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The quadratic convergence of approximate solutions for singular difference systems with "maxima"

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Abstract

This paper investigates the initial value problem of singular difference systems with maxima. An algorithm based on quasilinearization is suggested to solve the initial value problem for the nonlinear singular difference system with maxima, and the quadratic convergences of the sequence of successive approximations are obtained.

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1. Introduction

Difference equations with maxima are a special type of difference equations that contain the maximum of the unknown function over a previous interval(s). The presence of the maximum function in the equation requires not only more complicated calculations but also a development of new methods for qualitative investigations of the behavior of their solutions (see the monograph [5] and references cited therein). Some results of difference equations with maximum are presented in [4, 9, 10, 12, 15], in which, Hristova, Golev and Stefanova [10] discussed the initial value problem for difference equations with maximum by using the method of quasilinearization [6, 11] which has been used in the proof of the existence results for a wide variety of nonlinear problems.

Recently, much attention has been paid to singular discrete systems as they exist extensively in application fields (see [1, 7, 8]). For example, the discrete dynamic input-output system is a typical

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singular system [16], the mathematical modeling of which is

$$x(k) = Ax(k) + B[x(k+1) - x(k)] + d(k),$$

where $x(k)$ is a $n \times 1$ output vector, $d(k)$ is a $n \times 1$ final consume vector. $A = (a_{ij})_{n\times n}$ is a consume coefficient matrix; $B = (b_{ij})_{n\times n}$ is an investment coefficient matrix, which is usually singular. Hence, besides their theoretical interest, they are very important in terms of applications.

Up till now, there have been a few results for singular difference equations (SDEs). Anh and Loi [3] have studied the solvability of initial-value problems for SDEs; Wang and Zhang [14] have investigated the existence of extremal solutions for singular discrete systems by employing a monotone iterative technique combined with the method of upper and lower solutions; Anh and Hoang [2] have obtained some necessary and sufficient conditions for the stability properties of SDEs by employing Lyapunov functions and Wang and Kong [13] have analyzed the rapid convergence of solution of nonlinear singular difference system. However, we have not found any results for singular difference systems with maxima.

In this paper, we discuss the convergence of approximate solutions for nonlinear singular difference systems with maximum by the method of quasilinearization and prove the quadratic convergence of the successive approximations.

2. Preliminaries

First of all, we introduce some notations and definitions.

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{Z}[a, b] = \{z \in \mathbb{Z} : a \le z \le b\}$ for a, $b \in \mathbb{Z}$, $a < b$.

Consider the following nonlinear singular difference system with "maxima"

$$\begin{cases} Ax(k+1) = f(k, x(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} x(s)), & k \in J_1, \\ x(k) = \varphi(k), & k \in J_2, \end{cases} \tag{2.1}$$

where A is a singular $n \times n$ matrix, $x(k) \in R^n$ for all $k \in J$, $f : J_1 \times R^n \times R^n \to R^n$, $\varphi : J_2 \to R^n$, $J_1 = \mathbb{Z}[0, K]$, $J_2 = \mathbb{Z}[-h, 0]$, $J = J_1 \cup J_2$, h and K is any positive integer.

For $x(k)$, $y(k) \in R^n$, $k \in \mathbb{Z}$, $x(k) \le y(k)$ means $x_i(k) \le y_i(k)$, $i = 1, 2, \cdots, n$. $x(k)y(k) = (x_1(k)y_1(k), x_2(k)y_2(k), \cdots, x_n(k)y_n(k))^T$.

Let the functions α_0, $\beta_0 : J \to R^n$ be such that $\alpha_0(k) \le \beta_0(k)$, and define the following sets.

$\Omega(\alpha_0, \beta_0) = \{(k, x, y) \in J_1 \times R^n \times R^n \mid \alpha_0(k) \le x \le \beta_0(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s) \le y \le \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)\}$.

Definition 2.1. The function $\alpha_0 : J \to R^n$ is said to be a lower solution of (2.1) if it satisfies the following difference inequalities

$$\begin{cases} A\alpha_0(k+1) \le f(k, \alpha_0(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)), & k \in J_1, \\ \alpha_0(k) \le \varphi(k), & k \in J_2. \end{cases} \tag{2.2}$$

An upper solution of (2.1) is defined analogously by reversing the above inequalities.

In our further discussion, we will need some results on linear singular difference systems and inequalities.

