_2}, \mu_{H_2})$, *and* $H_3 = (\lambda_{H_3}, \mu_{H_3})$ *be three CR-FSs, then*

1. $H_1 \cap (H_2 \cap H_3) = (H_1 \cap H_2) \cap H_3$;
2. $H_1 \cup (H_2 \cup H_3) = (H_1 \cup H_2) \cup H_3$.

Proof. For the three CR-FSs H_1, H_2, and H_3, according to Definition 2.4, we can obtain:

1.

$$\begin{aligned}H_1 \cap (H_2 \cap H_3) &= (\lambda_{H_1}, \mu_{H_1}) \cap (\min\{\lambda_{H_2}, \lambda_{H_3}\}, \max\{\mu_{H_2}, \mu_{H_3}\}) \\ &= (\min\{\lambda_{H_1}, \min\{\lambda_{H_2}, \lambda_{H_3}\}\}, \max\{\mu_{H_1}, \max\{\mu_{H_2}, \mu_{H_3}\}\}) \\ &= (\min\{\min\{\lambda_{H_1}, \lambda_{H_2}\}, \lambda_{H_3}\}, \max\{\max\{\mu_{H_1}, \mu_{H_2}\}, \mu_{H_3}\}) \\ &= (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) \cap (\lambda_{H_3}, \mu_{H_3}) = (H_1 \cap H_2) \cap H_3.\end{aligned}$$

2. the proof is similar to 1. □

Theorem 2.10. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$ *and* $H_2 = (\lambda_{H_2}, \mu_{H_2})$ *be two CR-FSs, then*

1. $(H_1 \cap H_2)^c = H_1^c \cup H_2^c$;
2. $(H_1 \cup H_2)^c = H_1^c \cap H_2^c$.

Proof. For the two CR-FSs H_1 and H_2, according to Definition 2.4, we can obtain:

1.

$$\begin{aligned}(H_1 \cap H_2)^c &= (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\})^c \\ &= (\max\{\sqrt[9]{\mu_{H_1}}, \sqrt[9]{\mu_{H_2}}\}, \min\{(\lambda_{H_1})^9, (\lambda_{H_2})^9\}) = (\sqrt[9]{\mu_{H_1}}, (\lambda_{H_1})^9) \cup (\sqrt[9]{\mu_{H_2}}, (\lambda_{H_2})^9) = H_1^c \cup H_2^c;\end{aligned}$$

2. the proof is similar to 1. □

Definition 2.11. Let $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ be two CR-FSs, then

1. $H_1 = H_2$ if and only if $\lambda_{H_1} = \lambda_{H_2}$ and $\mu_{H_1} = \mu_{H_2}$;
2. $H_1 \geqslant H_2$ if and only if $\lambda_{H_1} \geqslant \lambda_{H_2}$ and $\mu_{H_1} \leqslant \mu_{H_2}$;

3. $H_2 \subset H_1$ or $H_1 \supset H_2$ if $H_1 \geqslant H_2$.

Example 2.12. Let $Y = \{y_1, y_2\}$, then

1. $H_1 = H_2$ for $H_1 = \{\langle y_1, 0.5, 0.6\rangle, \langle y_2, 0.4, 0.7\rangle\}$ and $H_2 = \{\langle y_1, 0.5, 0.6\rangle, \langle y_2, 0.4, 0.7\rangle\}$;
2. $H_2 \leqslant H_1$ and $H_2 \subset H_1$ for $H_1 = \{\langle y_1, 0.6, 0.4\rangle, \langle y_2, 0.5, 0.6\rangle\}$ and $H_2 = \{\langle y_1, 0.5, 0.6\rangle, \langle y_2, 0.4, 0.7\rangle\}$.

In order to rank CR-FSs, we define the score function and accuracy function of the CR-FS.

Definition 2.13.

1. The score function of an CR-FS $H = (\lambda_H, \mu_H)$ is defined as $\text{score}(H) = \lambda_H^3 - \sqrt[3]{\mu_H}$.
2. The accuracy function of an CR-FS $H = (\lambda_H, \mu_H)$ is defined as $\text{accuracy}(H) = \lambda_H^3 + \sqrt[3]{\mu_H}$.

