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New types of rough Pythagorean fuzzy UP-filters of UP-algebras

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Abstract

The aim of this paper is to introduce nine types of rough Pythagorean fuzzy sets in UP-algebras and nine types of rough sets in UP-algebras. Then we study relation of these rough Pythagorean fuzzy sets and new types of Pythagorean fuzzy UP-filter under equivalence (congruence) relation. Moreover, we will also discuss t-level subsets of rough Pythagorean fuzzy sets in UP-algebras to study the relationships between rough Pythagorean fuzzy sets and rough sets in UP-algebras which we defined them above. Finally, we discuss the concept of homomorphisms between IUP-algebras and also study the direct and inverse images of four special subsets.

Keywords: UP-algebra, rough set, rough Pythagorean fuzzy set, t-level subset.

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1. Introduction and preliminaries

The concept of fuzzy sets (FSs) was first considered by Zadeh [32] in 1965. Zadeh's and others' FS concepts have found numerous applications in mathematics and other fields. Following the introduction of the concept of FSs, various researchers were interviewed about generalizations of the concept of FSs, including: Atanassov [5] defined a new concept called an intuitionistic fuzzy set (IFS) which is a generalization of a FS, Yager [30] introduced a new class of non-standard fuzzy subsets called a Pythagorean fuzzy set (PFS) and the related idea of Pythagorean membership grades.

The concept of rough sets (RSs) was first considered by Pawlak [20] in 1982. After the introduction of the concept of RSs, several authors have applied the concept of RSs to the generalizations of the concept of

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FSs in many algebraic structures such as: Chen and Wang [6] combined RSs and fuzzy subalgebras (fuzzy ideals) fruitfully by defining rough fuzzy subalgebras (rough fuzzy ideals) of BCI-algebras, Moradiana et al. [19] presented a definition of the lower and upper approximation of subsets of BCK-algebras concerning a fuzzy ideal, Ahn and Kim [1] introduced the concept of rough fuzzy filters in BE-algebras, Ahn and Ko [2] introduced the concept of rough ideals and rough fuzzy ideals in BCK/BCI-algebras, Hussain et al. [10] introduced the concept of rough Pythagorean fuzzy ideals in semigroups, and Chinram and Panityakul [7] introduced rough Pythagorean fuzzy ideals in ternary semigroups and gave some remarkable properties. In 2022, Satirad et al. [22] combined notion of RSs and PFSs by defining rough Pythagorean fuzzy sets (RPFSs) in UP-algebras, studied relation between RPFSs and PFSs, discussed relation between PSFs and special subset of UP-algebras with t-level subset. By consideration of t-level subset, then discovered relation between RPSFs and RSs in UP-algebras under a congruence relation.

In this study, we extend concept of new types of Pythagorean fuzzy UP-filter of UP-algebras. Then apply concept of RSs into them and so define nine new types of rough Pythagorean fuzzy sets (RPFSs) in UP-algebras: upper rough Pythagorean fuzzy implicative UP-filters (UpRPFIUPFs), upper rough Pythagorean fuzzy comparative UP-filters (UpRPFCUPFs), upper rough Pythagorean fuzzy shift UP-filters (UpRPFSUPFs), lower rough Pythagorean fuzzy implicative UP-filters (LoRPFIUPFs), lower rough Pythagorean fuzzy comparative UP-filters (LoRPFCUPFs), lower rough Pythagorean fuzzy shift UP-filters (LoRPFSUPFs), rough Pythagorean fuzzy implicative UP-filters (RPFIUPFs), rough Pythagorean fuzzy comparative UP-filters (RPFCUPFs), and rough Pythagorean fuzzy shift UP-filters (RPFSUPFs) and new type of rough set (RS) in UP-algebras: upper rough implicative UP-filters (UpRIUPFs), upper rough comparative UP-filters (UpRCUPFs), upper rough shift UP-filters (UpRSUPFs), lower rough implicative UP-filters (LoRIUPFs), lower rough comparative UP-filters (LoRCUPFs), lower rough shift UP-filters (LoRSUPFs), rough implicative UP-filters (RIUPFs), rough comparative UP-filters (RCUPFs), and rough shift UP-filters (RSUPFs). Moreover, we verify their generalization of nine types of RPFSs in UP-algebras. Then, to investigate the relationships between new types of Pythagorean fuzzy UP-filter and new types of UP-filter of UP-algebras, we explore t-level subsets of PFSs. Finally, we study the relationships between nine types of RPFSs and nine types of RSs in UP-algebras under a congruence relation.

Let's go through the definition of UP-algebras first.

Definition 1.1 ([11]). A *UP-algebra* is one that has the algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$ of type $(2, 0)$, where $\mathcal{U}$ is a nonempty set, $\star$ is a binary operation on $\mathcal{U}$, and 0 is a fixed element of $\mathcal{U}$ if it meets the following axioms:

$$(\forall a, b, c \in \mathcal{U})((b \star c) \star ((a \star b) \star (a \star c)) = 0),$$
$$(\forall a \in \mathcal{U})(0 \star a = a),$$
$$(\forall a \in \mathcal{U})(a \star 0 = 0),$$
$$(\forall a, b \in \mathcal{U})(a \star b = 0, b \star a = 0 \Rightarrow a = b).$$

For more examples of UP-algebras, see [3, 4, 8, 12, 14, 25–28]. According to [11], we know that the concept of UP-algebras is a generalization of KU-algebras (see [21]).

Unless otherwise indicated, we will assume that $\mathcal{U} = (\mathcal{U}, \star, 0)$ is a UP-algebra.

In $\mathcal{U}$, the following assertions are valid (see [11, 12]).

$$(\forall a \in \mathcal{U})(a \star a = 0),$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0, b \star c = 0 \Rightarrow a \star c = 0),$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (c \star a) \star (c \star b) = 0),$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow (b \star c) \star (a \star c) = 0),$$
$$(\forall a, b \in \mathcal{U})(a \star (b \star a) = 0),$$
$$(\forall a, b \in \mathcal{U})((b \star a) \star a = 0 \Leftrightarrow a = b \star a),$$
$$(\forall a, b \in \mathcal{U})(a \star (b \star b) = 0),$$

$$(\forall u, a, b, c \in \mathcal{U})((a \star (b \star c)) \star (a \star ((u \star b) \star (u \star c))) = 0),$$
$$(\forall u, a, b, c \in \mathcal{U})((((u \star a) \star (u \star b)) \star c) \star ((a \star b) \star c) = 0),$$
$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star c) = 0),$$
$$(\forall a, b, c \in \mathcal{U})(a \star b = 0 \Rightarrow a \star (c \star b) = 0),$$
$$(\forall a, b, c \in \mathcal{U})(((a \star b) \star c) \star (a \star (b \star c)) = 0),$$
$$(\forall u, a, b, c \in \mathcal{U})(((a \star b) \star c) \star (b \star (u \star c)) = 0).$$

According to [11], the binary relation $\leqslant$ on $\mathcal{U}$ is defined as follows:

$$(\forall a, b \in \mathcal{U})(a \leqslant b \Leftrightarrow a \star b = 0).$$

Definition 1.2 ([9, 11, 13, 15–17, 29])**.** A nonempty subset S of a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$ is called

(1) a *UP-subalgebra* (UPS) of $\mathcal{U}$ if it satisfies the following condition:

$$(\forall a, b \in S)(a \star b \in S);$$

(2) a *near UP-filter* (NUPF) of $\mathcal{U}$ if it satisfies the following condition:

$$(\forall a, b \in \mathcal{U})(b \in S \Rightarrow a \star b \in S);$$

(3) a *UP-filter* (UPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } \mathcal{U} \text{ is in } S, \quad (\forall a, b \in \mathcal{U})(a \star b \in S, a \in S \Rightarrow b \in S); \tag{1.1}$$

(4) an *implicative UP-filter* (IUPF) of $\mathcal{U}$ if it satisfies the condition (1.1) and the following condition:

$$(\forall a, b, c \in \mathcal{U})(a \star (b \star c) \in S, a \star b \in S \Rightarrow a \star c \in S);$$

(5) a *comparative UP-filter* (CUPF) of $\mathcal{U}$ if it satisfies the condition (1.1) and the following condition:

$$(\forall a, b, c \in \mathcal{U})(a \star ((b \star c) \star b) \in S, a \in S \Rightarrow b \in S);$$

(6) a *shift UP-filter* (SUPF) of $\mathcal{U}$ if it satisfies the condition (1.1) and the following condition:

$$(\forall a, b, c \in \mathcal{U})(a \star (b \star c) \in S, a \in S \Rightarrow ((c \star b) \star b) \star c \in S);$$

(7) a *UP-ideal* (UPI) of $\mathcal{U}$ if it satisfies the condition (1.1) and the following condition:

$$(\forall a, b, c \in \mathcal{U})(a \star (b \star c) \in S, b \in S \Rightarrow a \star c \in S);$$

(8) a *strong UP-ideal* (SUPI) of $\mathcal{U}$ if it satisfies the condition (1.1) and the following condition:

$$(\forall a, b, c \in \mathcal{U})((c \star b) \star (c \star a) \in S, b \in S \Rightarrow a \in S).$$

We proved that the concept of UPSs is a generalization of NUPFs, NUPFs is a generalization of UPFs, UPFs is a generalization of UPIs, UPFs is a generalization of IUPFs, UPFs is a generalization of CUPFs, UPFs is a generalization of SUPFs, UPIs is a generalization of IUPFs, IUPFs is a generalization of SUPIs, CUPFs is a generalization of SUPIs, SUPFs is a generalization of SUPIs. They also proved that $\mathcal{U}$ is the only SUPI.