For the linear singular difference system

$$Ax(k+1) + M(k)x(k) = g(k), \quad x(0) = x_0, \quad k \in J_1, \tag{2.3}$$

where A and $M(k)$ are $n \times n$ matrices and $g(k)$ is a vector in R^n for all $k \in J$, we have the following result.

Lemma 2.2 ([7], Theorem 3.6.2)**.** *Assume that the following conditions hold.*

($H_{2.1}$) *There exists a constant λ such that $L(k) = [\lambda A + M(k)]^{-1} \geq 0$ exists, $\hat{A} = L(k)A$ is a constant matrix and $\hat{M} = L(k)M(k) = I - \lambda\hat{A}$ on J_1.*

($H_{2.2}$) *y_0 lies in the set $\{\hat{W} + R(\hat{A})\}$, where $\hat{W} = (I - \hat{A}\hat{A}^D)\hat{M}^D\hat{g}(0)$ and $\hat{g}(k) = L(k)g(k)$.*

Then the unique solution $x(k)$ of (2.3) *is given by*

$$x(k) = (-\hat{A}^D\hat{M})^k\hat{A}\hat{A}^D x_0 + \hat{A}^D\sum_{i=0}^{k-1}[-\hat{A}^D\hat{M}]^{k-i-1}g(i) + (I - \hat{A}\hat{A}^D)\hat{M}^D g(k),$$

where index$(A) = 1$, the notations $\hat{A}^D$ and $\hat{M}^D$ indicate the Drazin inverse of the matrices $\hat{A}$ and $\hat{M}$ respectively.

For the singular difference inequalities

$$Ax(k+1) + M(k)x(k) \leq 0, \; x(0) \leq 0, \quad k \in J_1, \tag{2.4}$$

where A, $M(k)$ are $n \times n$ matrices and A is singular on J_1, we have the following result.

Lemma 2.3 ([13], Lemma 1.1)**.** *Assume that the condition* ($H_{2.1}$) *of Lemma* 2.2 *holds, and*

($H_{2.3}$) *There exists a nonsingular matrix Q such that $[L(k)Q]^{-1}$ exists and Q^{-1}, $[L(k)Q]$, $[L(k)Q]^{-1} \geq 0$, satisfying*

$$Q^{-1}\hat{A}Q = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}, \; Q^{-1}\hat{M}Q = \begin{pmatrix} I_1 - \lambda C & 0 \\ 0 & I_2 \end{pmatrix},$$

where C is a diagonal matrix with $C^{-1} < 0$, $C^{-1}(I_1 - \lambda C) + I_1 < 0$.

Then $x(0) \leq 0$ implies $x(k) \leq 0$ on J_1.

Now, we will prove the existence result which is of vital importance for our further discussion.

Lemma 2.4. *Assume that the conditions* ($H_{2.1}$)-($H_{2.3}$) *hold, and*

($H_{2.4}$) *The functions α_0, $\beta_0 : J \to R^n$ are lower and upper solutions of* (2.1)*, respectively, and $\alpha_0(k) \leq \beta_0(k)$ on J;*

($H_{2.5}$) *The function $f : \Omega(\alpha_0, \beta_0) \to R^n$ is continuous with respect to its second and third arguments, the Fréchet derivative f_y exists and is nonnegative, and*

$$f(k, y, u) - f(k, x, u) \leq M(x - y),$$

where $\alpha_0(k) \leq y \leq x \leq \beta_0(k)$, $\max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s) \leq u \leq \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)$ and $M = M(k_0)$, $k \in J_1$, $k_0 \in J_1$.

Then (2.1) *has a solution $x(k)$ that satisfies $\alpha_0(k) \leq x(k) \leq \beta_0(k)$ on J.*

Proof. Let $\alpha_{n+1}(k)$ and $\beta_{n+1}(k)$ be the solutions of the following singular difference systems:

$$\begin{cases} A\alpha_{n+1}(k+1) = f(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) - M(\alpha_{n+1}(k) - \alpha_n(k)), \quad k \in J_1, \\ \alpha_{n+1}(k) = \varphi(k), \quad k \in J_2, \end{cases} \tag{2.5}$$

and

$$\begin{cases} A\beta_{n+1}(k+1) = f(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) - M(\beta_{n+1}(k) - \beta_n(k)), \quad k \in J_1, \\ \beta_{n+1}(k) = \varphi(k), \quad k \in J_2 \end{cases} \tag{2.6}$$

for $n = 0, 1, 2, \cdots$, which exist because of Lemma 2.2. Accordingly, we obtain the sequences $\{\alpha_n(k)\}$ and $\{\beta_n(k)\}$.