Example 2.14. Since $H = \{\langle y_1, 0.5, 0.6\rangle, \langle y_2, 0.4, 0.7\rangle\}$ is CR-FS in $Y = \{y_1, y_2\}$, then we find that $\text{score}(H)(y_1) \approx -0.718433$, $\text{score}(H)(y_2) \approx -0.823904$, $\text{accuracy}(H)(y_1) \approx 0.968433$, and $\text{accuracy}(H)(y_2) \approx 0.951904$.

Theorem 2.15. *Let $H = (\lambda_H, \mu_H)$ be any CR-FS, then the suggested score function $\text{score}(H) \in [-1, 1]$.*

Proof. Since for any CR-FS H, we have $\lambda_H^3 + \sqrt[3]{\mu_H} \leqslant 1$. Hence, $\lambda_H^3 - \sqrt[3]{\mu_H} \leqslant \lambda_H^3 \leqslant 1$ and $\lambda_H^3 - \sqrt[3]{\mu_H} \geqslant -\sqrt[3]{\mu_H} \geqslant -1$. Therefore, $-1 \leqslant \lambda_H^3 - \sqrt[3]{\mu_H} \leqslant 1$, namely $\text{score}(H) \in [-1, 1]$. In particular, if $H = (0, 1)$, then $\text{score}(H) = -1$ and if $H = (1, 0)$, then $\text{score}(H) = 1$. □

Remark 2.16. For any CR-FS $H = (\lambda_H, \mu_H)$, the suggested accuracy function $\text{accuracy}(H) \in [0, 1]$.

Definition 2.17. Let $H = (\lambda_H, \mu_H)$, $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ be three CR-FSs and δ be a positive real number ($\delta > 0$), then we define the following operations:

1. $H_1 \oplus H_2 = \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3}, \mu_{H_1}\mu_{H_2}\right)$;
2. $H_1 \otimes H_2 = \left(\lambda_{H_1}\lambda_{H_2}, (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3\right)$;
3. $\delta H = \left(\sqrt[3]{1 - (1 - \lambda_H^3)^\delta}, \mu_H^\delta\right)$;
4. $H^\delta = \left(\lambda_H^\delta, (1 - (1 - \sqrt[3]{\mu_H})^\delta)^3\right)$.

Example 2.18. Suppose that $H_1 = (\lambda_{H_1} = 0.52, \mu_{H_1} = 0.61)$ and $H_2 = (\lambda_{H_2} = 0.51, \mu_{H_2} = 0.63)$ are both CR-FSs for $Y = \{y\}$. Then,

1.
$$\begin{aligned} H_1 \oplus H_2 &= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3}, \mu_{H_1}\mu_{H_2}\right) \\ &= \left(\sqrt[3]{0.52^3 + 0.51^3 - (0.52)^3(0.51)^3}, (0.61)(0.63)\right) \approx (0.633807, 0.3843); \end{aligned}$$

2.
$$\begin{aligned} H_1 \otimes H_2 &= \left(\lambda_{H_1}\lambda_{H_2}, (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3\right) \\ &= \left((0.52)(0.51), (\sqrt[3]{0.61} + \sqrt[3]{0.63} - \sqrt[3]{0.61}\sqrt[3]{0.63})^3\right) = (0.2652, 0.936351); \end{aligned}$$

3. $\delta H_1 = \left(\sqrt[3]{1 - (1 - \lambda_{H_1}^3)^\delta}, \mu_{H_1}^\delta\right) = \left(\sqrt[3]{1 - (1 - 0.52^3)^2}, 0.61^2\right) \approx (0.639431, 0.3721)$, for $\delta = 2$;
4. $H_1^\delta = \left(\lambda_{H_1}^\delta, (1 - (1 - \sqrt[3]{\mu_{H_1}})^\delta)^3\right) = \left(0.52^2, (1 - (1 - \sqrt[3]{0.61})^2)^3\right) = (0.2704, 0.932358)$, for $\delta = 2$.