Definition 1.3 ([32]). A *fuzzy set* (FS) F in a nonempty set $\mathcal{U}$ is described by its membership function μ_F. To every point $a \in \mathcal{U}$, this function associates a real number $\mu_F(a)$ in the closed interval $[0,1]$. The real number $\mu_F(a)$ is interpreted for the point as a degree of membership of an object $a \in \mathcal{U}$ to the FS F, that is, $F := \{(a, \mu_F(a)) \mid a \in \mathcal{U}\}$. We say that a FS F in $\mathcal{U}$ is *constant fuzzy set* if its membership function μ_F is constant.

In 2013, Yager [30] and Yager and Abbasov [31] introduced the concept of PFSs for the first time.

Definition 1.4 ([30, 31]). A *Pythagorean fuzzy set* (PFS) P in a nonempty set $\mathcal{U}$ is described by their membership function μ_P and non-membership function ν_P. To every point $a \in \mathcal{U}$, these functions associate real numbers $\mu_P(a)$ and $\nu_P(a)$ in the closed interval $[0,1]$, with the following condition:

$$(\forall a \in \mathcal{U})(0 \leqslant \mu_P(a)^2 + \nu_P(a)^2 \leqslant 1).$$

The real numbers $\mu_P(a)$ and $\nu_P(a)$ are interpreted for the point as a degree of membership and non-membership of an object $a \in \mathcal{U}$, respectively, to the PFS P, that is, $P := \{(a, \mu_P(a), \nu_P(a)) \mid x \in \mathcal{U}\}$. For the sake of simplicity, a PFS P is denoted by $P = (\mu_P, \nu_P)$. We say that a PFS P in $\mathcal{U}$ is *constant Pythagorean fuzzy set* if their membership function μ_P and non-membership function ν_P are constant.

Definition 1.5 ([22, 23]). A PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ is called

(1) a *Pythagorean fuzzy UP-subalgebra* (PFUPS) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall a, b \in \mathcal{U})(\mu_P(a \star b) \geqslant \min\{\mu_P(a), \mu_P(b)\}), \quad (\forall a, b \in \mathcal{U})(\nu_P(a \star b) \leqslant \max\{\nu_P(a), \nu_P(b)\});$$

(2) a *Pythagorean fuzzy near UP-filter* (PFNUPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall a, b \in \mathcal{U})(\mu_P(a \star b) \geqslant \mu_P(b)), \quad (\forall a, b \in \mathcal{U})(\nu_P(a \star b) \leqslant \nu_P(b));$$

(3) a *Pythagorean fuzzy UP-filter* (PFUPF) of $\mathcal{U}$ if it satisfies the following conditions:

$$(\forall a \in \mathcal{U})(\mu_P(0) \geqslant \mu_P(a)), \tag{1.2}$$
$$(\forall a \in \mathcal{U})(\nu_P(0) \leqslant \nu_P(a)), \tag{1.3}$$
$$(\forall a, b \in \mathcal{U})(\mu_P(b) \geqslant \min\{\mu_P(a \star b), \mu_P(a)\}),$$
$$(\forall a, b \in \mathcal{U})(\nu_P(b) \leqslant \max\{\nu_P(a \star b), \nu_P(a)\});$$

(4) a *Pythagorean fuzzy implicative UP-filter* (PFIUPF) of $\mathcal{U}$ if it satisfies the conditions (1.2) and (1.3) and the following conditions:

$$(\forall a, b, c \in \mathcal{U})(\mu_P(a \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\}), \tag{1.4}$$
$$(\forall a, b, c \in \mathcal{U})(\nu_P(a \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\}); \tag{1.5}$$

(5) a *Pythagorean fuzzy comparative UP-filter* (PFCUPF) of $\mathcal{U}$ if it satisfies the conditions (1.2) and (1.3) and the following conditions:

$$(\forall a, b, c \in \mathcal{U})(\mu_P(b) \geqslant \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\}), \tag{1.6}$$
$$(\forall a, b, c \in \mathcal{U})(\nu_P(b) \leqslant \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\}); \tag{1.7}$$

(6) a *Pythagorean fuzzy shift UP-filter* (PFSUPF) of $\mathcal{U}$ if it satisfies the conditions (1.2) and (1.3) and the following conditions:

$$(\forall a, b, c \in \mathcal{U})(\mu_P(((c \star b) \star b) \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a)\}), \tag{1.8}$$
$$(\forall a, b, c \in \mathcal{U})(\nu_P(((c \star b) \star b) \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a)\}); \tag{1.9}$$

(7) a *Pythagorean fuzzy UP-ideal* (PFUPI) of $\mathfrak{U}$ if it satisfies the conditions (1.2) and (1.3) and the following conditions:

$$(\forall a, b, c \in \mathfrak{U})(\mu_P(a \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(b)\}),$$
$$(\forall a, b, c \in \mathfrak{U})(\nu_P(a \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(b)\});$$

(8) a *Pythagorean fuzzy strong UP-ideal* (PFSUPI) of $\mathfrak{U}$ if it satisfies the conditions (1.2) and (1.3) and the following conditions:

$$(\forall a, b, c \in \mathfrak{U})(\mu_P(a) \geqslant \min\{\mu_P((c \star b) \star (c \star a)), \mu_P(b)\}),$$
$$(\forall a, b, c \in \mathfrak{U})(\nu_P(a) \leqslant \max\{\nu_P((c \star b) \star (c \star a)), \nu_P(b)\}).$$

Satirad et al. [22, 23] proved that the concept of PFUPSs is a generalization of PFNUPFs, PFNUPFs is a generalization of PFUPFs, PFUPFs is a generalization of PFUPIs, PFUPFs is a generalization of PFCUPFs, PFUPFs is a generalization of PFSUPFs, PFUPIs is a generalization of PFIUPFs, PFIUPFs is a generalization of PFSUPIs, PFCUPFs is a generalization of PFSUPIs, and PFSUPFs is a generalization of PFSUPIs. Furthermore, they proved that PFSUPIs and constant PFSs coincide in $\mathfrak{U}$.

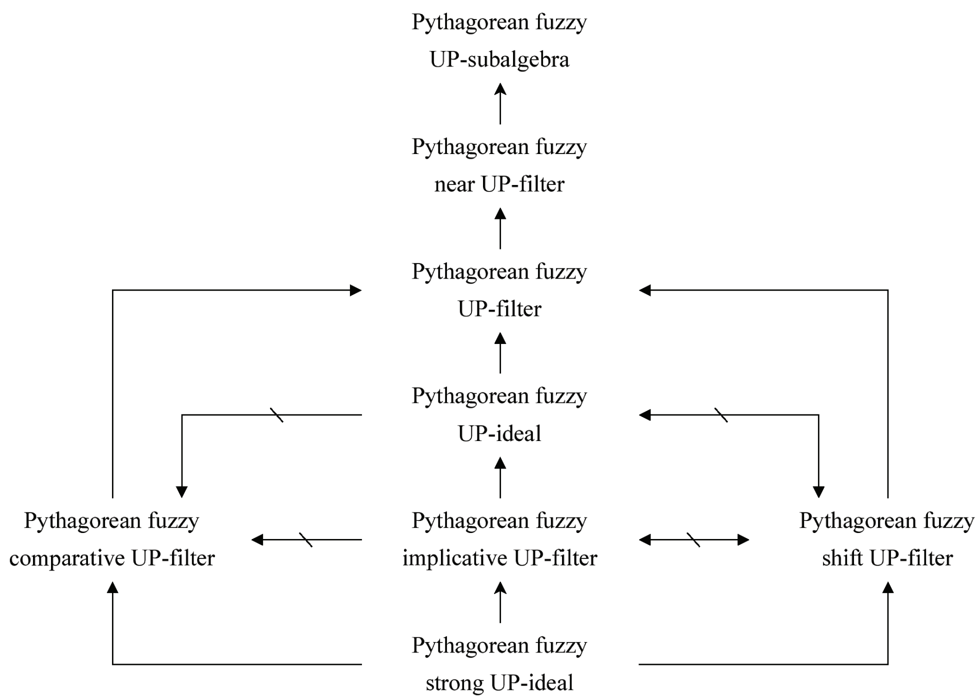

Figure 1: Pythagorean fuzzy sets in UP-algebras.

Let ρ be an equivalence relation (ER) on a set $\mathfrak{U}$. If $a \in \mathfrak{U}$, then the ρ-class of a is the set $(a)_\rho$ defined as follows:

$$(a)_\rho = \{b \in \mathfrak{U} \mid (a, b) \in \rho\}.$$

An ER ρ on $\mathfrak{U}$ is called a *congruence relation* (CR) if

$$(\forall a, b, c \in \mathfrak{U})((a, b) \in \rho \Rightarrow (a \star c, b \star c) \in \rho, (c \star a, c \star b) \in \rho).$$

Definition 1.6. For nonempty subsets A and B of $\mathfrak{U}$, we denote

$$AB = A \star B = \{u \star v \mid u \in A \text{ and } v \in B\}.$$

If ρ is a CR on $\mathcal{U}$, then

$$(\forall a, b \in \mathcal{U})((a)_\rho(b)_\rho \subseteq (a \star b)_\rho), \quad \text{see [18]}.$$

Definition 1.7. Let ρ be an ER on a nonempty set $\mathcal{U}$ and $S \in \mathcal{P}(\mathcal{U})$. The *upper approximation* of S is defined by

$$\rho^+(S) = \{a \in \mathcal{U} \mid (a)_\rho \subseteq S\},$$

the *lower approximation* of S is defined by

$$\rho^-(S) = \{a \in \mathcal{U} \mid (a)_\rho \cap S \neq \emptyset\}.$$

We know that $\rho^+(S)$ and $\rho^-(S)$ are subset of $\mathcal{U}$. Then we call S that a *rough set* (RS) of $\mathcal{U}$.