First, we show that $\alpha_0(k) \le \alpha_1(k) \le \beta_1(k) \le \beta_0(k)$ on J.

For this purpose, setting $m(k) = \alpha_0(k) - \alpha_1(k)$ so that $m(k) \le 0$ on J_2, using the condition $(H_{2.4})$, we obtain

$$\begin{aligned} Am(k+1) \le{} & f(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)) - [f(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)) \\ & - M(\alpha_1(k) - \alpha_0(k))] \\ \le{} & -Mm(k), \quad k \in J_1. \end{aligned}$$

By Lemma 2.3, we have $\alpha_0(k) \le \alpha_1(k)$ on J_1. Thus, we conclude that $\alpha_0(k) \le \alpha_1(k)$ on J. Analogously, we can prove that $\beta_1(k) \le \beta_0(k)$ on J.

We next prove that $\alpha_1(k) \le \beta_1(k)$ on J. Taking $m(k) = \alpha_1(k) - \beta_1(k)$ so that $m(k) = 0$ on J_2, and utilizing the condition $(H_{2.5})$, we have

$$\begin{aligned} Am(k+1) ={} & [f(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_0(s)) - M(\alpha_1(k) - \alpha_0(k))] \\ & - [f(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)) - M(\beta_1(k) - \beta_0(k))] \\ \le{} & [f(k, \alpha_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)) - M(\alpha_1(k) - \alpha_0(k))] \\ & - [f(k, \beta_0(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_0(s)) - M(\beta_1(k) - \beta_0(k))] \\ \le{} & M(\beta_0(k) - \alpha_0(k)) - M(\alpha_1(k) - \alpha_0(k)) + M(\beta_1(k) - \beta_0(k)) \\ ={} & -Mm(k), \quad k \in J_1. \end{aligned}$$

As before, we obtain that $\alpha_1(k) \le \beta_1(k)$ on J. Thus, we have

$$\alpha_0(k) \le \alpha_1(k) \le \beta_1(k) \le \beta_0(k), \quad k \in J.$$

Continuing with this process, by induction, we conclude that

$$\alpha_0(k) \le \alpha_1(k) \le ... \le \alpha_n(k) \le \beta_n(k) \le ... \le \beta_1(k) \le \beta_0(k), \quad k \in J.$$

Fixed any fixed $k \in J$, the sequence $\{\alpha_n(k)\}$ is monotone nondecreasing and bounded by $\beta_0(k)$. Therefore, the nondecreasing sequence $\{\alpha_n(k)\}$ converges pointwise to a function $x(k)$ that satisfies $\alpha_0(k) \le x(k) \le \beta_0(k)$. In view of (2.5), we can easily see that $x(k)$ is a solution of (2.1). Therefore, (2.1) has a solution $x(k)$ which satisfies $\alpha_0(k) \le x(k) \le \beta_0(k)$ on J. The proof is complete. □

3. Main results

In this section, we use the method of quasilinearization for nonlinear singular difference system with maxima. We will prove that the convergence of the sequence of successive approximations is quadratic.

Theorem 3.1. *Assume that*

($A_{3.1}$) *The function* $f : \Omega(\alpha_0, \beta_0) \to R^n$ *is continuous with respect to its second and third arguments, the Fréchet derivatives* f_{xx}, f_{xy}, f_{yy} *exist, and the following equalities are valid for* $(k, x, y) \in \Omega(\alpha_0, \beta_0)$*:*

$$f_{xx}(k, x, y) \geq 0, \;\; f_{xy}(k, x, y) \geq 0, \;\; f_{yy}(k, x, y) \geq 0;$$

($A_{3.2}$) *The conditions* $(H_{2.1})$*-*$(H_{2.5})$ *and the nonnegative matrix* $\bar{M}$ *for* $M(k) = -f_x(k, x, y)$, $N(k) = -f_y(k, x, y)$, $\hat{N}(k) = L(k)N(k)$, $(k, x, y) \in \Omega(\alpha_0, \beta_0)$ *hold, where*

$$\bar{M} = \Big\{ I - \max_{s\in\mathbb{Z}[0,K]} \big\{ -\hat{A}^D \sum_{i=0}^{s-1} [-\hat{A}^D \hat{M}]^{s-i-1} \hat{N}(i) - (I - \hat{A}\hat{A}^D)\hat{M}^D \hat{N}(s) \big\} \Big\}^{-1}.$$

