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A computational method on derivative variations of optimal control

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Abstract

In this paper, we propose an algorithm for solving optimal control problems in a class of continuously differentiable control functions with bounded derivatives. Based on derivative variations [R. Enkhbat, B. Barsbold, J. Mongolian Math. Soc., **17** (2013), 27–39], we derive new optimality conditions for the original problem. An algorithm has been constructed based on the optimality conditions. The convergence of the proposed algorithm has been proved. The algorithm was tested on some well known numerical examples.

Keywords: Calculus of variations, optimal control, derivative variation, optimality conditions.

2020 MSC: 49M05, 49K99.

1. Introduction

Optimal control problems subject to ordinary and partial differential equations (ODEs/ PDEs) arise in a wide range of applications, e.g., epidemiology, robotics, astronautics, active queue management, wireless networks, aeronautics and chemical engineering ([13, 17]). Optimal control problems are often solved numerically due to the complexity of most applications. Solving optimal control problems numerically date back the 1950s with the work of Bellman. Since that date to present, the complexity of solving optimal control problems has increased. Several previous works on optimal control have been published, provides a very comprehensive overview of history of the optimal control and numerical techniques for solving them. In order to solve the OCPs numerically, two basic approaches are found: direct methods and indirect methods. In the first method ([1, 2, 2, 9, 16]), the state and/or the control of the OCPs are discretized then converted into a discrete nonlinear optimization problems (NOP). The producing NOP can be solved numerically by well-know NOP solver such as JMODELICA [11], and SNOPT ([10]). A class of direct approaches is quite vast and involves different techniques. Particularly, there are two of the more common classes within direct approaches. Control parametrization, where the control is approximated

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and the dynamic systems are solved by numerical integration using control estimate. This class is called shooting method as well. The other class, the state and the control are discretized hence the dynamic systems are converted into algebraic constraints. Recently, considerable attention has been focused on the second class which is called pseudo spectral or orthogonal collocation methods as well. In the orthogonal collocation method, a finite basis of global interpolating polynomials is applied to estimate the control and state at a set of discretization points. In the other approach, a calculus of variations ([5]) are applied to derive the first order optimality conditions of the original OCPs based on Maximum Pontryagin Principle. This approach leads to a multiple point boundary value problem that is solved to compute the extremal optimal trajectories. Most common methods in this approach are are multiple shooting and collocation. In this paper, inspired by interior ([6]), we propose another approach to solving the OCPs in class of continuously differentiable control functions based on so-called derivative variations. Derivative variations allow us to formulate necessary conditions for the original problem. An algorithm constructed based on optimality conditions is shown to be convergent. We present some examples to illustrate the proposed algorithm.

2. Problem formulation and variations

Consider the following system of ordinary differential equation (ODE) on a fixed interval $[t_0, t_f]$,

$$\dot{x} = f(x(t), u(t), t), \quad \varphi(x(t_0)) = x(t_0) - x_0 = 0, \quad x_0 \in R^n, \tag{2.1}$$

where $t \in [t_0, t_f], x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is the state vector, $u(t) = (u_1(t), u_2(t), \dots, u_r(t))^T$ is the control variable vector and is selected from the class of admissible controls $U \in R^r$. The above equation (2.1) is called the equation of motion. Along with this process, we have a cost functional of the form:

$$J(u) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t))dt \leftarrow \min. \tag{2.2}$$

Here, the functional $L : R^n \times R^r \times R^1 \to R$ is continuous in all arguments and continuously differentiable in x and u. The function $f : R^n \times R^r \times R^1 \to R^n$ is a vector function which is continuous in its arguments (x, u, t) together with its partial derivative with respect to x and assumes to satisfy Lipshitz's condition in x with the same constant L, $\forall u \in R^r, t \in [t_0, t_f]$,

$$\| f(x + \Delta_\epsilon x, u, t) - f(x, u, t) \| \leqslant L \| x \| \tag{2.3}$$

with initial condition $x(t_0) = x_0$. Assume that $u(t) \in C^1_r[t_0, t_f], u(t) \not\equiv \text{cons}$, $\| \dot{u}(t) \| \leqslant M \leqslant \infty, t \in [t_0, t_f]$. By defining the set S as:

$$S = \{\delta \in C^1([t_0, t_f]) \mid \| \delta(t) \| \leqslant K, \delta(t_0) = \delta(t_f) = 0\},$$

the increment of the ODE (2.1) can be written as:

$$\Delta_\epsilon \dot{x} = \Delta_\epsilon f(x(t), u(t), t)), \ \Delta_\epsilon \varphi(x(t_0)) = 0,$$

where,

$$\Delta_\epsilon f(x(t), u(t), t) = f(\tilde{x}(t), \tilde{u}(t), t) - f(x(t), u(t), t).$$

The partial increment for function f with respect to the control can be written as:

$$\Delta_{\tilde{u}_\epsilon} f(x(t), u(t), t) = f(x(t), \tilde{u}, t) - f(x(t), u(t), t).$$

Hence, the increment of objective functional can be represented as:

$$\begin{aligned} \Delta_\epsilon J(u) = \Delta_\epsilon \phi(x(t_f)) &+ \int_{t_0}^{t_f} \Delta_\epsilon L(x(t), u(t), t) dt \\ &+ \int_{t_0}^{t_f} \langle \ \bar{x}, \Delta_\epsilon \dot{x}(t) - \Delta_\epsilon f(x(t), u(t), t) \rangle dt + \langle \ \mu, \Delta_\epsilon \varphi(x(t_0)) \rangle. \end{aligned} \tag{2.4}$$

Let,

$$H(x(t),u(t),\bar{x},t) = \langle \bar{x}(t), f(x(t),u(t),t) - L(x(t),u(t),t)\rangle \tag{2.5}$$

is the Hamiltonian function, $\langle .,.\rangle$ stands for inner product in the finite dimensional Euclidean space R^n and $\bar{x}(t) \in R^n$ is the so-called adjoint function that defines the 'co-state' of the dynamic system and satisfies the terminal value linear ODE system given by

$$\dot{\bar{x}}(t) = -\frac{\partial H(x,u,\bar{x},t)}{\partial x}, \quad \bar{x}(t_f) = \frac{\partial \phi(x(t_f))}{\partial x} \tag{2.6}$$

using Taylor expansion, the partial increment for the boundary conditions can be written as:

$$\Delta_\epsilon \phi(x(t_f)) = \langle \frac{\partial \phi}{\partial x(t_f)}, \Delta_\epsilon x(t_f) + O_\phi(\| \Delta_\epsilon x(t_f)) \|\rangle, \tag{2.7}$$

$$\Delta_\epsilon \varphi(x(t_0)) = \langle \frac{\partial \varphi}{\partial x(t_0)}, \Delta_\epsilon x(t_0) + O_\varphi(\| \Delta_\epsilon x(t_0)) \|\rangle. \tag{2.8}$$

By using an integrating by parts, we can write:

$$\int_{t_0}^{t_f} \langle \bar{x}(t), \Delta_\epsilon \dot{x}(t)\rangle dt = \langle \bar{x}(t_f), \Delta_\epsilon \dot{x}(t_f)\rangle - \langle \bar{x}(t_0), \Delta_\epsilon \dot{x}(t_0)\rangle - \int_{t_0}^{t_f} \langle \dot{\bar{x}}(t), \Delta_\epsilon x(t)\rangle dt. \tag{2.9}$$

Taking into account (2.5)-(2.9), Equation (2.4) becomes:

$$\begin{aligned}
\Delta_\epsilon J(u) &= \Delta_\epsilon \phi(x(t_f)) - \int_{t_0}^{t_f} \Delta_{\tilde{x}\tilde{u}} H(x(t),u(t),\bar{x},t)dt + \int_{t_0}^{t_f} \langle \bar{x}, \Delta_\epsilon \dot{x}(t)\rangle dt + \langle \mu, \Delta_\epsilon \varphi(x(t_0))\rangle \\
&= \int_{t_0}^{t_f} \langle \bar{x}, \Delta_\epsilon \dot{x}(t)\rangle dt - \int_{t_0}^{t_f} \Delta_{\tilde{x}\tilde{u}} H(x(t),u(t),\bar{x},t)dt + \langle \frac{\partial \phi}{\partial x(t_f)}, \Delta_\epsilon x(t_f) \\
&\quad + O_\phi(\| \Delta_\epsilon x(t_f)) \|\rangle + \langle \frac{\partial \varphi}{\partial x(t_0)}, \Delta_\epsilon x(t_0) + O_\varphi(\| \Delta_\epsilon x(t_0)) \|\rangle + \langle \mu, \Delta_\epsilon \varphi(x(t_0))\rangle \\
&= -\int_{t_0}^{t_f} \langle \dot{\bar{x}}, \Delta_\epsilon x(t)\rangle dt - \int_{t_0}^{t_f} \Delta_{\tilde{x}\tilde{u}} H(x(t),u(t),\bar{x},t)dt + \langle \frac{\partial \phi}{\partial x(t_f)} + \bar{x}(t_f), \Delta_\epsilon x(t_f)\rangle \\
&\quad + \langle -\bar{x}(t_0) + (\frac{\partial \varphi}{\partial x(t_f)})^T \mu, \Delta_\epsilon x(t_0)\rangle + O_\phi(\| \Delta_\epsilon x(t_f) \|) + \langle \mu, O_\varphi(\| \Delta_\epsilon x(t_0) \|)\rangle,
\end{aligned}$$

since,

$$\Delta_{\tilde{x}\tilde{u}} H(x(t),u(t),\bar{x},t) = H(\tilde{x},\tilde{u},\bar{x},t) - H(x,u,\bar{x},t) = \Delta_{\tilde{x}_\epsilon} H(x,\tilde{u},\bar{x},t) - \Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t)$$

and

$$\begin{aligned}
\Delta_{\tilde{x}_\epsilon} H(x,\tilde{u},\bar{x},t) &= \langle \frac{\partial H(x,\tilde{u},\bar{x},t)}{\partial x}, \Delta_\epsilon x(t)\rangle + O_H(\| \Delta_\epsilon x(t) \|), \\
\frac{\partial H(x,\tilde{u},\bar{x},t)}{\partial x} &= \Delta_{\tilde{u}_\epsilon} \frac{\partial H(x,u,\bar{x},t)}{\partial x} + \frac{\partial H(x,u,\bar{x},t)}{\partial x}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\Delta_\epsilon J(u) = &-\int_{t_0}^{t_f} \langle \dot{\bar{x}}, \Delta_\epsilon x(t)\rangle dt - \int_{t_0}^{t_f} \langle \frac{\partial H(x,\tilde{u},\bar{x},t)}{\partial x}, \Delta_\epsilon x(t)\rangle dt - \int_{t_0}^{t_f} O_H(\| \Delta_\epsilon x(t) \|)dt \\
&- \int_{t_0}^{t_f} \Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t)dt + O_\phi(\| \Delta_\epsilon x(t_f) \|) + \langle \mu, O_\varphi(\| \Delta_\epsilon x(t_0) \|)\rangle \\
= &-\int_{t_0}^{t_f} \langle \dot{\bar{x}}, \Delta_\epsilon x(t)\rangle dt - \int_{t_0}^{t_f} \langle \Delta_{\tilde{u}_\epsilon} \frac{\partial H(x,u,\bar{x},t)}{\partial x} + \frac{\partial H(x,u,\bar{x},t)}{\partial x}, \Delta_\epsilon x(t)\rangle dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{t_0}^{t_f} O_H(\| \Delta_\epsilon x(t) \|)dt - \int_{t_0}^{t_f} \Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t)dt + O_\phi(\| \Delta_\epsilon x(t_f) \|) + \langle \mu, O_\varphi(\| \Delta_\epsilon x(t_0) \|)\rangle \\
&= -\int_{t_0}^{t_f} \Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t)dt - \int_{t_0}^{t_f} \langle \Delta_{\tilde{u}_\epsilon} \frac{\partial H(x,u,\bar{x},t)}{\partial x}, \Delta_\epsilon x(t)\rangle dt \\
&\quad -\int_{t_0}^{t_f} O_H(\| \Delta_\epsilon x(t) \|)dt + O_\phi(\| \Delta_\epsilon x(t_f) \|) + O_\varphi(\langle \mu, \| \Delta_\epsilon x(t_0) \|\rangle) \\
&= -\int_{t_0}^{t_f} \Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t)dt + \eta_{\tilde{u}_\epsilon},
\end{aligned}
$$

