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Coefficient functionals for a class of bounded turning functions connected to three leaf function

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Abstract

In this article, we define the class of bounded turning functions connected with three leaf function to investigate results for the estimates of four initial coefficients, Fekete-Szegö functional, the second-order Hankel determinant and Zalcman conjecture and these results are shown to be sharp. Furthermore, we estimate the bounds of the third-order Hankel determinants for this class and for its 2-fold and 3-fold symmetric functions. Finally we evaluate the sharp Krushkal inequality for the functions in this class.

Keywords: Analytic functions, three leaf function, subordination, Hankel determinant, 2-fold and 3-fold symmetric function, Zalcman conjuncture, Krushkal inequality.

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1. Introduction and preliminary results

Let $\mathcal{A}$ represent the family of functions of the form

$$f(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \quad (\xi \in \mathbb{D}), \tag{1.1}$$

which are regular in open unit disc $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$. Also $\mathcal{S}$ denote the subfamily of $\mathcal{A}$ consisting of functions of the form (1.1), which are also univalent in $\mathbb{D}$.

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Let $\mathcal{P}$ represent the family of functions $h(\xi)$ that are regular with positive part in open unit disc $\mathbb{D}$ and of the form

$$h(\xi) = 1 + \sum_{n=1}^{\infty} c_n \xi^n \ (\xi \in \mathbb{D}). \tag{1.2}$$

Next we recall the definition of subordination. For two functions $g_1, g_2 \in \mathcal{A}$, we say that g_1 is subordinated to g_2 and symbolically written as $g_1 \prec g_2$ if there exists an analytic function w with the property $|w(\xi)| \leqslant |\xi|$ and $w(0) = 0$ such that $g_1(\xi) = g_2(w(\xi))$ for $\xi \in \mathbb{D}$. Further, if $g_2 \in \mathcal{S}$, then the condition becomes

$$g_1 \prec g_2 \Leftrightarrow g_1(0) = g_2(0) \text{ and } g_1(\mathbb{D}) \subset g_2(\mathbb{D}).$$

A function f is said to be starlike functions if it satisfies the condition:

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec \varphi(\xi) \right\}, \tag{1.3}$$

where $\varphi(\xi) = (1+\xi)/(1-\xi)$. If we contrast the function φ on the right hand side of (1.3), then we acquire some several subclasses of $\mathcal{S}$ whose image domains have some exciting geometrical configurations as follows.

1. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = 1 + \sin \xi$ was presented and studied by Cho et al. [5].
2. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = 1 + \xi - \frac{1}{3}\xi^3$, maps the open unit disc onto the interior of the nephroid, a 2-cusped kidney-shaped region was familiarized and investigated by Wani and Swaminathan [34].
3. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = \sqrt{1+\xi}$, which is bounded by lemniscate of Bernoulli in right half plan, was developed by Sokól and Stankiewicz [29].
4. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = 1 + \frac{4}{3}\xi + \frac{2}{3}\xi^2$ was presented by Sharma et al. [26].
5. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = e^{\xi}$ was introduced and deliberated by Mendiratta et al. [20].
6. The class $\mathcal{S}^*(\varphi)$ with $\varphi(\xi) = \xi + \sqrt{1+\xi^2}$, which maps open unit disk to crescent shaped region, was introduced by Raina and Sokól [24].

Also we note that lately various subclasses of starlike functions were introduced, see [6, 7, 11] by fixing some particular functions such as functions linked with Bell numbers, shell-like curve connected with Fibonacci numbers, functions associated with conic domains and rational functions instead of φ in (1.3).