Then there exist two monotone sequences $\{\alpha_n(k)\}$, $\{\beta_n(k)\}$ *which converge to the solution of* (2.1) *on* J *and the convergence is quadratic.*

Proof. It follows from the assumption $(A_{3.1})$ that the inequality

$$f(k, x_1, y_1) \geq f(k, x_2, y_2) + f_x(k, x_2, y_2)(x_1 - y_1) + f_y(k, x_2, y_2)(x_2 - y_2) \tag{3.1}$$

holds for (k, x_1, y_1), $(k, x_2, y_2) \in \Omega(\alpha_0, \beta_0)$, $x_1 \geq x_2$, $y_1 \geq y_2$.

Now, consider the following singular difference systems with maxima

$$\begin{cases} Ax(k+1) = f(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\ \qquad\qquad\quad + f_x(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(x(k) - \alpha_n(k)) \\ \qquad\qquad\quad + f_y(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(\max\limits_{s\in\mathbb{Z}[k-h,k]} x(s) - \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\ \qquad\qquad \equiv F_n(k, x(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} x(s)), \quad k \in J_1, \\ x(k) = \varphi(k), \quad k \in J_2. \end{cases} \tag{3.2}$$

and

$$\begin{cases} Ay(k+1) = f(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) \\ \qquad\qquad\quad + f_x(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(y(k) - \beta_n(k)) \\ \qquad\qquad\quad + f_y(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(\max\limits_{s\in\mathbb{Z}[k-h,k]} y(s) - \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) \\ \qquad\qquad \equiv G_n(k, y(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} y(s)), \quad k \in J_1, \\ y(k) = \varphi(k), \quad k \in J_2. \end{cases} \tag{3.3}$$

Let $n = 0$ in (3.2) and (3.3). Initially, we can prove that $\alpha_0(k)$ and $\beta_0(k)$ are lower and upper solutions of (3.2), respectively. Then, by Lemma 2.4, we conclude that there exists a solution $\alpha_1(k)$ of (3.2) with $\alpha_1(k) = \varphi(k)$ on J_2 such that $\alpha_0(k) \leq \alpha_1(k) \leq \beta_0(k)$ on J.

Similarly, we can also prove that $\alpha_1(k)$ and $\beta_0(k)$ are lower and upper solutions of (3.3) respectively. Then, it follows from Lemmas 2.4 that (3.3) has a solution $\beta_1(k)$ such that $\alpha_1(k) \le \beta_1(k) \le \beta_0(k)$ on J.

Furthermore, using the above results, we can prove that $\alpha_1(k)$ and $\beta_1(k)$ are lower and upper solutions of (2.1) respectively. Hence, we have

$$\alpha_0(k) \le \alpha_1(k) \le \beta_1(k) \le \beta_0(k), \quad k \in J.$$

The method of mathematical induction can be applied to prove that for all n

$$\alpha_0(k) \le \alpha_1(k) \le ... \le \alpha_n(k) \le \beta_n(k) \le ... \le \beta_1(k) \le \beta_0(k), \quad k \in J.$$

Since α_n, β_n are lower and upper solutions of (2.1) respectively, and all the assumptions of Lemma 2.4 are satisfied, we can conclude that there exists a solution $x(k)$ of (2.1) such that $\alpha_n(k) \le x(k) \le \beta_n(k)$ on J. Hence, we have

$$\alpha_0(k) \le \alpha_1(k) \le ... \le \alpha_n(k) \le x(k) \le \beta_n(k) \le ... \le \beta_1(k) \le \beta_0(k), \quad k \in J.$$

For any fixed $k \in J$, the sequences $\{\alpha_n(k)\}$, $\{\beta_n(k)\}$ are monotone nondreasing and monotone nonincreasing, and they are bounded by $\alpha_0(k)$, $\beta_0(k)$, respectively. Therefore, they are convergent on J, that is, there exist functions $\rho(k)$, $r(k)$ such that

$$\lim_{n\to\infty} \alpha_n(k) = \rho(k) \le x(k) \le r(k) = \lim_{n\to\infty} \beta_n(k).$$

To show the quadratic convergence. Define the function $a_{n+1}(k)$ as follows:

$$a_{n+1}(k) = x(k) - \alpha_{n+1}(k) \ge 0, \qquad k \in J.$$

Let $k \in J_2$. It is clear that $a_{n+1}(k) = 0$.