where,

$$
\begin{aligned}
\eta_{\tilde{u}_\epsilon} = &-\int_{t_0}^{t_f} O_H(\| \Delta_\epsilon x(t) \|)dt + O_\phi(\| \Delta_\epsilon x(t_f) \|) + O_\varphi(\langle \mu, \| \Delta_\epsilon x(t_0) \|\rangle) \\
&-\int_{t_0}^{t_f} \langle \Delta_{\tilde{u}_\epsilon} \frac{\partial H(x,u,\bar{x},t)}{\partial x}, \Delta_\epsilon x(t)\rangle dt.
\end{aligned}
$$

3. Derivative variations

By introducing the derivative variations of $u(t) \in C^1_r$ as:

$$\bar{u_\epsilon} = u(t) + \epsilon\delta\dot{u}(t), \tag{3.1}$$

for all $\delta \in S, \epsilon \in R$ and $t \in [t_0, t_f]$, we define a weak optimal control.

Definition 3.1. An admissible control u^* is said to be a weak optimal control to problem (2.1)-(2.2) if there exist a positive number β such that

$$J(u^*) \leqslant J(u)$$

holds for all $u \in C^1_r([t_0, t_f])$ satisfying

$$\| u - u^* \|_{C^1} \leqslant \beta.$$

Let a process $(x^*, u^*, \bar{x}^*)$ be a weak optimal process. Let $\Delta_\epsilon u(t)$ be an increment of u with respect to time t

$$\Delta_\epsilon u(t) = \bar{u_\epsilon}(t) - u(t) = \epsilon\dot{u}(t)\delta(t). \tag{3.2}$$

On the other hand, since $H(.)$ is differentiable with respect to u then there is an estimate by Vasilieva ([3, 8, 14, 18–21]) for $\Delta_\epsilon x(t)$ as

$$\| \Delta x_\epsilon(t) \| \leqslant K \int_{t_0}^{t_f} \| \Delta_\epsilon u(t) \| \, dt. \tag{3.3}$$

Using (3.2) and (3.3), we get:

$$\| \Delta_\epsilon x(t) \| \leqslant K \int_{t_0}^{t_f} \epsilon\dot{u}(t)\delta(t)dt \leqslant K \int_{t_0}^{t_f} |\epsilon||\dot{u}(t)||\delta(t)|dt \leqslant K|\epsilon|MK_1(t_f - t). \tag{3.4}$$

Thus we have

$$O_\phi(\| \Delta_\epsilon x(t_f) \|) \sim O(\epsilon) \text{ and } \int_{t_0}^{t_f} O_H(\| \Delta_\epsilon x(t_f) \|)dt \sim O(\epsilon). \tag{3.5}$$

We can write the following

$$\Delta_{\tilde{u}_\epsilon} H(x,u,\bar{x},t) = \langle \frac{\partial H(x,u,\bar{x},t)}{\partial u}, \Delta_\epsilon u(t)\rangle + O_H(\| \Delta_\epsilon u(t) \|).$$

Taking into account the Lipshitz condition (2.3) and (3.5) and substituting (3.1) in (3.3)-(3.5), we obtain

$$\eta_{u\bar{u}_\epsilon} \sim O(\epsilon).$$

Thus $\Delta_\epsilon J(u)$ reduces to:

$$\Delta_\epsilon J(u) = -\epsilon \int_{t_0}^{t_f} \langle \frac{\partial H(x, u, \bar{x}, t)}{\partial u}, \dot{u} \rangle \delta(t) dt + O(\epsilon), \tag{3.6}$$
$$\lim_{\epsilon \to 0} \frac{O(\epsilon)}{\epsilon} = 0.$$

In a class of continuously differentiable functions with bound derivatives based on derivative variation, the necessary conditions for problem (2.1)-(2.2) are ready to be described in the following theorem.