Definition 1.8 ([18])**.** Let ρ be an ER on $\mathcal{U}$. Then a nonempty subset S of $\mathcal{U}$ is called

(1) an *upper rough UP-subalgebra* (UpRUPS) of $\mathcal{U}$ if $\rho^+(S)$ is a UPS of $\mathcal{U}$;

(2) an *upper rough near UP-filter* (UpRNUPF) of $\mathcal{U}$ if $\rho^+(S)$ is a NUPF of $\mathcal{U}$;

(3) an *upper rough UP-filter* (UpRUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a UPF of $\mathcal{U}$;

(4) an *upper rough UP-ideal* (UpRUPI) of $\mathcal{U}$ if $\rho^+(S)$ is a UPI of $\mathcal{U}$;

(5) an *upper rough strong UP-ideal* (UpRSUPI) of $\mathcal{U}$ if $\rho^+(S)$ is a SUPI of $\mathcal{U}$;

(6) a *lower rough UP-subalgebra* (LoRUPS) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a UPS of $\mathcal{U}$;

(7) a *lower rough near UP-filter* (LoRNUPF) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a NUPF of $\mathcal{U}$;

(8) a *lower rough UP-filter* (LoRUPF) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a UPF of $\mathcal{U}$;

(9) a *lower rough UP-ideal* (LoRUPI) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a UPI of $\mathcal{U}$;

(10) a *lower rough strong UP-ideal* (LoRSUPI) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a SUPI of $\mathcal{U}$;

(11) a *rough UP-subalgebra* (RUPS) of $\mathcal{U}$ if it is both an UpRUPS and a LoRUPS of $\mathcal{U}$;

(12) a *rough near UP-filter* (RNUPF) of $\mathcal{U}$ if it is both an UpRNUPF and a LoRNUPF of $\mathcal{U}$;

(13) a *rough UP-filter* (RUPF) of $\mathcal{U}$ if it is both an UpRUPF and a LoRUPF of $\mathcal{U}$;

(14) a *rough UP-ideal* (RUPI) of $\mathcal{U}$ if it is both an UpRUPI and a LoRUPI of $\mathcal{U}$; and

(15) a *rough strong UP-ideal* (RSUPI) of $\mathcal{U}$ if it is both an UpRSUPI and a LoRSUPI of $\mathcal{U}$.

In this paper, we introduce necessary define for study rough Pythagorean fuzzy sets in UP-algebras.

Definition 1.9. Let ρ be an ER on $\mathcal{U}$. Then a nonempty subset S of $\mathcal{U}$ is called

(1) an *upper rough implicative UP-filter* (UpRIUPF) of $\mathcal{U}$ if $\rho^+(S)$ is an IUPF of $\mathcal{U}$;

(2) an *upper rough comparative UP-filter* (UpRCUPF) of $\mathcal{U}$ if $\rho^+(S)$ is a CUPF of $\mathcal{U}$;

(3) an *upper rough shift UP-filter* (UpRSUPF) of $\mathcal{U}$ if $\rho^+(S)$ is a SUPF of $\mathcal{U}$;

(4) a *lower rough implicative UP-filter* (LoRIUPF) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is an IUPF of $\mathcal{U}$;

(5) a *lower rough comparative UP-filter* (LoRCUPF) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a CUPF of $\mathcal{U}$;

(6) a *lower rough shift UP-filter* (LoRSUPF) of $\mathcal{U}$ if $\emptyset \neq \rho^-(S)$ is a SUPF of $\mathcal{U}$;

(7) a *rough implicative UP-filter* (RIUPF) of $\mathcal{U}$ if it is both an UpRIUPF and a LoRIUPF of $\mathcal{U}$;

(8) a *rough comparative UP-filter* (RCUPF) of $\mathcal{U}$ if it is both an UpRCUPF and a LoRCUPF of $\mathcal{U}$; and

(9) a *rough shift UP-filter* (RSUPF) of $\mathcal{U}$ if it is both an UpRSUPF and a LoRSUPF of $\mathcal{U}$.

2. RPFSs in UP-algebras

Definition 2.1 ([22]). Let ρ be an ER on a nonempty set $\mathcal{U}$ and $P = (\mu_P, \nu_P)$ a PFS in $\mathcal{U}$. The *upper approximation* of P is defined by

$$\rho^+(P) = \{(a, \overline{\mu}_P(a), \overline{\nu}_P(a)) \mid a \in \mathcal{U}\},$$

where $\overline{\mu}_P(a) = \sup_{u \in (a)_\rho} \{\mu_P(u)\}$ and $\overline{\nu}_P(a) = \inf_{u \in (a)_\rho} \{\nu_P(u)\}$. The *lower approximation* of P is defined by

$$\rho^-(P) = \{(a, \underline{\mu}_P(a), \underline{\nu}_P(a)) \mid a \in \mathcal{U}\},$$

where $\underline{\mu}_P(a) = \inf_{u \in (a)_\rho} \{\mu_P(u)\}$ and $\underline{\nu}_P(a) = \sup_{u \in (a)_\rho} \{\nu_P(u)\}$.

It is easy to proof that $\rho^+(P)$ and $\rho^-(P)$ are PFSs in $\mathcal{U}$. Then we call P a *rough Pythagorean fuzzy set* (RPFS) in $\mathcal{U}$. Thus we can denote the upper approximation and the lower approximation by $\rho^+(P) = (\overline{\mu}_P, \overline{\nu}_P)$ and $\rho^-(P) = (\underline{\mu}_P, \underline{\nu}_P)$, respectively.

Definition 2.2 ([24]). Let ρ be an ER on $\mathcal{U}$ and $P = (\mu_P, \nu_P)$ a PFS in $\mathcal{U}$. Then a RPFS P in $\mathcal{U}$ is called *constant rough Pythagorean fuzzy set* in $\mathcal{U}$ if their membership functions $\overline{\mu}_P, \underline{\mu}_P$ and non-membership functions $\overline{\nu}_P, \underline{\nu}_P$ are constant.

Definition 2.3 ([24]). Let ρ be an ER on $\mathcal{U}$. Then a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ is called

(1) an *upper rough Pythagorean fuzzy UP-subalgebra* (UpRPFUPS) of $\mathcal{U}$ if $\rho^+(P)$ is a PFUPS of $\mathcal{U}$;

(2) an *upper rough Pythagorean fuzzy near UP-filter* (UpRPFNUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a PFNUPF of $\mathcal{U}$;

(3) an *upper rough Pythagorean fuzzy UP-filter* (UpRPFUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a PFUPF of $\mathcal{U}$;

(4) an *upper rough Pythagorean fuzzy UP-ideal* (UpRPFUPI) of $\mathcal{U}$ if $\rho^+(P)$ is a PFUPI of $\mathcal{U}$;

(5) an *upper rough Pythagorean fuzzy strong UP-ideal* (UpRPFSUPI) of $\mathcal{U}$ if $\rho^+(P)$ is a PFSUPI of $\mathcal{U}$;

(6) a *lower rough Pythagorean fuzzy UP-subalgebra* (LoRPFUPS) of $\mathcal{U}$ if $\rho^-(P)$ is a PFUPS of $\mathcal{U}$;

(7) a *lower rough Pythagorean fuzzy near UP-filter* (LoRPFNUPF) of $\mathcal{U}$ if $\rho^-(P)$ is a PFNUPF of $\mathcal{U}$;

(8) a *lower rough Pythagorean fuzzy UP-filter* (LoRPFUPF) of $\mathcal{U}$ if $\rho^-(P)$ is a PFUPF of $\mathcal{U}$;

(9) a *lower rough Pythagorean fuzzy UP-ideal* (LoRPFUPI) of $\mathcal{U}$ if $\rho^-(P)$ is a PFUPI of $\mathcal{U}$;

(10) a *lower rough Pythagorean fuzzy strong UP-ideal* (LoRPFSUPI) of $\mathcal{U}$ if $\rho^-(P)$ is a PFSUPI of $\mathcal{U}$;

(11) a *rough Pythagorean fuzzy UP-subalgebra* (RPFUPS) of $\mathcal{U}$ if it is both an UpRPFUPS and a LoRPFUPS of $\mathcal{U}$;

(12) a *rough Pythagorean fuzzy near UP-filter* (RPFNUPF) of $\mathcal{U}$ if it is both an UpRPFNUPF and a LoRPFNUPF of $\mathcal{U}$;

(13) a *rough Pythagorean fuzzy UP-filter* (RPFUPF) of $\mathcal{U}$ if it is both an UpRPFUPF and a LoRPFUPF of $\mathcal{U}$;

(14) a *rough Pythagorean fuzzy UP-ideal* (RPFUPI) of $\mathcal{U}$ if it is both an UpRPFUPI and a LoRPFUPI of $\mathcal{U}$; and

(15) a *rough Pythagorean fuzzy strong UP-ideal* (RPFSUPI) of $\mathcal{U}$ if it is both an UpRPFSUPI and a LoRPF-SUPI of $\mathcal{U}$.

Next, we introduce nine notions of rough Pythagorean fuzzy sets in UP-algebras.