Pommerenke [22, 23] introduced the Hankel determinant $H_{q,n}(f)$, where the parameters $q, n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ for function $f \in \mathcal{S}$ of the form (1.1) as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The Hankel determinants for different orders are obtained for different values of q and n. When $q = 2$ and $n = 1$, the determinant is

$$|H_{2,1}(f)| = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left| a_3 - a_2^2 \right|, \text{ where } a_1 = 1.$$

Note that $H_{2,1}(f) = a_3 - a_2^2$, is the classical Fekete-Szegö functional. For various subclasses of class $\mathcal{A}$, the best conceivable value of the upper bound for $|H_{2,1}(f)|$ was explored by different authors (see [10, 13, 14, 27] for details). Furthermore, when $q = 2$ and $n = 2$, the second Hankel determinant is

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

The upper bound of $|H_{2,2}(f)|$ has been considered by several authors in last few decades. For instance, the readers may refer to the works of Hayman [9] and Noonan and Thomas [21]. Further, Babalola [2] studied the Hankel determinant $H_{3,1}(f)$ for some subclasses of analytic functions. For some current works on third order Hankel determinant we may mention (for example) [16, 28, 31]. The bound of the fourth Hankel determinant for a class of analytic functions with bounded turning associated with cardoid domain was approximated by Srivastava et al. in [33]. It should be remarked that a wide variety of applications of Hankel systems arise in linear filtering theory, discrete inverse scattering, and discretization of certain integral equations arising in mathematical physics [35].

Evaluating these Hankel determinants for various new subclasses has been an attracting area lately. One such field of interest is the Quantum Calculus (q-calculus), which is a generalization of classical calculus by replacing the limit by a parameter q. For the basics and preliminaries, the readers are advised to see the work in [30]. It is imperative to mention here the work on a q-differential operator by Srivastava et al. [32], in which they determined the upper bound of second Hankel determinant for a subclass of bi-univalent functions in q-analogue.

Lately in 2018, the class of starlike functions associated with three leaf function was defined by Gandhi [8], i.e,

$$\mathcal{S}^*_{3\mathcal{L}} = \left\{ f \in \mathcal{A} : \frac{\xi f'(\xi)}{f(\xi)} \prec 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4 \right\}, \quad (\xi \in \mathbb{D}),$$

and studied different geometric properties for this class. Further, Shi et al. [28] investigated coefficient estimates problems for three leaf-type class. Now recall the definition of class $\mathcal{R}$ of bounded turning functions

$$\mathcal{R} = \left\{ f \in \mathcal{A} : f'(\xi) \prec \frac{1+\xi}{1-\xi} \right\}, \quad (\xi \in \mathbb{D}).$$

Motivated by all works mentioned above and [15], in this article we introduce and investigate the class $\mathcal{R}_{3\mathcal{L}}$ which is defined as follows:

$$\mathcal{R}_{3\mathcal{L}} = \left\{ f \in \mathcal{A} : f'(\xi) \prec 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4 \right\}, \quad (\xi \in \mathbb{D}). \tag{1.4}$$

We also establish some sharp results such as coefficient bounds, Fekete-Szegö inequality, second-order determinant and Zalcman conjecture for functions belonging to the class $\mathcal{R}_{\mathcal{S}_G}$. Moreover, we estimate bounds of the third-order Hankel determinants for this class $\mathcal{R}_{3\mathcal{L}}$ and for the 2-fold and 3-fold symmetric functions.

For the proofs of our main findings, we need the following lemmas.

Lemma 1.1 ([18])**.** *Let $p \in \mathcal{P}$ have the series expansion of the form (1.2). Then, for x and σ with $|x| \leqslant 1, |\sigma| \leqslant 1$, such that*

$$2c_2 = c_1^2 + x(4 - c_1^2), \quad 4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma. \tag{1.5}$$

Lemma 1.2. *If $h \in \mathcal{P}$ and has the series form (1.2), then*

$$|c_{n+k} - \mu c_n c_k| \leqslant 2, \tag{1.6}$$

$$|c_n| \leqslant 2 \text{ for } n \geqslant 1, \tag{1.7}$$

$$|c_2 - \zeta c_1^2| \leqslant 2 \max\{1, |2\zeta - 1|\}, \ \zeta \in \mathbb{C}, \tag{1.8}$$

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leqslant 2|J| + |K - 2J| + 2|J - K + L|. \tag{1.9}$$

We note that the inequalities (1.6), (1.7), and (1.9) in the above can be found in [23] and (1.8) are from [12], and also (1.9) was evaluated in [1].