Theorem 2.19. *If $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ are two CR-FSs, then $H_1 \oplus H_2$ and $H_1 \otimes H_2$ are also CR-FSs.*

Proof. For CR-FSs $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ the following relations are evident:

$$0 \leqslant \lambda^3_{H_1} \leqslant 1, \;\; 0 \leqslant \sqrt[3]{\mu_{H_1}} \leqslant 1, \;\; 0 \leqslant (\lambda_{H_1})^3 + \sqrt[3]{\mu_{H_1}} \leqslant 1;$$

and

$$0 \leqslant \lambda^3_{H_2} \leqslant 1, \;\; 0 \leqslant \sqrt[3]{\mu_{H_2}} \leqslant 1, \;\; 0 \leqslant (\lambda_{H_2})^3 + \sqrt[3]{\mu_{H_2}} \leqslant 1.$$

Then, we have

$$\lambda^3_{H_1} \geqslant \lambda^3_{H_1}\lambda^3_{H_2}, \;\; \lambda^3_{H_2} \geqslant \lambda^3_{H_1}\lambda^3_{H_2}, \;\; 1 \geqslant \lambda^3_{H_1}\lambda^3_{H_2} \geqslant 0$$

and

$$\sqrt[3]{\mu_{H_1}} \geqslant \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}}, \;\; \sqrt[3]{\mu_{H_2}} \geqslant \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}}, \;\; 1 \geqslant \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}} \geqslant 0,$$

which indicates that

$$\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2} \geqslant 0 \;\text{ implies }\; \sqrt[3]{\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2}} \geqslant 0,$$

and

$$\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}} \geqslant 0 \;\text{ implies }\; (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3 \geqslant 0.$$

Since $\lambda^3_{H_2} \leqslant 1$ and $0 \leqslant 1 - \lambda^3_{H_1}$, then $\lambda^3_{H_2}(1 - \lambda^3_{H_1}) \leqslant (1 - \lambda^3_{H_1})$ and we get $\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2} \leqslant 1$ and hence $\sqrt[3]{\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2}} \leqslant 1$.

Similarly, we can get

$$(\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3 \leqslant 1.$$

It is obvious that

$$0 \leqslant \sqrt[3]{\mu_{H_1}} \leqslant 1 - \lambda^3_{H_1} \;\text{ and }\; 0 \leqslant \sqrt[3]{\mu_{H_2}} \leqslant 1 - \lambda^3_{H_2},$$

then we can get

$$(\sqrt[3]{\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2}})^3 + \sqrt[3]{\mu_{H_1}\mu_{H_2}} \leqslant \lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2} + (1 - \lambda^3_{H_1})(1 - \lambda^3_{H_2}) = 1.$$

Therefore,

$$0 \leqslant \sqrt[3]{\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2}} \leqslant 1, \;\; 0 \leqslant \mu_{H_1}\mu_{H_2} \leqslant 1,$$
$$0 \leqslant (\sqrt[3]{\lambda^3_{H_1} + \lambda^3_{H_2} - \lambda^3_{H_1}\lambda^3_{H_2}})^3 + \sqrt[3]{\mu_{H_1}\mu_{H_2}} \leqslant 1.$$

Similarly, we have

$$0 \leqslant \lambda_{H_1}\lambda_{H_2} \leqslant 1, \;\; 0 \leqslant (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3 \leqslant 1,$$
$$0 \leqslant (\lambda_{H_1}\lambda_{H_2})^3 + \sqrt[3]{(\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3} \leqslant 1.$$

These indicate that both of $H_1 \oplus H_2$ and $H_1 \otimes H_2$ are CR-FSs. □

Theorem 2.20. *Let* $H = (\lambda_H, \mu_H)$ *be a CR-FS and* δ *be a positive real number. Then,* δH *and* H^δ *are also CR-FSs.*

Proof. Since $0 \leqslant \lambda^3_H \leqslant 1$, $0 \leqslant \sqrt[3]{\mu_H} \leqslant 1$ and $0 \leqslant (\lambda_H)^3 + \sqrt[3]{\mu_H} \leqslant 1$, then

$$\begin{aligned} 0 \leqslant \sqrt[3]{\mu_H} \leqslant 1 - \lambda^3_H \;\; &\Rightarrow \;\; 0 \leqslant (1 - \lambda^3_H)^\delta \\ &\Rightarrow \;\; 1 - (1 - \lambda^3_H)^\delta \leqslant 1 \\ &\Rightarrow \;\; 0 \leqslant \sqrt[3]{1 - (1 - \lambda^3_H)^\delta} \leqslant \sqrt[3]{1} = 1. \end{aligned}$$