Let $k \in J_1$. Using the mean value theorem and the assumption $f_y \ge 0$, we arrive at

$$\begin{aligned}
Aa_{n+1}(k+1) &= f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(x(k) - \alpha_{n+1}(k)) \\
&\quad + f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(\max_{s\in\mathbb{Z}[k-h,k]} x(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_{n+1}(s)) \\
&\quad - f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(x(k) - \alpha_n(k)) \\
&\quad - f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(\max_{s\in\mathbb{Z}[k-h,k]} x(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\
&\quad + \Big(\int_0^1 f_x(k, \sigma x(k) + (1-\sigma)\alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)) d\sigma\Big)(x(k) - \alpha_n(k)) \\
&\quad + \Big(\int_0^1 f_y(k, \alpha_n(k), \sigma \max_{s\in\mathbb{Z}[k-h,k]} x(s) + (1-\sigma) \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) d\sigma\Big) \\
&\qquad \times (\max_{s\in\mathbb{Z}[k-h,k]} x(s) - \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\
&\le f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(x(k) - \alpha_{n+1}(k)) \\
&\quad + f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \max_{s\in\mathbb{Z}[k-h,k]} (x(s) - \alpha_{n+1}(s)) \\
&\quad + [f_x(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} x(s)) - f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))](x(k) - \alpha_n(k))
\end{aligned}$$

$$
\begin{aligned}
&+[f_y(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s))-f_y(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))]\\
&\quad\times(\max_{s\in\mathbb{Z}[k-h,k]}x(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))\\
&\le -M(k)a_{n+1}(k)-N(k)\max_{s\in\mathbb{Z}[-h,T]}a_{n+1}(s)+A_1+B_1,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1&=[f_x(k,x(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s))-f_x(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))](x(k)-\alpha_n(k)),\\
B_1&=[f_y(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s))-f_y(k,\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))]\times(\max_{s\in\mathbb{Z}[k-h,k]}x(s)-\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s)).
\end{aligned}
$$

Using the mean value theorem and the condition $(A_{3.1})$, we get

$$
\begin{aligned}
A_1\le{}&(a_n(k))^T\Big(\int_0^1 f_{xx}(k,\sigma x(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s))d\sigma\Big)a_n(k)\\
&+(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^T\Big(\int_0^1 f_{xy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}x(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))d\sigma\Big)\\
&\times a_n(k)\\
\le{}&\Big(\sum_{j=1}^n|f_{x_jx}(k,\sigma x(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}x(s))|\Big)(a_n(k))^2\\
&+\frac{1}{2}\Big(\sum_{j=1}^n|f_{x_jy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}x(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)(a_n(k))^2\\
&+\frac{1}{2}\Big(\sum_{j=1}^n|f_{y_jx}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}x(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2\\
\le{}&M_{11}(a_n(k))^2+\frac{1}{2}M_{12}(a_n(k))^2+\frac{1}{2}M_{13}(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2\\
\le{}&[M_{11}+\frac{1}{2}(M_{12}+M_{13})]\max_{s\in\mathbb{Z}[-h,K]}|a_n(s)|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
B_1\le{}&(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^T\Big(\int_0^1 f_{yy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}x(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))d\sigma\Big)\\
&\quad\times\max_{s\in\mathbb{Z}[k-h,k]}a_n(s)\\
\le{}&\Big(\sum_{j=1}^n|f_{y_jy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}x(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2\\
\le{}&M_{14}\max_{s\in\mathbb{Z}[-h,K]}|a_n(s)|^2,
\end{aligned}
$$

where $\sum_{j=1}^n|f_{x_jx}(k,x,y)|\le M_{11}$, $\sum_{j=1}^n|f_{x_jy}(k,x,y)|\le M_{12}$, $\sum_{j=1}^n|f_{y_jx}(k,x,y)|\le M_{13}$, $\sum_{j=1}^n|f_{y_jy}(k,x,y)|\le M_{14}$ for $(k,x,y)\in\Omega(\alpha_0,\beta_0)$, M_{11} M_{12}, M_{13} and M_{14} are positive matrices. Then, from the above discussion, we have

$$
\begin{aligned}
&Aa_{n+1}(k+1)+M(k)a_{n+1}(k)\le -N(k)\max_{s\in\mathbb{Z}[-h,K]}a_{n+1}(s)+M_1\max_{s\in\mathbb{Z}[-h,K]}|a_n(s)|^2,\quad k\in J_1,\\
&a_{n+1}(k)=0,\quad k\in J_2,
\end{aligned}
$$

where $M_1 = M_{11} + \frac{1}{2}(M_{12} + M_{13}) + M_{14}$. According to Lemma 2.3, we find that $a_{n+1}(k) \le x(k)$ on J_1, where $u(k)$ is a solution of

$$\begin{aligned} &Au(k+1) + M(k)u(k) = -N(k) \max_{s\in\mathbb{Z}[-h,K]} a_{n+1}(s) + M_1 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2, \quad k \in J_1,\\ &u(k) = 0, \quad k \in J_2. \end{aligned}$$