Definition 2.4. Let ρ be an ER on $\mathcal{U}$. Then a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ is called

(1) an *upper rough Pythagorean fuzzy implicative UP-filter* (UpRPFIUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a PFIUPF of $\mathcal{U}$;

(2) an *upper rough Pythagorean fuzzy comparative UP-filter* (UpRPFCUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a PFCUPF of $\mathcal{U}$;

(3) an *upper rough Pythagorean fuzzy shift UP-filter* (UpRPFSUPF) of $\mathcal{U}$ if $\rho^+(P)$ is a PFSUPF of $\mathcal{U}$;

(4) a *lower rough Pythagorean fuzzy implicative UP-filter* (LoRPFIUPF) of $\mathcal{U}$ if $\rho^-(P)$ is a PFIUPF of $\mathcal{U}$;

(5) a *lower rough Pythagorean fuzzy comparative UP-filter* (LoRPFCUPF) of $\mathcal{U}$ if $\rho^-(P)$ is a PFCUPF of $\mathcal{U}$;

(6) a *lower rough Pythagorean fuzzy shift UP-filter* (LoRPFSUPF) of $\mathcal{U}$ if $\rho^-(P)$ is a PFSUPF of $\mathcal{U}$;

(7) a *rough Pythagorean fuzzy implicative UP-filter* (RPFIUPF) of $\mathcal{U}$ if it is both an UpRPFIUPF and a LoRPFIUPF of $\mathcal{U}$;

(8) a *rough Pythagorean fuzzy comparative UP-filter* (RPFCUPF) of $\mathcal{U}$ if it is both an UpRPFCUPF and a LoRPFCUPF of $\mathcal{U}$; and

(9) a *rough Pythagorean fuzzy shift UP-filter* (RPFSUPF) of $\mathcal{U}$ if it is both an UpRPFSUPF and a LoRPFSUPF of $\mathcal{U}$.

It is simple to verify the generalizations of RPFSs in UP-algebras. As a result, we obtain the diagram of the generalization of RPFSs in UP-algebras, which is shown in Figures 2, 3, and 4.

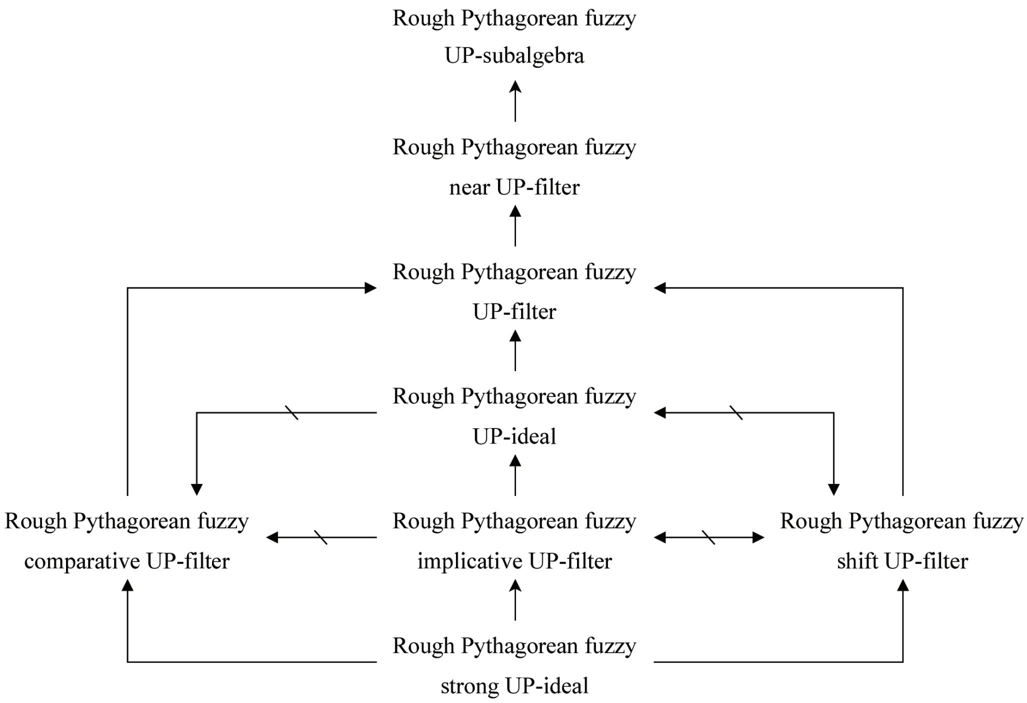

Figure 2: Rough Pythagorean fuzzy sets in UP-algebras.

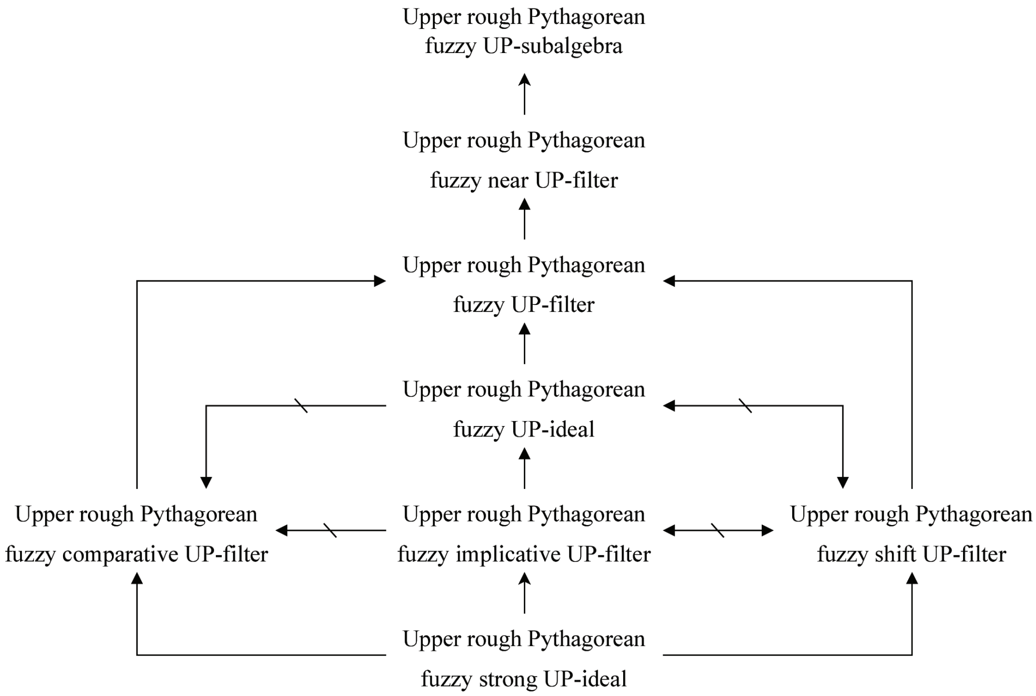

Figure 3: Upper rough Pythagorean fuzzy sets in UP-algebras.

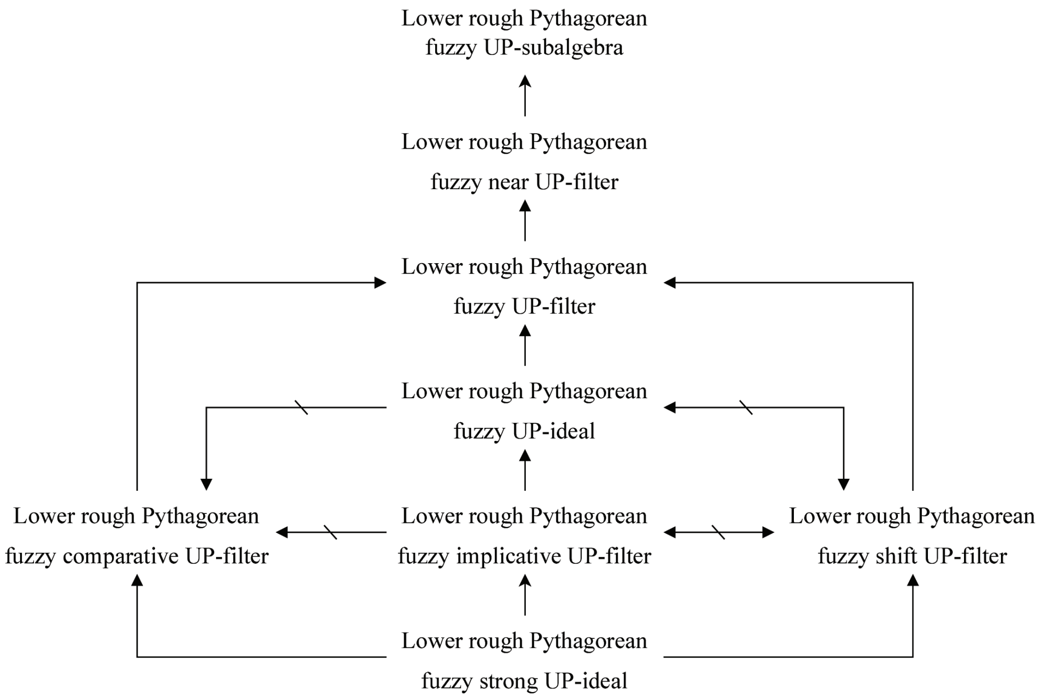

Figure 4: Lower rough Pythagorean fuzzy sets in UP-algebras.

Theorem 2.5 ([24]). *Let ρ be an ER (CR) on $\mathcal{U}$ and $\mathrm{P} = (\mu_{\mathrm{P}}, \nu_{\mathrm{P}})$ a PFS in $\mathcal{U}$. If P is a PFSUPI of $\mathcal{U}$, then P is a RPFSUPI of $\mathcal{U}$.*

The following examples show the relationships between PFSs in $\mathcal{U}$ and RPFSs in $\mathcal{U}$ with ρ is an ER on $\mathcal{U}$.