It is obvious that $0 \leqslant \mu_H^\delta \leqslant 1$, then we can get

$$0 \leqslant (\sqrt[3]{1-(1-\lambda_H^3)^\delta})^3 + \sqrt[3]{\mu_H^\delta} \leqslant 1-(1-\lambda_H^3)^\delta + (1-\lambda_H^3)^\delta = 1.$$

Similarly, we can also get

$$0 \leqslant (\lambda_H^\delta)^3 + \sqrt[3]{(1-(1-\sqrt[3]{\mu_H})^\delta)^3} \leqslant 1.$$

Therefore, δH and H^δ are CR-FSs. □

Theorem 2.21. *Let $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ be two CR-FSs, then the following properties hold:*

1. $H_1 \oplus H_2 = H_2 \oplus H_1$;
2. $H_1 \otimes H_2 = H_2 \otimes H_1$.

Proof. From Definition 2.17, we can obtain:

1. $H_1 \oplus H_2 = \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3}, \mu_{H_1}\mu_{H_2}\right) \left(\sqrt[3]{\lambda_{H_2}^3 + \lambda_{H_1}^3 - \lambda_{H_2}^3\lambda_{H_1}^3}, \mu_{H_2}\mu_{H_1}\right) = H_2 \oplus H_1$;
2. $H_1 \otimes H_2 = \left(\lambda_{H_1}\lambda_{H_2}, (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3\right) = \left(\lambda_{H_2}\lambda_{H_1}, (\sqrt[3]{\mu_{H_2}} + \sqrt[3]{\mu_{H_1}} - \sqrt[3]{\mu_{H_2}}\sqrt[3]{\mu_{H_1}})^3\right) = H_2 \otimes H_1$. □

Theorem 2.22. *Let $H = (\lambda_H, \mu_H)$, $H_1 = (\lambda_{H_1}, \mu_{H_1})$ and $H_2 = (\lambda_{H_2}, \mu_{H_2})$ be three CR-FSs, then*

1. $\delta(H_1 \oplus H_2) = \delta H_1 \oplus \delta H_2$, *for* $\delta > 0$;
2. $(\delta_1 + \delta_2)H = \delta_1 H \oplus \delta_2 H$, *for* $\delta_1, \delta_2 > 0$;
3. $(H_1 \otimes H_2)^\delta = H_1^\delta \otimes H_2^\delta$, *for* $\delta > 0$;
4. $H^{\delta_1} \otimes H^{\delta_2} = H^{(\delta_1+\delta_2)}$, *for* $\delta_1, \delta_2 > 0$.

Proof. For the three CR-FSs H, H_1 and H_2, and $\delta, \delta_1, \delta_2 > 0$, according to Definition 2.17, we can obtain:

1.

$$\begin{aligned}
\delta(H_1 \oplus H_2) &= \delta\left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3}, \mu_{H_1}\mu_{H_2}\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3 - \lambda_{H_2}^3 + \lambda_{H_1}^3\lambda_{H_2}^3)^\delta}, (\mu_{H_1}\mu_{H_2})^\delta\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta(1-\lambda_{H_2}^3)^\delta}, \mu_{H_1}^\delta\mu_{H_2}^\delta\right), \\
\delta H_1 \oplus \delta H_2 &= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta}, \mu_{H_1}^\delta\right) \oplus \left(\sqrt[3]{1-(1-\lambda_{H_2}^3)^\delta}, \mu_{H_2}^\delta\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta + 1-(1-\lambda_{H_2}^3)^\delta - (1-(1-\lambda_{H_1}^3)^\delta)(1-(1-\lambda_{H_2}^3)^\delta)}, \mu_{H_1}^\delta\mu_{H_2}^\delta\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta(1-\lambda_{H_2}^3)^\delta}, \mu_{H_1}^\delta\mu_{H_2}^\delta\right) = \delta(H_1 \oplus H_2);
\end{aligned}$$

2.