Furthermore, using the expression of $x(k)$ in Lemma 2.2, we have

$$\begin{aligned} \max_{s\in\mathbb{Z}[-h,K]} a_{n+1}(s) \le \Big\{ I - \max_{s\in\mathbb{Z}[0,K]} \big\{ &- \hat{A}^D \sum_{i=0}^{s-1} [-\hat{A}^D\hat{M}]^{s-i-1}[\lambda A + M(i)]^{-1}N(i) \\ &- (I - \hat{A}\hat{A}^D)\hat{M}^D[\lambda A + M(s)]^{-1}N(s)\big\}\Big\}^{-1} \\ &\times \max_{s\in\mathbb{Z}[0,K]} \Big\{ \hat{A}^D \sum_{i=0}^{s-1} [-\hat{A}^D\hat{M}]^{s-i-1}[\lambda A + M(i)]^{-1}M_1 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 \\ &+ (I - \hat{A}\hat{A}^D)\hat{M}^D[\lambda A + M(s)]^{-1}M_1 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 \Big\}. \end{aligned}$$

Then, by suitable estimates, we can get

$$|a_{n+1}|_0 \le K_1 |a_n|_0^2,$$

where K_1 is a positive constant matrix and $|a|_0 = \max\limits_{s\in J} |a(s)| = (\max\limits_{s\in J} |a_1(s)|, \cdots, \max\limits_{s\in J} |a_n(s)|)^T$. The convergence of $\{\alpha_n(k)\}$ is quadratic.

Similarly, we define the function $b_{n+1}(k)$ as follows:

$$b_{n+1}(k) = \beta_{n+1}(k) - x(k) \ge 0, \qquad t \in J.$$

Let $k \in J_2$. We can see that $b_{n+1}(k) = 0$.

Let $k \in J_1$. In view of the mean value theorem and the condition $f_y \ge 0$, it can be deduced that

$$\begin{aligned} Ab_{n+1}(k+1) \le\ & f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))(\beta_{n+1}(k) - x(k)) \\ &+ f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \max_{s\in\mathbb{Z}[k-h,k]} (\beta_{n+1}(s) - x(s)) \\ &+ [f_x(k, \beta_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) - f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))](\beta_n(k) - x(k)) \\ &+ [f_y(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) - f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))] \\ &\times (\max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s) - \max_{s\in\mathbb{Z}[k-h,k]} x(s)) \\ \le\ & -M(k)b_{n+1}(k) - N(k) \max_{s\in\mathbb{Z}[-h,T]} b_{n+1}(s) + A_2 + B_2, \end{aligned}$$

where

$$\begin{aligned} A_2 &= [f_x(k, \beta_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) - f_x(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))](\beta_n(k) - x(k)), \\ B_2 &= [f_y(k, x(k), \max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) - f_y(k, \alpha_n(k), \max_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s))] \times (\max_{s\in\mathbb{Z}[k-h,k]} \beta_n(s) - \max_{s\in\mathbb{Z}[k-h,k]} x(s)). \end{aligned}$$