Example 2.6. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	0
3	0	1	2	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	1	0.4	0.4	0.4
ν_P	0	0.3	0.3	0.3

Then P is a PFIUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (0,1), (1,0), (2,3), (3,2)\}.$$

Then ρ is an ER on $\mathcal{U}$. But $\rho^+(P)$ is not PFIUPF of $\mathcal{U}$.

Example 2.7. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	0	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.6	0.6	0.6	0.3
ν_P	0	0	0	0.1

Then P is a PFCUPF (resp., PFSUPF) of $\mathcal{U}$. Let

$$\rho = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}.$$

Then ρ is an ER on $\mathcal{U}$. But $\rho^-(P)$ is not PFCUPF (resp., PFSUPF) of $\mathcal{U}$.

From Examples 2.6 and 2.7, we get the results that if P is a PFIUPF (resp., PFCUPF and PFSUPF), then it may not be a RPFIUPF (resp., RPFCUPF and RPFSUPF).

Example 2.8. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0, 1, 2, 3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	1	2	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.9	0.1	0.2	0.1
ν_P	0.1	0.5	0.4	0.5

Then P is not a PFIUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}.$$

Then ρ is an ER on $\mathcal{U}$. But $\rho^+(P)$ and $\rho^-(P)$ are PFIUPFs of $\mathcal{U}$.

Example 2.9. By Example 2.8, we define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.6	0.5	0.4	0.4
ν_P	0.5	0.7	0.8	0.8

Then P is not a PFCUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,0),(0,1),(2,3),(3,2)\}.$$

Then ρ is an ER on $\mathcal{U}$. But $\rho^+(P)$ and $\rho^-(P)$ are PFCUPFs of $\mathcal{U}$.

Example 2.10. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	0	0	2
3	0	0	0	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.6	0.5	0.2	0.5
ν_P	0	0.1	0.7	0.1

Then P is not a PFSUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}.$$

Then ρ is an ER on $\mathcal{U}$. But $\rho^+(P)$ and $\rho^-(P)$ are PFSUPFs of $\mathcal{U}$.

From Examples 2.8, 2.9, and 2.10, we get the results that if P is a RPFIUPF (resp., RPFCUPF, and RPFSUPF), then it may not be a PFSUPF (resp., PFCUPF, and PFSUPF).

Example 2.11. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	2	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.9	0.3	0.2	0.1
ν_P	0.1	0.3	0.4	0.5

Then P is a PFIUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}.$$

Then ρ is an ER on $\mathcal{U}$. Thus $\rho^+(P)$ and $\rho^-(P)$ are PFIUPFs of $\mathcal{U}$.

Example 2.12. By Example 2.11, we define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.6	0.6	0.3	0.1
ν_P	0.5	0.5	0.6	0.7

Then P is a PFCUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(2,3),(3,2)\}.$$

Then ρ is an ER on $\mathcal{U}$. Thus $\rho^+(P)$ and $\rho^-(P)$ are PFCUPFs of $\mathcal{U}$.

Example 2.13. By Example 2.10, we define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	1	0.5	0.4	0.4
ν_P	0	0.1	0.5	0.5

Then P is a PFSUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,0),(0,1)\}.$$

Then ρ is an ER on $\mathcal{U}$. Thus $\rho^+(P)$ and $\rho^-(P)$ are PFSUPFs of $\mathcal{U}$.

From Examples 2.11, 2.12, and 2.13, we get the results that P can be a RPFIUPF (resp., RPFCUPF, and RPFSUPF) and a PFIUPF (resp., PFCUPF, and PFSUPF) in the same time.

The following examples show the relationships between PFSs in $\mathcal{U}$ and RPFSs in $\mathcal{U}$ with ρ is a CR on $\mathcal{U}$.

Example 2.14. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.7	0.2	0.6	0.6
ν_P	0.3	0.6	0.5	0.5

Then P is a PFIUPF (resp., PFCUPF, and PFSUPF) of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0)\}.$$

Then ρ is a CR on $\mathcal{U}$. But $\rho^-(P)$ is not a PFSUPF (resp., PFCUPF, and PFSUPF) of $\mathcal{U}$.

From Example 2.14, we get the results that if P is a PFIUPF (resp., PFCUPF, and PFSUPF), then it may not be a RPFIUPF (resp., RPFCUPF, and RPFSUPF).

Example 2.15. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	0	0	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.9	0.5	0.5	0.1
ν_P	0.3	0.5	0.5	0.6

Then P is not a PFIUPF (resp., PFCUPF) of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(0,2),(2,0),(1,2),(2,1)\}.$$

Then ρ is a CR on $\mathcal{U}$. But $\rho^-(P)$ and $\rho^-(P)$ are PFIUPFs (resp., PFCUPFs) of $\mathcal{U}$.

Example 2.16. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.8	0.4	0.2	0.2
ν_P	0.2	0.3	0.6	0.6

Then P is not a PFSUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(0,1),(1,0),(0,2),(2,0),(1,2),(2,1)\}.$$

Then ρ is a CR on $\mathcal{U}$. But $\rho^+(P)$ and $\rho^-(P)$ are PFSUPFs of $\mathcal{U}$.

From Examples 2.15 and 2.16, we get the results that if P is a RPFIUPF (resp., RPFCUPF, and RPFSUPF), then it may not be a PFSUPF (resp., PFCUPF, and PFSUPF).

Example 2.17. By Example 2.16, we have P is a PFIUPF of $\mathcal{U}$ and $\rho^+(P), \rho^-(P)$ are PFIUPFs of $\mathcal{U}$.

Example 2.18. Consider a UP-algebra $\mathcal{U} = (\mathcal{U}, \star, 0)$, where $\mathcal{U} = \{0,1,2,3\}$ is defined in the Cayley table below.

$\star$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	2	0

We define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.6	0.6	0.6	0.4
ν_P	0.5	0.5	0.5	0.8

Then P is a PFCUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,0),(0,1),(2,0),(0,2),(1,2),(2,1)\}.$$

Then ρ is a CR on $\mathcal{U}$. Thus $\rho^+(P)$ and $\rho^-(P)$ are PFCUPFs of $\mathcal{U}$.

Example 2.19. By Example 2.18, we define a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$ as follows:

$\mathcal{U}$	0	1	2	3
μ_P	0.8	0.4	0.4	0.2
ν_P	0.3	0.5	0.5	0.7

Then P is a PFSUPF of $\mathcal{U}$. Let

$$\rho = \{(0,0),(1,1),(2,2),(3,3),(1,0),(0,1),(2,0),(0,2),(1,2),(2,1)\}.$$

Then ρ is a CR on $\mathcal{U}$. Thus $\rho^+(P)$ and $\rho^-(P)$ are PFSUPFs of $\mathcal{U}$.

From Examples 2.17, 2.18, and 2.19, we get the results that P can be a RPFIUPF (resp., RPFCUPF, and RPFSUPF) and a PFIUPF (resp., PFCUPF, and PFSUPF) in the same time.

3. t-Level Subsets of a PFS

In this section, we shall let P be a PFS $P = (\mu_P, \nu_P)$ in $\mathcal{U}$. We shall discuss the relationships between PFIUPFs (resp., PFCUPFs, PFSUPFs, RPFIUPFs, RPFCUPFs, and RPFSUPFs addUP LOW) of UP-algebras and their t-level subsets.

Definition 3.1 ([29]). Let F be a FS with the membership function μ_F in $\mathcal{U}$. The sets

$$\begin{aligned}
U(\mu_F, t) &= \{a \in \mathcal{U} \mid \mu_F(a) \geqslant t\},\\
U^+(\mu_F, t) &= \{a \in \mathcal{U} \mid \mu_F(a) > t\},\\
L(\mu_F, t) &= \{a \in \mathcal{U} \mid \mu_F(a) \leqslant t\},\\
L^-(\mu_F, t) &= \{a \in \mathcal{U} \mid \mu_F(a) < t\},\\
E(\mu_F, t) &= \{a \in \mathcal{U} \mid \mu_F(a) = t\}
\end{aligned}$$

are referred to as an *upper* t-*level subset*, an *upper* t-*strong level subset*, a *lower* t-*level subset*, a *lower* t-*strong level subset*, and an *equal* t-*level subset* of F, respectively, for any $t \in [0, 1]$.

Theorem 3.2. P *is a PFIUPFs of* $\mathcal{U}$ *if and only if* $U(\mu_P, t)$ *and* $L(\nu_P, t)$ *are, if the sets are nonempty, IUPFs for every* $t \in [0, 1]$.