$$\begin{aligned}
(\delta_1 + \delta_2)H &= (\delta_1 + \delta_2)(\lambda_H, \mu_H) \\
&= \left(\sqrt[3]{1-(1-\lambda_H^3)^{\delta_1+\delta_2}}, \mu_H^{\delta_1+\delta_2}\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_H^3)^{\delta_1}(1-\lambda_H^3)^{\delta_2}}, \mu_H^{\delta_1+\delta_2}\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_H^3)^{\delta_1} + 1-(1-\lambda_H^3)^{\delta_2} - (1-(1-\lambda_H^3)^{\delta_1})(1-(1-\lambda_H^3)^{\delta_2})}, \mu_H^{\delta_1}\mu_H^{\delta_2}\right) \\
&= \left(\sqrt[3]{1-(1-\lambda_H^3)^{\delta_1}}, \mu_H^{\delta_1}\right) \oplus \left(\sqrt[3]{1-(1-\lambda_H^3)^{\delta_2}}, \mu_H^{\delta_2}\right) = \delta_1 H \oplus \delta_2 H;
\end{aligned}$$

3.

$$\begin{aligned}(H_1\otimes H_2)^\delta &= \left(\lambda_{H_1}\lambda_{H_2},(\sqrt[3]{\mu_{H_1}}+\sqrt[3]{\mu_{H_2}}-\sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3\right)^\delta\\ &= \left((\lambda_{H_1}\lambda_{H_2})^\delta,(1-(1-\sqrt[3]{\mu_{H_1}}-\sqrt[3]{\mu_{H_2}}+\sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^\delta)^3\right)\\ &= \left(\lambda_{H_1}^\delta\lambda_{H_2}^\delta,(1-(1-\sqrt[3]{\mu_{H_1}})^\delta(1-\sqrt[3]{\mu_{H_2}})^\delta)^3\right)\\ &= \left(\lambda_{H_1}^\delta,(1-(1-\sqrt[3]{\mu_{H_1}})^\delta)^3\right)\otimes\left(\lambda_{H_2}^\delta,(1-(1-\sqrt[3]{\mu_{H_2}})^\delta)^3\right)=H_1^\delta\otimes H_2^\delta;\end{aligned}$$

4.

$$\begin{aligned}H^{\delta_1}\otimes H^{\delta_2} &= \left(\lambda_H^{\delta_1},(1-(1-\sqrt[3]{\mu_H})^{\delta_1})^3\right)\otimes\left(\lambda_H^{\delta_2},(1-(1-\sqrt[3]{\mu_H})^{\delta_2})^3\right)\\ &= \left(\lambda_H^{\delta_1+\delta_2},1-(1-\sqrt[3]{\mu_H})^{\delta_1}+1-(1-\sqrt[3]{\mu_H})^{\delta_2}-(1-(1-\sqrt[3]{\mu_H})^{\delta_1})(1-(1-\sqrt[3]{\mu_H})^{\delta_2})\right)\\ &= \left(\lambda_H^{\delta_1+\delta_2},(1-(1-\sqrt[3]{\mu_H})^{\delta_1+\delta_2})^3\right)=H^{(\delta_1+\delta_2)}.\end{aligned}$$

□

Theorem 2.23. *Let* $H_1=(\lambda_{H_1},\mu_{H_1})$ *and* $H_2=(\lambda_{H_2},\mu_{H_2})$ *be two CR-FSs and* $\delta>0$*, then*

1. $\delta(H_1\cup H_2)=\delta H_1\cup\delta H_2$;
2. $(H_1\cup H_2)^\delta=H_1^\delta\cup H_2^\delta$.

Proof. For the two CR-FSs H_1 and H_2, and $\delta>0$, according to Definitions 2.4 and 2.17, we can obtain:

1.

$$\delta(H_1\cup H_2)=\delta(\max\{\lambda_{H_1},\lambda_{H_2}\},\min\{\mu_{H_1},\mu_{H_2}\})=\left(\sqrt[3]{1-(1-\max\{\lambda_{H_1}^3,\lambda_{H_2}^3\})^\delta},\min\{\mu_{H_1}^\delta,\mu_{H_2}^\delta\}\right),$$

$$\begin{aligned}\delta H_1\cup\delta H_2 &= \left(\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta},\mu_{H_1}^\delta\right)\cup\left(\sqrt[3]{1-(1-\lambda_{H_2}^3)^\delta},\mu_{H_2}^\delta\right)\\ &= \left(\max\{\sqrt[3]{1-(1-\lambda_{H_1}^3)^\delta},\sqrt[3]{1-(1-\lambda_{H_2}^3)^\delta}\},\min\{\mu_{H_1}^\delta,\mu_{H_2}^\delta\}\right)\\ &= \left(\sqrt[3]{1-(1-\max\{\lambda_{H_1}^3,\lambda_{H_2}^3\})^\delta},\min\{\mu_{H_1}^\delta,\mu_{H_2}^\delta\}\right)=\delta(H_1\cup H_2);\end{aligned}$$