Theorem 3.2. *Assume that* u^* *is optimal of the problem* (2.1)-(2.2) *and that* x^* *and* $\bar{x}^*$ *are the corresponding feasible trajectories and the adjoint systems* (2.6)*, respectively. Further, suppose that* $(x^*, u^*, \bar{x}^*)$ *is a weak optimal to problem* (2.1)-(2.2), (2.6)*. Then the following conditions*

$$\langle \frac{\partial (x^*, u^*, \bar{x}^*, t)}{\partial u}, \dot{u}^*(t) \rangle = 0,$$

are satisfied, $\forall t \in [t_0, t_f]$.

Proof. Let $\{x^*, u^*, \bar{x}^*\}$ be a weak optimal process, then the formula (3.6) at the optimal control has the form:

$$\Delta_\epsilon J(u^*) = J(u_\epsilon(t)) - J(u^*) = -\epsilon \int_{t_0}^{t_f} \langle \frac{\partial H(x^*, u^*, \bar{x}^*, t)}{\partial u}, \dot{u}^* \rangle \delta(t) dt + O(\epsilon), \tag{3.7}$$

where $\delta(t) \in S$ and $\frac{O(\epsilon)}{\epsilon} \to_{\epsilon \to 0} \to 0$, $t \in [t_0, t_f]$. Moreover, $\parallel O(\epsilon) \parallel \leqslant Q \parallel \epsilon^2 \parallel$, $Q = \text{const} > 0$. Denote by γ the main term in (3.7). That is

$$\gamma = \langle \frac{\partial H(x^*, u^*, \bar{x}^*, t)}{\partial u}, \dot{u}^* \rangle \delta(t) dt.$$

Then we have

$$\Delta_\epsilon J(u^*) = -\epsilon\gamma + O(\epsilon),$$

where $\lim_{\epsilon \to 0} \frac{O(\epsilon)}{\epsilon} = 0$. To show $\gamma = 0$, assume the contrary, that is $\gamma \neq 0$. Now we can write $\Delta_\epsilon J(u)$ as

$$\Delta_\epsilon J(u^*) = -\epsilon(\gamma + \frac{O(\epsilon)}{\epsilon}).$$

For sufficiently small ϵ, the sign of $\Delta_\epsilon J(u^*)$ is determined by the sign of $\epsilon\gamma$. By taking ϵ as $\epsilon = +\epsilon' \text{sign}(\gamma)$ for small $\epsilon' > 0$, we have $\Delta_\epsilon J(u^*) = J(u_\epsilon(t)) - J(u^*) < 0$ which contradicts the definition of the weak control. Consequently, $\gamma = 0$ or equivalently

$$\int_{t_0}^{t_f} \langle \frac{\partial H(x^*, u^*, \bar{x}^*, t)}{\partial u}, \dot{u}^* \rangle \delta(t) dt = 0,$$

for all $\delta(t) \in S$. The above equality can be written as ([22]):

$$\langle \frac{\partial H(x^*, u^*, \bar{x}^*, t)}{\partial u}, \dot{u}^* \rangle \delta(t) dt = 0,$$

which completes the proof. □

3.1. Derivative variation based algorithm

For the purpose of constructing an algorithm for solving problem (2.1)-(2.2), we additionally assume that $u(t) \in C_r^\infty([t_0, t_f])$ and rewrite the increment formula $\Delta_\varepsilon J(u)$ as:

$$J(u_\varepsilon) - J(u) = -\varepsilon\gamma(u) + O(\varepsilon),$$

where $\gamma(u) = \int_{t_0}^{t_f} \langle \frac{\partial H(x,u,\bar{x},t)}{\partial u}, \dot{u}\rangle \delta(t)dt$. Introduce the function $A(u,t)$ at (u,t) as:

$$A(u,t) = \langle \frac{\partial H(x,u,\bar{x},t)}{\partial u}, \dot{u}\rangle. \tag{3.8}$$