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFIUPFs of $\mathcal{U}$. Let $t \in [0, 1]$ be such that $U(\mu_P, t), L(\nu_P, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$a \in U(\mu_P, t) \Rightarrow \mu_P(a) \geqslant t \Rightarrow \mu_P(0) \geqslant \mu_P(a) \geqslant t \Rightarrow 0 \in U(\mu_P, t), \quad \text{(by (1.2))}$$

$$\begin{aligned}
a \star (b \star c), a \star b \in U(\mu_P, t) &\Rightarrow \mu_P(a \star (b \star c)) \geqslant t, \mu_P(a \star b) \geqslant t\\
&\Rightarrow \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\} \geqslant t\\
&\Rightarrow \mu_P(a \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\} \geqslant t \quad \text{(by (1.4))}\\
&\Rightarrow a \star c \in U(\mu_P, t),
\end{aligned}$$

$$a \in L(\nu_P, t) \Rightarrow \nu_P(a) \leqslant t \Rightarrow \nu_P(0) \leqslant \nu_P(a) \leqslant t \Rightarrow 0 \in L(\nu_P, t), \quad \text{(by (1.3))}$$

and

$$\begin{aligned}
a \star (b \star c), a \star b \in L(\nu_P, t) &\Rightarrow \nu_P(a \star (b \star c)) \leqslant t, \nu_P(a \star b) \leqslant t\\
&\Rightarrow \max\{\mu_P(a \star (b \star c)), \nu_P(a \star b)\} \leqslant t
\end{aligned}$$

$$\begin{aligned} &\Rightarrow \nu_P(a \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\} \leqslant t && \text{(by (1.5))} \\ &\Rightarrow a \star c \in L(\nu_P, t). \end{aligned}$$

Hence, $U(\mu_P, t)$ and $L(\nu_P, t)$ are IUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U(\mu_P, t)$ and $L(\nu_P, t)$ are IUPFs of $\mathcal{U}$ if the sets are nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_P(a) \in [0,1]$. Then $\mu_P(a) \geqslant t$. Thus $a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is an IUPF of $\mathcal{U}$ and so $0 \in U(\mu_P, t)$. Thus $\mu_P(0) \geqslant t = \mu_P(a)$.

Choose $t = \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\} \in [0,1]$. Then $\mu_P(a \star (b \star c)) \geqslant t$ and $\mu_P(a \star b) \geqslant t$. Thus $a \star (b \star c), a \star b \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is an IUPF of $\mathcal{U}$ and so $a \star c \in U(\mu_P, t)$. Thus $\mu_P(a \star c) \geqslant t = \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\}$.

Choose $t = \nu_P(a) \in [0,1]$. The $\nu_P(a) \leqslant t$. Thus $a \in L(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_P, t)$ is an IUPF of $\mathcal{U}$ and so $0 \in U(\nu_P, t)$. Thus $\nu_P(0) \leqslant t = \nu_P(a)$.

Choose $t = \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\} \in [0,1]$. Then $\nu_P(a \star (b \star c)) \leqslant t$ and $\nu_P(a \star b) \leqslant t$. Thus $a \star (b \star c), a \star b \in L(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\mu_P, t)$ is an IUPF of $\mathcal{U}$ and so $a \star c \in L(\mu_P, t)$. Thus $\nu_P(a \star c) \leqslant t = \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\}$.

Hence, P is a PFIUPF of $\mathcal{U}$. □

Theorem 3.3. *P is a PFIUPF of $\mathcal{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, IUPFs of $\mathcal{U}$ for every $t \in [0,1]$.*

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFIUPF of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U^+(\mu_P, t), L^-(\nu_P, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$a \in U^+(\mu_P, t) \Rightarrow \mu_P(a) > t \Rightarrow \mu_P(0) \geqslant \mu_P(a) > t \Rightarrow 0 \in U^+(\mu_P, t), \quad \text{(by (1.2))}$$

$$\begin{aligned} a \star (b \star c), a \star b \in U^+(\mu_P, t) &\Rightarrow \mu_P(a \star (b \star c)) > t, \mu_P(a \star b) > t \\ &\Rightarrow \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\} > t \\ &\Rightarrow \mu_P(a \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\} > t && \text{(by (1.4))} \\ &\Rightarrow a \star c \in U^+(\mu_P, t), \end{aligned}$$

$$a \in L^-(\nu_P, t) \Rightarrow \nu_P(a) < t \Rightarrow \nu_P(0) \leqslant \nu_P(a) < t \Rightarrow 0 \in L^-(\nu_P, t), \quad \text{(by (1.3))}$$

and

$$\begin{aligned} a \star (b \star c), a \star b \in L^-(\nu_P, t) &\Rightarrow \nu_P(a \star (b \star c)) < t, \nu_P(a \star b) < t \\ &\Rightarrow \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\} < t \\ &\Rightarrow \nu_P(a \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\} < t && \text{(by (1.5))} \\ &\Rightarrow a \star c \in L^-(\nu_P, t). \end{aligned}$$

Hence, $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are IUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are IUPFs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a \in \mathcal{U}$ such that $\mu_P(0) < \mu_P(a)$. Choose $t = \mu_P(0) \in [0,1]$. Then $\mu_P(a) > t$. Thus $a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is an IUPF of $\mathcal{U}$ and so $0 \in U^+(\mu_P, t)$. Thus $\mu_P(0) > t = \mu_P(0)$, a contradiction. Hence, $\mu_P(0) \geqslant \mu_P(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_P(a \star c) < \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\}$. Choose $t = \mu_P(a \star c) \in [0,1]$. Then $\mu_P(a \star (b \star c)) > t$ and $\mu_P(a \star b) > t$. Thus $a \star (b \star c), a \star b \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is an IUPF of $\mathcal{U}$ and so $a \star c \in U^+(\mu_P, t)$. Thus $\mu_P(a \star c) > t = \mu_P(a \star c)$, a contradiction. Hence, $\mu_P(a \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a \star b)\}$ for all $a, b \in \mathcal{U}$.

Suppose there exists $b \in \mathcal{U}$ such that $\nu_P(0) > \nu_P(a)$. Choose $t = \nu_P(0) \in [0,1]$. Then $\nu_P(a) < t$. Thus $a \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is an IUPF of $\mathcal{U}$ and so $0 \in L^-(\nu_P, t)$. Thus $\nu_P(0) < t = \nu_P(0)$, a contradiction. Hence, $\nu_P(0) \leqslant \nu_P(a)$ for all $a, b \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\nu_P(a \star c) > \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\}$. Choose $t = \nu_P(a \star c) \in [0,1]$. Then $\nu_P(a \star (b \star c)) < t$ and $\nu_P(a \star b) < t$. Thus $a \star (b \star c), a \star b \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is an IUPF of $\mathcal{U}$ and so $a \star c \in L^-(\nu_P, t)$. Thus $\nu_P(a \star c) < t = \nu_P(a \star c)$, a contradiction. Hence, $\nu_P(a \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a \star b)\}$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFIUPF of $\mathcal{U}$. □

Theorem 3.4. *P is a PFCUPF of $\mathcal{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, CUPFs for every $t \in [0,1]$.*

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFCUPFs of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U(\mu_P, t), L(\nu_P, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$a \in U(\mu_P, t) \Rightarrow \mu_P(a) \geqslant t \Rightarrow \mu_P(0) \geqslant \mu_P(a) \geqslant t \Rightarrow 0 \in U(\mu_P, t), \qquad \text{(by (1.2))}$$

$$\begin{aligned} a \star ((b \star c) \star b), a \in U(\mu_P, t) &\Rightarrow \mu_P(a \star ((b \star c) \star b)) \geqslant t, \mu_P(a) \geqslant t \\ &\Rightarrow \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\} \geqslant t \\ &\Rightarrow \mu_P(b) \geqslant \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\} \geqslant t \qquad \text{(by (1.6))} \\ &\Rightarrow b \in U(\mu_P, t), \end{aligned}$$

$$a \in L(\nu_P, t) \Rightarrow \nu_P(a) \leqslant t \Rightarrow \nu_P(0) \leqslant \nu_P(a) \leqslant t \Rightarrow 0 \in L(\nu_P, t), \qquad \text{(by (1.3))}$$

and

$$\begin{aligned} a \star ((b \star c) \star b), a \in L(\nu_P, t) &\Rightarrow \nu_P(a \star ((b \star c) \star b)) \leqslant t, \nu_P(a) \leqslant t \\ &\Rightarrow \max\{\mu_P(a \star ((b \star c) \star b)), \nu_P(a)\} \leqslant t \\ &\Rightarrow \nu_P(b) \leqslant \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\} \leqslant t \qquad \text{(by (1.7))} \\ &\Rightarrow b \in L(\nu_P, t). \end{aligned}$$

Hence, $U(\mu_P, t)$ and $L(\nu_P, t)$ are CUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1], U(\mu_P, t)$ and $L(\nu_P, t)$ are CUPFs of $\mathcal{U}$ if the sets are nonempty. Let $a, b \in \mathcal{U}$.

Choose $t = \mu_P(a) \in [0,1]$. Then $\mu_P(a) \geqslant t$. Thus $a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is a CUPF of $\mathcal{U}$ and so $0 \in U(\mu_P, t)$. Thus $\mu_P(0) \geqslant t = \mu_P(a)$.

Choose $t = \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\} \in [0,1]$. Then $\mu_P(a \star ((b \star c) \star b)) \geqslant t$ and $\mu_P(a) \geqslant t$. Thus $a \star ((b \star c) \star b), a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is a CUPF of $\mathcal{U}$ and so $b \in U(\mu_P, t)$. Thus $\mu_P(b) \geqslant t = \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\}$.

Choose $t = \nu_P(a) \in [0,1]$. The $\nu_P(a) \leqslant t$. Thus $a \in L(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_P, t)$ is a CUPF of $\mathcal{U}$ and so $0 \in U(\nu_P, t)$. Thus $\nu_P(0) \leqslant t = \nu_P(a)$.

Choose $t = \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\} \in [0,1]$. Then $\nu_P(a \star ((b \star c) \star b)) \leqslant t$ and $\nu_P(a) \leqslant t$. Thus $a \star ((b \star c) \star b), a \in L(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\mu_P, t)$ is a CUPF of $\mathcal{U}$ and so $b \in L(\mu_P, t)$. Thus $\nu_P(b) \leqslant t = \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\}$.