2. the proof is similar to 1. □

Theorem 2.24. *Let* $H=(\lambda_H,\mu_H)$, $H_1=(\lambda_{H_1},\mu_{H_1})$ *and* $H_2=(\lambda_{H_2},\mu_{H_2})$ *be three CR-FSs, and* $\delta>0$*, then*

1. $(H_1\oplus H_2)^c=H_1^c\otimes H_2^c$;
2. $(H_1\otimes H_2)^c=H_1^c\oplus H_2^c$;
3. $(H^c)^\delta=(\delta H)^c$;
4. $\delta(H)^c=(H^\delta)^c$.

Proof. For the three CR-FSs H, H_1 and H_2, and $\delta>0$, according to Definitions 2.4 (3) and 2.17, we can obtain:

1.

$$\begin{aligned}(H_1\oplus H_2)^c &= \left(\sqrt[3]{\lambda_{H_1}^3+\lambda_{H_2}^3-\lambda_{H_1}^3\lambda_{H_2}^3},\mu_{H_1}\mu_{H_2}\right)^c\\ &= \left(\sqrt[9]{\mu_{H_1}\mu_{H_2}},(\sqrt[3]{\lambda_{H_1}^3+\lambda_{H_2}^3-\lambda_{H_1}^3\lambda_{H_2}^3})^9\right)\\ &= \left(\sqrt[9]{\mu_{H_1}}\sqrt[9]{\mu_{H_2}},(\lambda_{H_1}^3+\lambda_{H_2}^3-\lambda_{H_1}^3\lambda_{H_2}^3)^3\right)\\ &= (\sqrt[9]{\mu_{H_1}},(\lambda_{H_1})^9)\otimes(\sqrt[9]{\mu_{H_2}},(\lambda_{H_2})^9)=H_1^c\otimes H_2^c;\end{aligned}$$

2.

$$\begin{aligned}(H_1 \otimes H_2)^c &= \left(\lambda_{H_1}\lambda_{H_2}, (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3\right)^c \\ &= \left(\sqrt[9]{(\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})^3}, (\lambda_{H_1}\lambda_{H_2})^9\right) \\ &= \left(\sqrt[3]{(\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_2}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_2}})}, (\lambda_{H_1})^9(\lambda_{H_2})^9\right) \\ &= (\sqrt[9]{\mu_{H_1}}, (\lambda_{H_1})^9) \oplus (\sqrt[9]{\mu_{H_2}}, (\lambda_{H_2})^9) = H_1^c \oplus H_2^c;\end{aligned}$$

3.

$$(H^c)^\delta = (\sqrt[9]{\mu_H}, (\lambda_H)^9)^\delta = \left((\sqrt[9]{\mu_H})^\delta, (1-(1-\lambda_H^3)^\delta)^3\right) = \left(\sqrt[3]{1-(1-\lambda_H^3)^\delta}, \mu_H^\delta\right)^c = (\delta H)^c;$$

4.

$$\delta(H)^c = \delta(\sqrt[9]{\mu_H}, (\lambda_H)^9) = \left(\sqrt[3]{1-(1-\sqrt[3]{\mu_H})^\delta}, ((\lambda_H)^9)^\delta\right) = \left(\lambda_H^\delta, (1-(1-\sqrt[3]{\mu_H})^\delta)^3\right)^c = (H^\delta)^c.$$

□

Theorem 2.25. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$, $H_2 = (\lambda_{H_2}, \mu_{H_2})$, *and* $H_3 = (\lambda_{H_3}, \mu_{H_3})$ *be three CR-FSs, then*

1. $(H_1 \cap H_2) \oplus H_3 = (H_1 \oplus H_3) \cap (H_2 \oplus H_3)$;
2. $(H_1 \cup H_2) \oplus H_3 = (H_1 \oplus H_3) \cup (H_2 \oplus H_3)$;
3. $(H_1 \cap H_2) \otimes H_3 = (H_1 \otimes H_3) \cap (H_2 \otimes H_3)$;
4. $(H_1 \cup H_2) \otimes H_3 = (H_1 \otimes H_3) \cup (H_2 \otimes H_3)$.