Denote by C the maximum value of $A(t,u)$ on $[t_0, t_f]$, i.e.,

$$C = \max_{t\in[t_0,t_f]} \parallel A(u,t) \parallel .$$

Since parameter ε and $\delta(t)$ are arbitrarily, we can specify these parameters as follows:

- $\varepsilon \in (0, \varepsilon_0), \varepsilon_0 > 0$ is a given small number;
-
$$\delta(t) = \frac{(t-t_0)(t_f-t_0)}{(t_f-t_0)} A(u,t), \forall u \in R^r. \tag{3.9}$$

It is clear that $\parallel \delta(t) \parallel \leqslant t_f - t_0$. Then the increment formula is:

$$J(u_\varepsilon) - J(u) = -\varepsilon \int_{t_0}^{t_f} \frac{(t-t_0)(t_f-t_0)}{(t_f-t_0)} A^2(u,t)dt + O(\varepsilon),\ \varepsilon \in (0, \varepsilon_0).$$

From here, we conclude that there exists ε^* such that:

$$J(u_\varepsilon) \leqslant J(u),\ \forall \varepsilon \in (0, \varepsilon^*).$$

Clearly, if u^* is a weak optimal, then $A(u^*, t) = 0, \forall t \in [t_0, t_f]$.

Theorem 3.3. *Assume that* $\inf_{u\in C_r^\infty} J(u) > -\infty$, *and conditions* (2.3) *and* (3.9) *hold. Then for a sequence* $\{u^i\}$ *generated by the Algorithm* 1 *the condition:*

$$\lim_{i\to\infty} \gamma_i = 0$$

is satisfied.

Proof. By constructing of u^i, we have

$$J(u_\varepsilon^i) - J(u^i) = -\varepsilon P(u^i) + O(\varepsilon).$$

Taking into account of (2.3) and (3.9), we get:

$$J(u^{i+1}) - J(u^i) \leqslant -\varepsilon\gamma_i + Q|\varepsilon|^2, \forall \varepsilon > 0,$$
$$J(u^i) - J(u^{i+1}) \geqslant \varepsilon\gamma_i - Q\varepsilon^2,$$
$$J(u^i) - J(u^{i+1}) \geqslant \max_{\varepsilon>0}[\varepsilon\gamma_i - Q\varepsilon^2] = \frac{\gamma_i^2}{4Q} > 0.$$

Since $J(.)$ is bounded below, there exist

$$\lim_{i\to\infty} J(u^i) = A > -\infty, \qquad J(u^i) - J(u^{i+1}) > \frac{\gamma_i^2}{4Q} > 0.$$

Hence, we obtain $\lim_{i\to\infty} \gamma_i = 0$ which completes the proof. □

Algorithm 1 Derivative variation based algorithm

Step1. $i := 0$ and $u^i \in C_r^\infty$ be an admissible control and $x^i = x^i(u^i, t), \bar{x}^i = \bar{x}^i(x^i, u^i, t)$ are the solution of (2.6) for $u = u^i$ and $x = x^i$.
Step2. Compute $A(u^i, t)$ by the formula (3.8), i.e.,

$$A(u, t) = \langle \frac{\partial H(x, u, \bar{x}, t)}{\partial u}, \dot{u} \rangle, \forall t \in [t_0, t_f].$$

Step3. If $A(u^i, t) = 0, \forall t \in [t_0, t_f]$ then stop and $u^* = u^i$ is a weak optimal control.
Step4. Choose $\delta_i(t) \in S$ such that

$$\delta_i(t) = \frac{(t - t_0)(t_f - t_0)}{(t_f - t_0)C_i} A(u^i, t), \text{ where } C_i = \max_{t \in [t_0, t_f]} A(u^i, t).$$

Step5. Calculate u^{i+1} as:

$$u^{i+1} = u^i + \epsilon_i \delta_k \dot{u}^i, \epsilon_i > 0,$$

where ϵ_i satisfies the condition

$$J(u^i + \epsilon \delta_i \dot{u}^i) < J(u^i), \epsilon > 0.$$

Step6. Set $i = i + 1$ and go to Step 1.