Hence, P is a PFCUPF of $\mathcal{U}$. □

Theorem 3.5. *P is a PFCUPF of $\mathcal{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, CUPFs of $\mathcal{U}$ for every $t \in [0,1]$.*

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFCUPF of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U^+(\mu_P, t), L^-(\nu_P, t) \neq \emptyset$. Let $a, b \in \mathcal{U}$. Then

$$a \in U^+(\mu_P, t) \Rightarrow \mu_P(a) > t \Rightarrow \mu_P(0) \geqslant \mu_P(a) > t \Rightarrow 0 \in U^+(\mu_P, t), \qquad \text{(by (1.2))}$$

$$\begin{aligned} a \star ((b \star c) \star b), a \in U^+(\mu_P, t) &\Rightarrow \mu_P(a \star ((b \star c) \star b)) > t, \mu_P(a) > t \\ &\Rightarrow \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\} > t \\ &\Rightarrow \mu_P(b) \geqslant \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\} > t \qquad \text{(by (1.6))} \\ &\Rightarrow b \in U^+(\mu_P, t), \end{aligned}$$

$$a \in L^-(\nu_P, t) \Rightarrow \nu_P(a) < t \Rightarrow \nu_P(0) \leqslant \nu_P(a) < t \Rightarrow 0 \in L^-(\nu_P, t), \qquad \text{(by (1.3))}$$

and

$$\begin{aligned} a \star ((b \star c) \star b), a \in L^-(\nu_P, t) &\Rightarrow \nu_P(a \star ((b \star c) \star b)) < t, \nu_P(a) < t \\ &\Rightarrow \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\} < t \\ &\Rightarrow \nu_P(b) \leqslant \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\} < t \qquad \text{(by (1.7))} \\ &\Rightarrow b \in L^-(\nu_P, t). \end{aligned}$$

Hence, $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are CUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0,1]$, $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are CUPFs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a \in \mathcal{U}$ such that $\mu_P(0) < \mu_P(a)$. Choose $t = \mu_P(0) \in [0,1]$. Then $\mu_P(a) > t$. Thus $a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is a CUPF of $\mathcal{U}$ and so $0 \in U^+(\mu_P, t)$. Thus $\mu_P(0) > t = \mu_P(0)$, a contradiction. Hence, $\mu_P(0) \geqslant \mu_P(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\mu_P(b) < \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\}$. Choose $t = \mu_P(b) \in [0,1]$. Then $\mu_P(a \star ((b \star c) \star b)) > t$ and $\mu_P(a) > t$. Thus $a \star ((b \star c) \star b), a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is a CUPF of $\mathcal{U}$ and so $b \in U^+(\mu_P, t)$. Thus $\mu_P(b) > t = \mu_P(b)$, a contradiction. Hence, $\mu_P(b) \geqslant \min\{\mu_P(a \star ((b \star c) \star b)), \mu_P(a)\}$ for all $a, b \in \mathcal{U}$.

Suppose there exists $b \in \mathcal{U}$ such that $\nu_P(0) > \nu_P(a)$. Choose $t = \nu_P(0) \in [0,1]$. Then $\nu_P(a) < t$. Thus $a \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is a CUPF of $\mathcal{U}$ and so $0 \in L^-(\nu_P, t)$. Thus $\nu_P(0) < t = \nu_P(0)$, a contradiction. Hence, $\nu_P(0) \leqslant \nu_P(a)$ for all $a, b \in \mathcal{U}$.

Suppose there exist $a, b \in \mathcal{U}$ such that $\nu_P(b) > \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\}$. Choose $t = \nu_P(b) \in [0,1]$. Then $\nu_P(a \star ((b \star c) \star b)) < t$ and $\nu_P(a) < t$. Thus $a \star ((b \star c) \star b), a \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is a CUPF of $\mathcal{U}$ and so $b \in L^-(\nu_P, t)$. Thus $\nu_P(b) < t = \nu_P(b)$, a contradiction. Hence, $\nu_P(b) \leqslant \max\{\nu_P(a \star ((b \star c) \star b)), \nu_P(a)\}$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFCUPF of $\mathcal{U}$. □

Theorem 3.6. *P is a PFSUPF of $\mathcal{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, SUPFs for every $t \in [0,1]$.*

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFSUPF of $\mathcal{U}$. Let $t \in [0,1]$ be such that $U(\mu_P, t), L(\nu_P, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$a \in U(\mu_P, t) \Rightarrow \mu_P(a) \geqslant t \Rightarrow \mu_P(0) \geqslant \mu_P(a) \geqslant t \Rightarrow 0 \in U(\mu_P, t), \qquad \text{(by (1.2))}$$

$$\begin{aligned} a \star (b \star c), a \in U(\mu_P, t) &\Rightarrow \mu_P(a \star (b \star c)) \geqslant t, \mu_P(a) \geqslant t \\ &\Rightarrow \min\{\mu_P(a \star (b \star c)), \mu_P(a)\} \geqslant t \\ &\Rightarrow \mu_P(((c \star b) \star b) \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a)\} \geqslant t \qquad \text{(by (1.8))} \\ &\Rightarrow ((c \star b) \star b) \star c \in U(\mu_P, t), \end{aligned}$$

$$a \in L(\nu_P, t) \Rightarrow \nu_P(a) \leqslant t \Rightarrow \nu_P(0) \leqslant \nu_P(a) \leqslant t \Rightarrow 0 \in L(\nu_P, t), \quad \text{(by (1.3))}$$

and

$$\begin{aligned} a \star (b \star c), a \in L(\nu_P, t) &\Rightarrow \nu_P(a \star (b \star c)) \leqslant t, \nu_P(a) \leqslant t \\ &\Rightarrow \max\{\mu_P(a \star (b \star c)), \nu_P(a)\} \leqslant t \\ &\Rightarrow \nu_P(((c \star b) \star b) \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a)\} \leqslant t \quad \text{(by (1.9))} \\ &\Rightarrow ((c \star b) \star b) \star c \in L(\nu_P, t). \end{aligned}$$

Hence, $U(\mu_P, t)$ and $L(\nu_P, t)$ are SUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0, 1], U(\mu_P, t)$ and $L(\nu_P, t)$ are SUPFs of $\mathcal{U}$ if the sets are nonempty. Let $a, b, c \in \mathcal{U}$.

Choose $t = \mu_P(a) \in [0, 1]$. Then $\mu_P(a) \geqslant t$. Thus $a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is a SUPF of $\mathcal{U}$ and so $0 \in U(\mu_P, t)$. Thus $\mu_P(0) \geqslant t = \mu_P(a)$.

Choose $t = \min\{\mu_P(a \star (b \star c)), \mu_P(a)\} \in [0, 1]$. Then $\mu_P(a \star (b \star c)) \geqslant t$ and $\mu_P(a) \geqslant t$. Thus $a \star (b \star c), a \in U(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U(\mu_P, t)$ is a SUPF of $\mathcal{U}$ and so $((c \star b) \star b) \star c \in U(\mu_P, t)$. Thus $\mu_P(((c \star b) \star b) \star c) \geqslant t = \min\{\mu_P(a \star (b \star c)), \mu_P(a)\}$.

Choose $t = \nu_P(a) \in [0, 1]$. The $\nu_P(a) \leqslant t$. Thus $a \in L(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\nu_P, t)$ is a SUPF of $\mathcal{U}$ and so $0 \in U(\nu_P, t)$. Thus $\nu_P(0) \leqslant t = \nu_P(a)$.

Choose $t = \max\{\nu_P(a \star (b \star c)), \nu_P(a)\} \in [0, 1]$. Then $\nu_P(a \star (b \star c)) \leqslant t$ and $\nu_P(a) \leqslant t$. Thus $a \star (b \star c), a \in L(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $L(\mu_P, t)$ is a SUPF of $\mathcal{U}$ and so $((c \star b) \star b) \star c \in L(\mu_P, t)$. Thus $\nu_P(((c \star b) \star b) \star c) \geqslant t = \max\{\nu_P(a \star (b \star c)), \nu_P(a)\}$.

Hence, P is a PFSUPF of $\mathcal{U}$. □

Theorem 3.7. *P is a PFSUPF of $\mathcal{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, SUPFs of $\mathcal{U}$ for every $t \in [0, 1]$.*

Proof. Assume $P = (\mu_P, \nu_P)$ is a PFSUPF of $\mathcal{U}$. Let $t \in [0, 1]$ be such that $U^+(\mu_P, t), L^-(\nu_P, t) \neq \emptyset$. Let $a, b, c \in \mathcal{U}$. Then

$$a \in U^+(\mu_P, t) \Rightarrow \mu_P(a) > t \Rightarrow \mu_P(0) \geqslant \mu_P(a) > t \Rightarrow 0 \in U^+(\mu_P, t), \quad \text{(by (1.2))}$$

$$\begin{aligned} a \star (b \star c), a \in U^+(\mu_P, t) &\Rightarrow \mu_P(a \star (b \star c)) > t, \mu_P(a) > t \\ &\Rightarrow \min\{\mu_P(a \star (b \star c)), \mu_P(a)\} > t \\ &\Rightarrow \mu_P(((c \star b) \star b) \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a)\} > t \quad \text{(by (1.8))} \\ &\Rightarrow ((c \star b) \star b) \star c \in U^+(\mu_P, t), \end{aligned}$$

$$a \in L^-(\nu_P, t) \Rightarrow \nu_P(a) < t \Rightarrow \nu_P(0) \leqslant \nu_P(a) < t \Rightarrow 0 \in L^-(\nu_P, t), \quad \text{(by (1.3))}$$

and

$$\begin{aligned} a \star (b \star c), a \in L^-(\nu_P, t) &\Rightarrow \nu_P(a \star (b \star c)) < t, \nu_P(a) < t \\ &\Rightarrow \max\{\nu_P(a \star (b \star c)), \nu_P(a)\} < t \\ &\Rightarrow \nu_P(((c \star b) \star b) \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a)\} < t \quad \text{(by (1.9))} \\ &\Rightarrow ((c \star b) \star b) \star c \in L^-(\nu_P, t). \end{aligned}$$

Hence, $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are SUPFs of $\mathcal{U}$.