Proof. By Definitions 2.4 and 2.17, we can obtain:

1.

$$\begin{aligned}(H_1 \cap H_2) \oplus H_3 &= (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) \oplus (\lambda_{H_3}, \mu_{H_3}) \\ &= \left(\sqrt[3]{\min\{\lambda_{H_1}^3, \lambda_{H_2}^3\} + \lambda_{H_3}^3 - \lambda_{H_3}^3 \min\{\lambda_{H_1}^3, \lambda_{H_2}^3\}}, \max\{\mu_{H_1}, \mu_{H_2}\}\mu_{H_3}\right) \\ &= \left(\sqrt[3]{(1-\lambda_{H_3}^3)\min\{\lambda_{H_1}^3, \lambda_{H_2}^3\} + \lambda_{H_3}^3}, \max\{\mu_{H_1}\mu_{H_3}, \mu_{H_2}\mu_{H_3}\}\right), \\ (H_1 \oplus H_3) \cap (H_2 \oplus H_3) &= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_3}^3}, \mu_{H_1}\mu_{H_3}\right) \cap \left(\sqrt[3]{\lambda_{H_2}^3 + \lambda_{H_3}^3 - \lambda_{H_2}^3\lambda_{H_3}^3}, \mu_{H_2}\mu_{H_3}\right) \\ &= \left(\min\{\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_3}^3}, \sqrt[3]{\lambda_{H_2}^3 + \lambda_{H_3}^3 - \lambda_{H_2}^3\lambda_{H_3}^3}\}, \max\{\mu_{H_1}\mu_{H_3}, \mu_{H_2}\mu_{H_3}\}\right) \\ &= \left(\min\{\sqrt[3]{(1-\lambda_{H_3}^3)\lambda_{H_1}^3 + \lambda_{H_3}^3}, \sqrt[3]{(1-\lambda_{H_3}^3)\lambda_{H_2}^3 + \lambda_{H_3}^3}\}, \max\{\mu_{H_1}\mu_{H_3}, \mu_{H_2}\mu_{H_3}\}\right) \\ &= \left(\sqrt[3]{(1-\lambda_{H_3}^3)\min\{\lambda_{H_1}^3, \lambda_{H_2}^3\} + \lambda_{H_3}^3}, \max\{\mu_{H_1}\mu_{H_3}, \mu_{H_2}\mu_{H_3}\}\right),\end{aligned}$$

thus, $(H_1 \cap H_2) \oplus H_3 = (H_1 \oplus H_3) \cap (H_2 \oplus H_3)$;

2. the proof is similar to 1;

3.

$$\begin{aligned}(H_1 \cap H_2) \otimes H_3 &= (\min\{\lambda_{H_1}, \lambda_{H_2}\}, \max\{\mu_{H_1}, \mu_{H_2}\}) \otimes H_3 \\ &= \left(\min\{\lambda_{H_1}, \lambda_{H_2}\}\lambda_{H_3}, (\max\{\sqrt[3]{\mu_{H_1}}, \sqrt[3]{\mu_{H_2}}\} + \sqrt[3]{\mu_{H_3}} - \sqrt[3]{\mu_{H_3}}\max\{\sqrt[3]{\mu_{H_1}}, \sqrt[3]{\mu_{H_2}}\})^3\right) \\ &= \left(\min\{\lambda_{H_1}\lambda_{H_3}, \lambda_{H_2}\lambda_{H_3}\}, ((1-\sqrt[3]{\mu_{H_3}})\max\{\sqrt[3]{\mu_{H_1}}, \sqrt[3]{\mu_{H_2}}\} + \sqrt[3]{\mu_{H_3}})^3\right), \\ (H_1 \otimes H_3) \cap (H_2 \otimes H_3) &= \left(\lambda_{H_1}\lambda_{H_3}, (\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_3}} - \sqrt[3]{\mu_{H_1}}\sqrt[3]{\mu_{H_3}})^3\right)\end{aligned}$$