Lemma 1.3 ([25]). *Let m, n, l, and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and*

$$8r\left(1-r\right)\left[\left(mn-2l\right)^2+\left(m\left(r+m\right)-n\right)^2\right]+m\left(1-m\right)\left(n-2rm\right)^2 \leqslant 4m^2\left(1-m\right)^2 r\left(1-r\right).$$

If $h \in \mathcal{P}$ and has power series (1.2), then

$$\left|lc_1^4+rc_2^2+2mc_1c_3-\frac{3}{2}nc_1^2c_2-c_4\right| \leqslant 2.$$

2. Bounds of $H_{3,1}\left(f\right)$ for the class $\mathcal{R}_{3\mathcal{L}}$

Theorem 2.1. *Let $f \in \mathcal{R}_{3\mathcal{L}}$ and be of the form (1.1). Then*

$$|a_2| \leqslant \frac{2}{5},$$

the result is sharp for function

$$f_1(\xi)=\int_0^{\xi}\left(1+\frac{4}{5}t+\frac{1}{5}t^4\right)dt=\xi+\frac{2}{5}\xi^2+\cdots, \quad |a_3| \leqslant \frac{4}{15}, \tag{2.1}$$

the result is sharp for function

$$f_2(\xi)=\int_0^{\xi}\left(1+\frac{4}{5}t^2+\frac{1}{5}t^8\right)dt=\xi+\frac{4}{15}\xi^3+\cdots, \quad |a_4| \leqslant \frac{1}{5}, \tag{2.2}$$

the result is sharp for function

$$f_3(\xi)=\int_0^{\xi}\left(1+\frac{4}{5}t^3+\frac{1}{5}t^{12}\right)dt=\xi+\frac{1}{5}\xi^4+\cdots, \quad |a_5| \leqslant \frac{4}{25}, \tag{2.3}$$

the result is sharp for function

$$f_3(\xi)=\int_0^{\xi}\left(1+\frac{4}{5}t^3+\frac{1}{5}t^{12}\right)dt=\xi+\frac{4}{25}\xi^5+\cdots.$$

Proof. Let $f \in \mathcal{R}_{3\mathcal{L}}$. Then, (1.4) can be put in the form of Schwarz function w as

$$f'(\xi)=1+\frac{4}{5}w(\xi)+\frac{1}{5}\left(w(\xi)\right)^4, \quad (\xi \in \mathbb{D}). \tag{2.4}$$

Also, if $h \in \mathcal{P}$, then it may be written in terms of the Schwarz function w as

$$h(\xi)=1+c_1\xi+c_2\xi^2+c_3\xi^3+\cdots=\frac{1+w(\xi)}{1-w(\xi)},$$

equivalently,

$$w(\xi)=\frac{h(\xi)-1}{h(\xi)+1}=\frac{c_1\xi+c_2\xi^2+c_3\xi^3+\cdots}{2+c_1\xi+c_2\xi^2+c_3\xi^3+\cdots}. \tag{2.5}$$

From (2.4), we easily get

$$f'(\xi)=1+2a_2\xi+3a_3\xi^2+4a_4\xi^3+5a_5\xi^4+\cdots. \tag{2.6}$$

By a simplification and using the series expansion (2.5), we have

$$
\begin{aligned}
1+\frac{4}{5}w(\xi)+\frac{1}{5}\left(w(\xi)\right)^4 &= 1+\frac{2}{5}c_1\xi+\left(\frac{2}{5}c_2-\frac{1}{5}c_1^2\right)\xi^2+\left(\frac{1}{10}c_1^3-\frac{2}{5}c_2c_1+\frac{2}{5}c_3\right)\xi^3 \\
&\quad+\left(-\frac{3}{80}c_1^4+\frac{3}{10}c_1^2c_2-\frac{2}{5}c_3c_1-\frac{1}{5}c_2^2+\frac{2}{5}c_4\right)\xi^4+\cdots .
\end{aligned} \tag{2.7}
$$