Conversely, assume for all $t \in [0, 1], U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are SUPFs of $\mathcal{U}$ if the sets are nonempty.

Suppose there exists $a \in \mathcal{U}$ such that $\mu_P(0) < \mu_P(a)$. Choose $t = \mu_P(0) \in [0,1]$. Then $\mu_P(a) > t$. Thus $a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is a SUPF of $\mathcal{U}$ and so $0 \in U^+(\mu_P, t)$. Thus $\mu_P(0) > t = \mu_P(0)$, a contradiction. Hence, $\mu_P(0) \geqslant \mu_P(a)$ for all $a \in \mathcal{U}$.

Suppose there exist $a, b, c \in \mathcal{U}$ such that $\mu_P(((c \star b) \star b) \star c) < \min\{\mu_P(a \star (b \star c)), \mu_P(a)\}$. Choose $t = \mu_P(((c \star b) \star b) \star c) \in [0,1]$. Then $\mu_P(a \star (b \star c)) > t$ and $\mu_P(a) > t$. Thus $a \star (b \star c), a \in U^+(\mu_P, t) \neq \emptyset$. As a hypothesis, we get $U^+(\mu_P, t)$ is a SUPF of $\mathcal{U}$ and so $((c \star b) \star b) \star c \in U^+(\mu_P, t)$. Thus $\mu_P(((c \star b) \star b) \star c) > t = \mu_P(((c \star b) \star b) \star c)$, a contradiction. Hence, $\mu_P(((c \star b) \star b) \star c) \geqslant \min\{\mu_P(a \star (b \star c)), \mu_P(a)\}$ for all $a, b \in \mathcal{U}$.

Suppose there exists $b \in \mathcal{U}$ such that $\nu_P(0) > \nu_P(a)$. Choose $t = \nu_P(0) \in [0,1]$. Then $\nu_P(a) < t$. Thus $a \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is a SUPF of $\mathcal{U}$ and so $0 \in L^-(\nu_P, t)$. Thus $\nu_P(0) < t = \nu_P(0)$, a contradiction. Hence, $\nu_P(0) \leqslant \nu_P(a)$ for all $a, b \in \mathcal{U}$.

Suppose there exist $a, b, c \in \mathcal{U}$ such that $\nu_P(((c \star b) \star b) \star c) > \max\{\nu_P(a \star (b \star c)), \nu_P(a)\}$. Choose $t = \nu_P(((c \star b) \star b) \star c) \in [0,1]$. Then $\nu_P(a \star (b \star c)) < t$ and $\nu_P(a) < t$. Thus $a \star (b \star c), a \in L^-(\nu_P, t) \neq \emptyset$. As a hypothesis, we get $L^-(\nu_P, t)$ is a SUPF of $\mathcal{U}$ and so $((c \star b) \star b) \star c \in L^-(\nu_P, t)$. Thus $\nu_P(((c \star b) \star b) \star c) < t = \nu_P(((c \star b) \star b) \star c)$, a contradiction. Hence, $\nu_P(((c \star b) \star b) \star c) \leqslant \max\{\nu_P(a \star (b \star c)), \nu_P(a)\}$ for all $a, b \in \mathcal{U}$.

Therefore, P is a PFSUPF of $\mathcal{U}$. □

Lemma 3.8 ([24]). *Let ρ be a CR on $\mathcal{U}$ and $t \in [0,1]$. Then the following statements hold:*

(1) $U(\overline{\mu}_P, t) = \rho^-(U(\mu_P, t))$;

(2) $U^+(\overline{\mu}_P, t) = \rho^-(U^+(\mu_P, t))$;

(3) $L(\overline{\nu}_P, t) = \rho^+(L(\nu_P, t))$;

(4) $L^-(\overline{\nu}_P, t) = \rho^+(L^-(\nu_P, t))$;

(5) $U(\underline{\mu}_P, t) = \rho^+(U(\mu_P, t))$;

(6) $U^+(\underline{\mu}_P, t) = \rho^+(U^+(\mu_P, t))$;

(7) $L(\underline{\nu}_P, t) = \rho^-(L(\nu_P, t))$; *and*

(8) $L^-(\underline{\nu}_P, t) = \rho^-(L^-(\nu_P, t))$.

By Lemma 3.8, we shall show the relationships between RPFSs and their t-level subsets.

Theorem 3.9. *Let ρ be a CR on $\mathcal{U}$. Then P is an UpRPFIUPF of $\mathcal{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRIUPF and a LoRIUPF of $\mathcal{U}$ for every $t \in [0,1]$, respectively.*

Proof. It is straightforward by Theorem 3.2 and Lemma 3.8 (1) and (3). □

Theorem 3.10. *Let ρ be a CR on $\mathcal{U}$. Then P is an UpRPFIUPF of $\mathcal{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRIUPF and a LoRIUPF of $\mathcal{U}$ for every $t \in [0,1]$, respectively.*

Proof. It is straightforward by Theorem 3.3 and Lemma 3.8 (2) and (4). □

Theorem 3.11. *Let ρ be a CR on $\mathcal{U}$. Then P is an UpRPFCUPF of $\mathcal{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRCUPF and a LoRCUPF of $\mathcal{U}$ for every $t \in [0,1]$, respectively.*

Proof. It is straightforward by Theorem 3.4 and Lemma 3.8 (1) and (3). □

Theorem 3.12. *Let ρ be a CR on $\mathcal{U}$. Then P is an UpRPFCUPF of $\mathcal{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRCUPF and a LoRCUPF of $\mathcal{U}$ for every $t \in [0,1]$, respectively.*

Proof. It is straightforward by Theorem 3.5 and Lemma 3.8 (2) and (4). □

Theorem 3.13. *Let ρ be a CR on $\mathfrak{U}$. Then P is an UpRPFSUPF of $\mathfrak{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRSUPF and a LoRSUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.6 and Lemma 3.8 (1) and (3). □

Theorem 3.14. *Let ρ be a CR on $\mathfrak{U}$. Then P is an UpRPFSUPF of $\mathfrak{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRSUPF and a LoRSUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.7 and Lemma 3.8 (2) and (4). □

Theorem 3.15. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFIUPF of $\mathfrak{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRIUPF and a LoRIUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.2 and Lemma 3.8 (5) and (7). □

Theorem 3.16. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFIUPF of $\mathfrak{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRIUPF and a LoRIUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.3 and Lemma 3.8 (6) and (8). □

Theorem 3.17. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFCUPF of $\mathfrak{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRCUPF and a LoRCUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.4 and Lemma 3.8 (5) and (7). □

Theorem 3.18. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFCUPF of $\mathfrak{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRCUPF and a LoRCUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.5 and Lemma 3.8 (6) and (8). □

Theorem 3.19. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFSUPF of $\mathfrak{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, an UpRSUPF and a LoRSUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.6 and Lemma 3.8 (5) and (7). □

Theorem 3.20. *Let ρ be a CR on $\mathfrak{U}$. Then P is a LoRPFSUPF of $\mathfrak{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, an UpRSUPF and a LoRSUPF of $\mathfrak{U}$ for every $t \in [0, 1]$, respectively.*

Proof. It is straightforward by Theorem 3.7 and Lemma 3.8 (6) and (8). □

Theorem 3.21. *Let ρ be a CR on $\mathfrak{U}$. Then P is a RPFIUPF (resp., RPFCUPF, and RPFSUPF) of $\mathfrak{U}$ if and only if $U(\mu_P, t)$ and $L(\nu_P, t)$ are, if the sets are nonempty, RIUPFs (resp., RCUPFs, and RSUPFs) of $\mathfrak{U}$ for every $t \in [0, 1]$.*

Proof. It is straightforward by Theorems 3.9 (resp., Theorems 3.11 and 3.13) and 3.15 (resp., Theorems 3.17 and 3.19). □

Theorem 3.22. *Let ρ be a CR on $\mathfrak{U}$. Then P is a RPFIUPF (resp., RPFCUPF, and RPFSUPF) of $\mathfrak{U}$ if and only if $U^+(\mu_P, t)$ and $L^-(\nu_P, t)$ are, if the sets are nonempty, RIUPFs (resp., RCUPFs, and RSUPFs) of $\mathfrak{U}$ for every $t \in [0, 1]$.*

Proof. It is straightforward by Theorems 3.10 (resp., Theorems 3.12 and 3.14) and 3.16 (resp., Theorems 3.18 and 3.20). □

4. Conclusions and Future Work

In this paper, we have introduced nine types of RPFSs in UP-algebras, namely, UpRPFIUPFs, UpRPFCUPFs, UpRPFSUPFs, LoRPFIUPFs, LoRPFCUPFs, LoRPFSUPFs, RPFIUPFs, RPFCUPFs, and RPFSUPFs. Then we get UpRPFIUPFs (resp., UpRPFCUPFs, and UpRPFSUPFs) and LoRPFIUPFs (resp., LoRPFCUPFs, and LoRPFSUPFs) are generalizations of RPFIUPFs (resp., RPFCUPFs, and RPFSUPFs). In addition, we investigated t-level subsets of RPFSs in UP-algebras in order to discuss the relationships between RPFSs and RSs in UP-algebras under a congruence relation.

The following are some essential subjects for our future research of UP-algebras: applying the soft set theory to PFSs and studying Fermatean fuzzy set in UP-algebra.

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