$$
\begin{aligned}
&\cap \left(\lambda_{H_2}\lambda_{H_3}, (\sqrt[3]{\mu_{H_2}} + \sqrt[3]{\mu_{H_3}} - \sqrt[3]{\mu_{H_2}}\sqrt[3]{\mu_{H_3}})^3\right) \\
&= \left(\lambda_{H_1}\lambda_{H_3}, ((1-\sqrt[3]{\mu_{H_3}})\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_3}})^3\right) \cap \left(\lambda_{H_2}\lambda_{H_3}, ((1-\sqrt[3]{\mu_{H_3}})\sqrt[3]{\mu_{H_2}} + \sqrt[3]{\mu_{H_3}})^3\right) \\
&= \Big(\min\{\lambda_{H_1}\lambda_{H_3}, \lambda_{H_2}\lambda_{H_3}\}, \max\Big\{((1-\sqrt[3]{\mu_{H_3}})\sqrt[3]{\mu_{H_1}} + \sqrt[3]{\mu_{H_3}})^3, ((1 \\
&\quad - \sqrt[3]{\mu_{H_3}})\sqrt[3]{\mu_{H_2}} + \sqrt[3]{\mu_{H_3}})^3\Big\}\Big) \\
&= \left(\min\{\lambda_{H_1}\lambda_{H_3}, \lambda_{H_2}\lambda_{H_3}\}, ((1-\sqrt[3]{\mu_{H_3}})\max\{\sqrt[3]{\mu_{H_1}}, \sqrt[3]{\mu_{H_2}}\} + \sqrt[3]{\mu_{H_3}})^3\right),
\end{aligned}
$$

thus, $(H_1 \cap H_2) \otimes H_3 = (H_1 \otimes H_3) \cap (H_2 \otimes H_3)$;

4. the proof is similar to 3. □

Theorem 2.26. *Let* $H_1 = (\lambda_{H_1}, \mu_{H_1})$, $H_2 = (\lambda_{H_2}, \mu_{H_2})$, *and* $H_3 = (\lambda_{H_3}, \mu_{H_3})$ *be three CR-FSs, then*

1. $H_1 \oplus H_2 \oplus H_3 = H_1 \oplus H_3 \oplus H_2$;
2. $H_1 \otimes H_2 \otimes H_3 = H_1 \otimes H_3 \otimes H_2$.

Proof.

1.

$$
\begin{aligned}
H_1 \oplus H_2 \oplus H_3 &= (\lambda_{H_1}, \mu_{H_1}) \oplus (\lambda_{H_2}, \mu_{H_2}) \oplus (\lambda_{H_3}, \mu_{H_3}) \\
&= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3}, \mu_{H_1}\mu_{H_2}\right) \oplus (\lambda_{H_3}, \mu_{H_3}) \\
&= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3 + \lambda_{H_3}^3 - \lambda_{H_3}^3(\lambda_{H_1}^3 + \lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_2}^3)}, \mu_{H_1}\mu_{H_2}\mu_{H_3}\right) \\
&= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_2}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_2}^3 - \lambda_{H_1}^3\lambda_{H_3}^3 - \lambda_{H_2}^3\lambda_{H_3}^3 + \lambda_{H_1}^3\lambda_{H_2}^3\lambda_{H_3}^3}, \mu_{H_1}\mu_{H_2}\mu_{H_3}\right) \\
&= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_3}^3 + \lambda_{H_2}^3 - \lambda_{H_2}^3(\lambda_{H_1}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_3}^3)}, \mu_{H_1}\mu_{H_2}\mu_{H_3}\right) \\
&= \left(\sqrt[3]{\lambda_{H_1}^3 + \lambda_{H_3}^3 - \lambda_{H_1}^3\lambda_{H_3}^3}, \mu_{H_1}\mu_{H_3}\right) \oplus (\lambda_{H_2}, \mu_{H_2}) \\
&= H_1 \oplus H_3 \oplus H_2.
\end{aligned}
$$

2. The proof is similar to 1. □

3. Conclusions

In this paper, we have defined CR-fuzzy sets and compared their relationship with other types of the generalized fuzzy sets. In addition, some operators on CR-fuzzy sets are introduced and their relationship have been studied.

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