Equating (2.6) and (2.7), we get

$$a_2=\frac{1}{5}c_1, \tag{2.8}$$

$$a_3=\frac{2}{15}c_2-\frac{1}{15}c_1^2, \tag{2.9}$$

$$a_4=\frac{1}{40}c_1^3-\frac{1}{10}c_1c_2+\frac{1}{10}c_3, \tag{2.10}$$

$$a_5=-\frac{2}{25}\left(\frac{3}{32}c_1^4-\frac{3}{4}c_1^2c_2+c_3c_1+\frac{1}{2}c_2^2-c_4\right). \tag{2.11}$$

Using(1.7) in (2.8), we have

$$|a_2|\leqslant\frac{2}{5}.$$

From (2.9), we get

$$a_3=\frac{2}{15}\left(c_2-\frac{1}{2}c_1^2\right),$$

and applying (1.6), we have

$$|a_3|\leqslant\frac{4}{15}.$$

Using (2.10), we obtain

$$|a_4|=\left|\frac{1}{40}c_1^3-\frac{1}{10}c_1c_2+\frac{1}{10}c_3\right|.$$

By applying (1.9), we get

$$|a_4|\leqslant\left[2\left|\frac{1}{40}\right|+2\left|\frac{1}{10}-2\left(\frac{1}{40}\right)\right|+2\left|\frac{1}{40}-\frac{1}{10}+\frac{1}{10}\right|\right]=\frac{1}{5}.$$

By applying Lemma 1.3 to (2.11), we get

$$|a_5|\leqslant\frac{4}{25}.$$

□

Next, we consider Fekete-Szegö problem and Hankel determinants for the class $\mathcal{R}_{3\mathcal{L}}$.

Theorem 2.2 (Fekete-Szegö inequality). *If* f *of the form* (1.1) *belongs to* $\mathcal{R}_{3\mathcal{L}}$, *then*

$$|a_3-\zeta a_2^2|\leqslant\frac{4}{15}\max\left\{1,\frac{3|\zeta|}{5}\right\},\quad(\zeta\in\mathbb{C}).$$

The result is sharp for

$$f_2(\xi)=\int_0^{\xi}\left(1+\frac{4}{5}t^2+\frac{1}{5}t^8\right)dt=\xi+\frac{4}{15}\xi^3+\frac{1}{45}\xi^9+\cdots .$$

Proof. Using (2.8) and (2.9), we can write

$$|a_3 - \zeta a_2^2| = \left| \frac{2}{15}c_2 - \frac{1}{15}c_1^2 - \zeta \frac{c_1^2}{25} \right|.$$

By rearranging we have

$$|a_3 - \zeta a_2^2| = \frac{2}{15} \left| c_2 - \left(\frac{3\zeta + 5}{10} \right) c_1^2 \right|.$$

Applying (1.8) we get

$$|a_3 - \zeta a_2^2| \leqslant \frac{2}{15} \max \left\{ 2, 2 \left| 2 \left(\frac{3\zeta + 5}{10} \right) - 1 \right| \right\}.$$

Then with simple calculations, we obtain

$$|a_3 - \zeta a_2^2| \leqslant \frac{4}{15} \max \left\{ 1, \frac{3|\zeta|}{5} \right\}.$$

□

If we put $\zeta = 1$, then the above result becomes as follows.

Corollary 2.3. *If* f *of the form* (1.1) *belongs to* $\mathcal{R}_{3\mathcal{L}}$, *then*

$$|a_3 - a_2^2| \leqslant \frac{4}{15}. \tag{2.12}$$

The result is sharp.