By using the mean value theorem and the assumption $(A_{3.1})$, we can get

$$\begin{aligned}
A_2 \leq{} & (b_n(k)+a_n(k))^T\Big(\int_0^1 f_{xx}(k,\sigma\beta_n(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s))d\sigma\Big)b_n(k)\\
&+(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s)+\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^T\\
&\times\Big(\int_0^1 f_{xy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))d\sigma\Big)b_n(k)\\
\leq{} & \Big(\sum_{j=1}^n|f_{x_jx}(k,\sigma\beta_n(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s))|\Big)(b_n(k)+a_n(k))b_n(k)\\
&+\Big(\sum_{j=1}^n|f_{x_jy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)(b_n(k))^2\\
&+\frac{1}{2}\Big(\sum_{j=1}^n|f_{y_jx}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)\\
&\times[(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2+(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s))^2]\\
\leq{} & M_{11}(b_n(k))^2+\frac{1}{2}M_{11}(a_n(k))^2+\frac{1}{2}M_{11}(b_n(k))^2+M_{12}(b_n(k))^2\\
&+\frac{1}{2}M_{13}(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2+\frac{1}{2}M_{13}(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s))^2\\
\leq{} & \frac{1}{2}(M_{11}+M_{13})\max_{s\in\mathbb{Z}[-h,K]}|a_n(s)|^2+\frac{1}{2}(3M_{11}+2M_{12}+M_{13})\max_{s\in\mathbb{Z}[-h,K]}|b_n(s)|^2
\end{aligned}$$

and

$$\begin{aligned}
B_2 \leq{} & (a_n(k))^T\Big(\int_0^1 f_{yx}(k,\sigma x(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s))d\sigma\Big)\max_{s\in\mathbb{Z}[k-h,k]}(b_n(s))\\
&+(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s)+\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^T\\
&\times\Big(\int_0^1 f_{yy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))d\sigma\Big)\max_{s\in\mathbb{Z}[k-h,k]}b_n(s)\\
\leq{} & \frac{1}{2}\Big(\sum_{j=1}^n|f_{x_jy}(k,\sigma x(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s))|\Big)(a_n(k))^2\\
&+\frac{1}{2}\Big(\sum_{j=1}^n|f_{y_jx}(k,\sigma x(k)+(1-\sigma)\alpha_n(k),\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s))|\Big)(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s))^2\\
&+\Big(\sum_{j=1}^n|f_{y_jy}(k,\alpha_n(k),\sigma\max_{s\in\mathbb{Z}[k-h,k]}\beta_n(s)+(1-\sigma)\max_{s\in\mathbb{Z}[k-h,k]}\alpha_n(s))|\Big)\\
&\times(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s)+\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))\max_{s\in\mathbb{Z}[k-h,k]}b_n(s)\\
\leq{} & \frac{1}{2}M_{12}(a_n(t))^2+\frac{1}{2}M_{14}(\max_{s\in\mathbb{Z}[k-h,k]}a_n(s))^2+\frac{1}{2}(M_{13}+3M_{14})(\max_{s\in\mathbb{Z}[k-h,k]}b_n(s))^2\\
\leq{} & \frac{1}{2}(M_{12}+M_{14})\max_{s\in\mathbb{Z}[-h,K]}|a_n(s)|^2+\frac{1}{2}(M_{13}+3M_{14})\max_{s\in\mathbb{Z}[-h,K]}|b_n(s)|^2.
\end{aligned}$$

Then, we conclude from above discussion that

$$\begin{aligned} Ab_{n+1}(k+1) + M(k)b_{n+1}(k) \le & -N(k) \max_{s\in\mathbb{Z}[-h,K]} b_{n+1}(s) \\ & + M_2 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 + M_3 \max_{s\in\mathbb{Z}[-h,K]} |b_n(s)|^2, \quad k \in J_1, \\ b_{n+1}(k) = 0, & \quad k \in J_2, \end{aligned}$$

where $M_2 = \frac{1}{2}(M_{11} + M_{12} + M_{13} + M_{14})$, and $M_3 = \frac{1}{2}(3M_{11} + 2M_{12} + 2M_{13} + 3M_{14})$. Hence, by Lemma 2.3, we obtain that $b_{n+1}(k) \le u(k)$ on J_1, where $u(k)$ is a solution of

$$\begin{aligned} Au(k+1) + M(k)u(k) = & -N(k) \max_{s\in\mathbb{Z}[-h,K]} b_{n+1}(s) \\ & + M_2 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 + M_3 \max_{s\in\mathbb{Z}[-h,K]} |b_n(s)|^2, \quad k \in J_1, \\ u(k) = 0, & \quad k \in J_2. \end{aligned}$$