4. Numerical experiments

In this section, for showing the efficiency of the proposed method, three examples are introduced.

4.1. *Example 1: Feldbaum problem*

In this application, we consider the optimal control for the Feldbaum problem [4, 7, 12, 15].

Problem Formulation: The dynamic system for the Feldbaum problem is:

$$\dot{x}(t) = -x(t) + u(t),$$

with initial condition at $t_0 = 0$ being $x(0) = 1$. The problem is to minimize:

$$J = \frac{1}{2} \int_0^1 (x^2(t) + u^2(t)) dx.$$

The adjoint equations are determined from the Hamiltonian function,

$$H(x(t), u(t), \bar{x}(t)) = \frac{1}{2}(x^2(t) + u^2(t)) + \bar{x}(t)(-x(t) + u(t))$$

as

$$\dot{\bar{x}}(t) = -x(t) + \bar{x}(t), \quad \bar{\bar{x}}(t_f) = 0.0.$$

The algebraic relation that must be satisfied is

$$\frac{\partial H}{\partial u} = u(t) + \bar{x}(t) = 0.$$

The ODE was solved on a equidistant discretization with 300 discretization points using the Euler method. The optimal control and optimal state are depicted in Figure 1. The cost functional is reduced to $J^* = 0.19368096635$, the running time is 0.002s, and $|\delta(t)| = 9.75251580493e-05$.

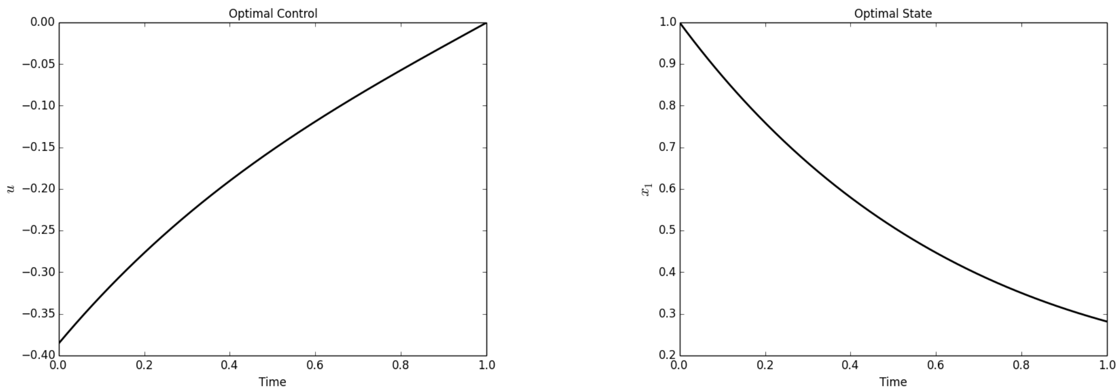

Figure 1: Program data for the Feldbaum Example using Algorithm 1.

4.2. Example 2

Consider the following optimal control problem ([9]):

$$J = -x(2)$$

subject to the dynamic system:

$$\dot{x}(t) = \frac{5}{2}(-x + xu - u^2),$$

with initial condition at $t_0 = 0$ being $x(0) = 1$.

The adjoint equations are determined from the Hamiltonian function,

$$H(x(t), u(t), \bar{x}(t)) = \frac{5}{2}\bar{x}(t)(-x + xu - u^2)$$

as

$$\dot{\bar{x}}(t) = -\frac{5}{2}\bar{x}(1 + u), \quad \bar{x}(\pi) = 1.0.$$

The ODE was solved on a equidistant discretization with 300 discretization points using the Euler method. The optimal control and optimal state are depicted in Figure 2. The cost functional is reduced to $J^* = -0.00863510491982$ and $|\delta(t)| = 0.0180760435862$.