Theorem 2.4. *If* f *of the form* (1.1) *belongs to* $\mathcal{R}_{3\mathcal{L}}$, *then*

$$|a_2 a_3 - a_4| \leqslant \frac{1}{5}. \tag{2.13}$$

The result is sharp for

$$f_3(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^3 + \frac{1}{5}t^{12} \right) dt = \xi + \frac{1}{5}\xi^4 + \cdots.$$

Proof. From (2.8), (2.9), and (2.10), we get

$$|a_2 a_3 - a_4| = \left| \frac{23}{600}c_1^3 - \frac{19}{150}c_2 c_1 + \frac{1}{10}c_3 \right|.$$

By applying (1.9), we get

$$|a_2 a_3 - a_4| \leqslant \left[2\left|\frac{23}{600}\right| + 2\left|\frac{19}{150} - 2\left(\frac{23}{600}\right)\right| + 2\left|\frac{23}{600} - \frac{19}{150} + \frac{1}{10}\right| \right] = \frac{1}{5}.$$

□

Theorem 2.5. *If* f *of the form* (1.1) *belongs to* $\mathcal{R}_{3\mathcal{L}}$, *then*

$$|H_{2,2}(f)| = |a_2 a_4 - a_3^2| \leqslant \frac{16}{225}. \tag{2.14}$$

The result is sharp for

$$f_2(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^2 + \frac{1}{5}t^8 \right) dt = \xi + \frac{4}{15}\xi^3 + \frac{1}{45}\xi^9 + \cdots.$$

Proof. From (2.8), (2.9), and (2.10), we have

$$H_{2,2}(f) = \frac{1}{1800}c_1^4 - \frac{1}{450}c_1^2c_2 + \frac{1}{50}c_1c_3 - \frac{4}{225}c_2^2.$$

Applying (1.5) to express c_2 and c_3 in terms of $c_1 = c$, with $0 \leqslant c \leqslant 2$, we get

$$H_{2,2}(f) = -\tfrac{1}{200}c^2(4-c^2)x^2 - \tfrac{1}{225}(4-c^2)^2x^2 + \tfrac{1}{100}c(4-c^2)(1-|x|^2)\sigma,$$

with the aid of the triangle inequality and replacing $|\sigma| \leqslant 1$, $|x| = b$, with $b \leqslant 1$. So, we obtain

$$|H_{2,2}(f)| \leqslant \frac{1}{200}c^2(4-c^2)b^2 + \frac{1}{225}(4-c^2)^2b^2 + \frac{1}{100}c(4-c^2)(1-b^2) := \varphi(c,b).$$

It is a simple calculation to show that $\varphi'(c,b) \geqslant 0$ on $[0,1]$, so that $\varphi(c,b) \leqslant \varphi(c,1)$. Putting $b = 1$ gives

$$|H_{2,2}(f)| \leqslant \frac{1}{200}c^2(4-c^2) + \frac{1}{225}(4-c^2)^2 := \varphi(c,1).$$

Also, $\varphi'(c,1) < 0$, and so $\varphi(c,1)$ is a decreasing function. Thus, the maximum value at $c = 0$ is

$$|H_{2,2}(f)| \leqslant \frac{16}{225}.$$

□

We will now determine bound of the third Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{R}_{3\mathcal{L}}$.

Theorem 2.6. *If* f *as assumed in* (1.1) *and belongs to* $\mathcal{R}_{3\mathcal{L}}$, *then*

$$|H_{3,1}(f)| \leqslant \frac{343}{3375} \simeq 0.101\,63.$$

Proof. We have the third Hankel determinant form as follows:

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

where $a_1 = 1$. This yields

$$|H_{3,1}(f)| \leqslant |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

By using (2.1), (2.2), (2.3), (2.12), (2.13), (2.14), we obtain our desired result. □

3. Bounds of $H_{3,1}(f)$ for the 2-fold and 3-fold symmetric functions

A function f is said to be m-fold symmetric if the following condition holds true for $\varepsilon = \exp\left(\frac{2\pi i}{m}\right)$,

$$f(\varepsilon\xi) = \varepsilon f(\xi), \quad (\xi \in \mathbb{D}).$$

The set of all m-fold symmetric functions belonging to the familiar class of univalent functions $\mathcal{S}$ is denoted by $\mathcal{S}^{(m)}$ which is represented as the following series expansion

$$f(\xi) = \xi + \sum_{n=1}^{\infty} a_{mn+1}\xi^{mn+1} \quad (\xi \in \mathbb{D}). \tag{3.1}$$