Then, using the expression of $x(k)$ in Lemma 2.2, we have

$$\begin{aligned} & \max_{s\in\mathbb{Z}[-h,K]} b_{n+1}(s) \\ & \le \Big\{ I - \max_{s\in\mathbb{Z}[0,K]} \big\{ -\hat{A}^D \sum_{i=0}^{s-1} [-\hat{A}^D\hat{M}]^{s-i-1}[\lambda A + M(i)]^{-1}N(i) \\ & \quad - (I - \hat{A}\hat{A}^D)\hat{M}^D[\lambda A + M(s)]^{-1}N(s) \big\} \Big\}^{-1} \max_{s\in\mathbb{Z}[0,K]} \Big\{ \hat{A}^D \sum_{i=0}^{s-1} [-\hat{A}^D\hat{M}]^{s-i-1}[\lambda A + M(i)]^{-1} \\ & \quad \times (M_2 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 + M_3 \max_{s\in\mathbb{Z}[-h,K]} |b_n(s)|^2) \\ & \quad + (I - \hat{A}\hat{A}^D)\hat{M}^D[\lambda A + M(s)]^{-1}(M_2 \max_{s\in\mathbb{Z}[-h,K]} |a_n(s)|^2 + M_3 \max_{s\in\mathbb{Z}[-h,K]} |b_n(s)|^2) \Big\}. \end{aligned}$$

Taking suitable computation, we obtain

$$|b_{n+1}|_0 \le K_2|b_n|_0^2 + K_3|a_n|_0^2,$$

where K_2 and K_3 are positive constant matrices. Thus, the convergence of $\{\beta_n(k)\}$ is quadratic. The proof is complete. □

Theorem 3.2. *Assume that the condition $(A_{3.2})$ hold, and*

$(A_{3.3})$ *The function $f : \Omega(\alpha_0, \beta_0) \to R^n$ is continuous with respect to its second and third arguments, the Fréchet derivatives f_{xx}, f_{xy}, f_{yy} exist, and the following equalities are valid for $k \in J_1$, $(k, x, y) \in \Omega(\alpha_0, \beta_0)$:*

$$f_{xx}(k, x, y) \le 0, \;\; f_{xy}(k, x, y) \le 0, \;\; f_{yy}(k, x, y) \le 0.$$

Then there exist two monotone sequences $\{\alpha_n(k)\}$, $\{\beta_n(k)\}$ which converge to the solution of (2.1) on J and the convergence is quadratic.

Proof. Consider the following singular difference systems with maxima

$$\begin{cases} Ax(k+1) = f(k, \alpha_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\ \qquad\qquad + f_x(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s))(x(k) - \alpha_n(k)) \\ \qquad\qquad + f_y(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s))(\max\limits_{s\in\mathbb{Z}[k-h,k]} x(s) - \max\limits_{s\in\mathbb{Z}[k-h,k]} \alpha_n(s)) \\ \qquad \equiv F_n(k, x(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} x(s)), \quad k \in J_1, \\ x(k) = \varphi(k), \quad k \in J_2, \end{cases} \tag{3.4}$$

and

$$\begin{cases} Ay(k+1) = f(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) \\ \qquad\qquad + f_x(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s))(y(k) - \beta_n(k)) \\ \qquad\qquad + f_y(k, \beta_n(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s))(\max\limits_{s\in\mathbb{Z}[k-h,k]} y(s) - \max\limits_{s\in\mathbb{Z}[k-h,k]} \beta_n(s)) \\ \qquad \equiv G_n(k, y(k), \max\limits_{s\in\mathbb{Z}[k-h,k]} y(s)), \quad k \in J_1, \\ y(k) = \varphi(k), \quad k \in J_2. \end{cases} \tag{3.5}$$

Analogous to the proof of Theorem 3.1, we can get the convergence is quadratic. □

Remark 3.3. The above result can be extended to the situation where $f(k, x, y) = F(k, x, y) - g(k, x, y)$, and $F(k, x, y)$ and $g(k, x, y)$ satisfy

$$F_{xx}(k, x, y) \geq 0, \ F_{xy}(k, x, y) \geq 0, \ F_{yy}(k, x, y) \geq 0,$$

$$g_{xx}(k, x, y) \geq 0, \ g_{xy}(k, x, y) \geq 0, \ g_{yy}(k, x, y) \geq 0.$$

By using the method of generalized quasilinearization, we can obtain the result that there exist two monotone sequences which converge quadratically to the solution of (2.1). Similarly, when $F(k, x, y)$ and $g(k, x, y)$ satisfy

$$F_{xx}(k, x, y) \leq 0, \ F_{xy}(k, x, y) \leq 0, \ F_{yy}(k, x, y) \leq 0,$$

$$g_{xx}(k, x, y) \leq 0, \ g_{xy}(k, x, y) \leq 0, \ g_{yy}(k, x, y) \leq 0,$$

we can also obtain the quadratic convergence. We omit the details.

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