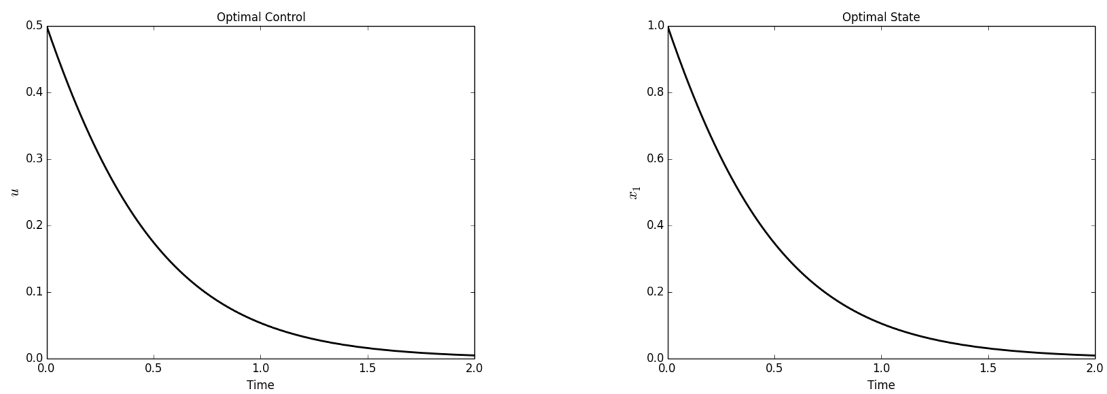

Figure 2: Program data for the Example 2 using Algorithm 1.

4.3. Example 3

Consider the following optimal control problem ([6]):

$$J = \int_0^{\pi} x \sin(x) dx - x(\pi),$$

subject to the dynamic system:

$$\dot{x}(t) = u(t),$$

with initial condition at $t_0 = 0$ being $x(0) = 0$.

The adjoint equations are determined from the Hamiltonian function,

$$H(x(t), u(t), \bar{x}(t)) = x \sin(x) + \bar{x}(t) u(t)$$

as

$$\dot{\bar{x}}(t) = -\sin(t), \quad \bar{x}(\pi) = -1.0.$$

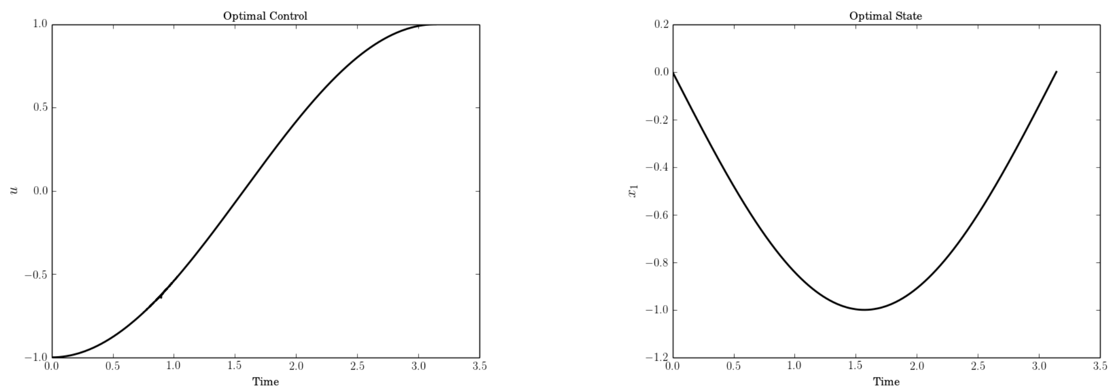

Figure 3: Program data for the Example 3 using Algorithm 1.

The ODE was solved on a equidistant discretization with 300 discretization points using the Euler method. The optimal control and optimal state are depicted in Figure 3. The cost functional is reduced to $J^* = -1.57111862433$ and $|\delta(t)| = 0.0134330255315$.

5. Conclusions

We have introduced a new algorithm using derivative variation approach for solving some class of optimal control problem with bounded control derivatives. Under certain assumptions, we prove that the proposed algorithm converges to a weak solution. The proposed approach has been numerically illustrated on some well know examples. The approach can be used also in solving engineering problems such as robot control and signal processing control problems where only continuous control variables are required.

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