An analytic functions f of the form (3.1) belong to class $\mathcal{R}_{3\mathcal{L}}^{(m)}$ if and only if

$$f'(\xi) = 1 + \frac{4}{5}\left(\frac{h(\xi)-1}{h(\xi)+1}\right) + \frac{1}{5}\left(\frac{h(\xi)-1}{h(\xi)+1}\right)^4, \quad (\xi \in \mathbb{D}), \tag{3.2}$$

where the class $\mathcal{P}^{(m)}$ is defined as follows

$$\mathcal{P}^{(m)} = \left\{ h \in \mathcal{P} : h(\xi) = 1 + \sum_{n=1}^{\infty} c_{mn} \xi^{mn} \right\}. \tag{3.3}$$

If a function f belong to $\mathcal{S}^{(2)}$, then its series representation is

$$f(\xi) = \xi + a_3\xi^3 + a_5\xi^5 + \cdots \quad \text{and} \quad H_{3,1}(f) = a_3\left(a_5 - a_3^2\right).$$

Further, if f belong to $\mathcal{S}^{(3)}$ then it series representation is

$$f(\xi) = \xi + a_4\xi^4 + a_7\xi^7 + \cdots \quad \text{and} \quad H_{3,1}(f) = -a_4^2.$$

Theorem 3.1. *If $f \in \mathcal{R}_{3\mathcal{L}}^{(2)}$, then*

$$|H_{3,1}(f)| \leqslant \frac{16}{375}.$$

Proof. Let $f \in \mathcal{R}_{3\mathcal{L}}^{(2)}$, then by series (3.1), (3.2), and (3.3), for $m = 2$, we have

$$f(\xi) = 1 + 3a_3\xi^2 + 5a_5\xi^4 + \cdots = 1 + \frac{2}{5}c_2\xi^2 + \left(\frac{2}{5}c_4 - \frac{1}{5}c_2^2\right)\xi^4 + \cdots.$$

After comparing, we get

$$a_3 = \frac{2}{15}c_2, \quad a_5 = \frac{2}{25}c_4 - \frac{1}{25}c_2^2.$$

This implies that

$$H_{3,1}(f) = a_3\left(a_5 - a_3^2\right).$$

Substituting above values, we get

$$H_{3,1}(f) = \frac{2}{15}c_2\left(\frac{2}{25}c_4 - \frac{13}{225}c_2^2\right) = \frac{4}{375}c_2\left(c_4 - \frac{13}{18}c_2^2\right).$$

Now using triangle inequality along with (1.6) and (1.7), we get the required result. □

Theorem 3.2. *If $f \in \mathcal{R}_{3\mathcal{L}}^{(3)}$, then*

$$|H_{3,1}(f)| \leqslant \frac{1}{25}.$$

The result is sharp for function

$$f_3(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^3 + \frac{1}{5}t^{12}\right) dt = \xi + \frac{1}{5}\xi^4 + \cdots.$$

Proof. Let $f \in \mathcal{R}_{3\mathcal{L}}^{(3)}$. Then by (3.1), (3.2), and (3.3) for $m = 3$, we have

$$f'(\xi) = 1 + 4a_4\xi^3 + \cdots = 1 + \frac{2}{5}c_3\xi^3 + \cdots.$$

After comparing, we get

$$a_4 = \frac{1}{10}c_3.$$

This implies that

$$H_{3,1}(f) = -a_4^2$$

and substituting above values, we get

$$H_{3,1}(f) = -\frac{1}{100}c_3^2.$$

Now using triangle inequality along with (1.7), we get the required result. □

4. Zalcman functional for $f \in \mathcal{R}_{3\mathcal{L}}$

One of the main conjectures in Geometric function theory, suggested by Lawrence Zalcman in 1960, is that the coefficients of class $\mathcal{S}$ satisfy the inequality,

$$|a_n^2 - a_{2n-1}| \leqslant (n-1)^2.$$

Only the well-known Koebe function $k(\xi) = \frac{\xi}{(1-\xi)^2}$ and its rotations have equality in the above form. For the popular Fekete-Szego inequality, when $n = 2$, the equality holds. Many researchers have researched Zalcman functional in the literature [3, 4, 19].

Theorem 4.1. *Let* f *be given by* (1.1) *and belongs to* $\mathcal{R}_{3\mathcal{L}}$. *Then*

$$|a_3^2 - a_5| \leqslant \frac{4}{25}.$$

The result is sharp for function

$$f_4(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^4 + \frac{1}{5}t^{16}\right) dt = \xi + \frac{4}{25}\xi^5 + \cdots .$$

Proof. We use the equations (2.9) and (2.11) to get Zalcman functional, and then we have

$$\left|a_3^2 - a_5\right| = \frac{2}{25}\left|\frac{43}{288}c_1^4 - \frac{35}{36}c_1^2c_2 + c_3c_1 + \frac{13}{18}c_2^2 - c_4\right|.$$

Using Lemma 1.3, we can get the necessary result for the last expression. □

5. Krushkal inequality for $f \in \mathcal{R}_{3\mathcal{L}}$

In this section we will give direct proof of the inequality

$$\left|a_n^p - a_2^{p(n-1)}\right| \leqslant 2^{p(n-1)} - n^p$$

over the class $\mathcal{R}_{3\mathcal{L}}$ for choice of $n = 4$, $p = 1$, and for $n = 5$, $p = 1$. Krushkal introduced and proved this inequality for whole class of univalent functions in [17].

Theorem 5.1. *Let* f *be given by* (1.1) *and belongs to* $\mathcal{R}_{3\mathcal{L}}$. *Then*

$$|a_4 - a_2^3| \leqslant \frac{1}{5}.$$

The result is sharp for function

$$f_3(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^3 + \frac{1}{5}t^{12}\right) dt = \xi + \frac{1}{5}\xi^4 + \cdots .$$

Proof. From equations (2.8) and (2.10), we get

$$|a_4 - a_2^3| = \left|\frac{17}{1000}c_1^3 - \frac{1}{10}c_2c_1 + \frac{1}{10}c_3\right|,$$

applying (1.9) to above, we get the required result. □

Theorem 5.2. *Let* f *be given by* (1.1) *and belongs to* $\mathcal{R}_{3\mathcal{L}}$. *Then*

$$\left|a_5 - a_2^4\right| \leqslant \frac{4}{25}.$$

The result is sharp for function

$$f_4(\xi) = \int_0^{\xi} \left(1 + \frac{4}{5}t^4 + \frac{1}{5}t^{16}\right) dt = \xi + \frac{4}{25}\xi^5 + \cdots .$$

Proof. From equations (2.8) and (2.10), we get

$$\left|a_5 - a_2^4\right| = \frac{2}{25}\left|\frac{91}{800}c_1^4 - \frac{3}{4}c_1^2c_2 + c_3c_1 + \frac{1}{2}c_2^2 - c_4\right|.$$

Using Lemma 1.3, we can get the necessary result for the last expression. □

6. Conclusion

A class of bounded turning functions is defined in connection with three leaf functions. Various important properties are evaluated for these functions like the estimates of four initial coefficients, Fekete-Szegö functional, the second-order Hankel determinant, Zalcman conjecture and Krushkal inequality. The sharpness is also discussed for these results. In addition to these results we not only estimated the bounds of the third-order Hankel determinants for this class but also for its 2-fold and 3-fold symmetric functions. The idea and means of characterizing these functions with various important properties is novel and can be extended to various other areas like multivlent functions, meromorphic functions and Harmonic functions.

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