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Expressions and dynamical behavior of solutions of eighteenth-order of a class of rational difference equations

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Abstract

The aim of this work is to obtain the forms of the solutions of the following nonlinear eighteenth-order difference equations

$$x_{n+1} = \frac{x_{n-17}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions $x_{-17}, x_{-16}, \ldots, x_0$ are arbitrary real numbers. Moreover, we investigate stability, boundedness, oscillation, and the periodic character of these solutions. Finally, we confirm the results with some numerical examples and graphs by using Matlab program.

Keywords: Recursive sequence, oscillation, semicycles, stability, periodicity, solutions of difference equations.

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1. Introduction

The study of difference equations is a very rich research field, and difference equations have been applied in several mathematical models in biology, economics, genetics, population dynamics, medicine, and so forth. Solving difference equations and studying the asymptotic behavior of their solutions has attracted the attention of many authors, see for results in this area, for example,

Ahmed et al. [4] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-14}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions are arbitrary real numbers.

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Elsayed et al. [14] obtained the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-11}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions are arbitrary real numbers.

Elsayed [13] studied the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-2}x_{n-5}}.$$

For other related papers, see [1–3, 5–12, 15–18].

In this paper, we obtain the solutions of the following nonlinear difference equations

$$x_{n+1} = \frac{x_{n-17}}{\pm 1 \pm x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \ldots,$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 17$.

Moreover, we investigate stability, boundedness, oscillation and the periodic character of these solutions. Finally, we confirm the results with some numerical examples and graphs by using Matlab program.

Throughout this paper, we define $\bmod(k, 3) = k - 3\left[\frac{k}{3}\right]$, where $[x]$ be the greatest integer less than or equal to the real number x.

2. The difference equation $x_{n+1} = \frac{x_{n-17}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$

In this section, we give a specific form of the solutions of the first equation in the form

$$x_{n+1} = \frac{x_{n-17}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 17$. Also, we investigate the stability and boundedness of these solutions.

Theorem 2.1. *Let $\{x_n\}_{n=-17}^{\infty}$ be a solution of the difference (2.1). Then for $n = 0, 1, 2, \ldots$,*

$$x_{18n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{1 + (6i + M_k - 1) P_k}{1 + (6i + M_k) P_k} \right), \tag{2.2}$$

where $P_k = \prod_{j=0}^{5} a_{\bmod(k,3)+3j}$, $M_k = 6 - \left[\frac{k}{3}\right]$ and $x_{-k} = a_k$, with $rP_k \neq -1$ such that $r \in \{1, 2, 3, \ldots\}$, $k = 0, 1, 2, \ldots, 17$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{18n-18-k} = a_k \prod_{i=0}^{n-2} \left(\frac{1 + (6i + M_k - 1) P_k}{1 + (6i + M_k) P_k} \right). \tag{2.3}$$

Now, it follows from (2.1) and using (2.3) that

$$x_{18n-17} = \frac{x_{18n-35}}{1 + x_{18n-20}x_{18n-23}x_{18n-26}x_{18n-29}x_{18n-32}x_{18n-35}}$$

$$= \frac{a_{17} \prod_{i=0}^{n-2} \left(\frac{1+(6i+M_{17}-1)P_{17}}{1+(6i+M_{17})P_{17}} \right)}{1 + \prod_{j=0}^{5} \left(a_{3j+2} \prod_{i=0}^{n-2} \left(\frac{1+\left(6i+M_{3j+2}-1\right)P_{3j+2}}{1+\left(6i+M_{3j+2}\right)P_{3j+2}} \right) \right)}$$

$$= \frac{a_{17} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_2a_5a_8a_{11}a_{14}a_{17}}{1+(6i+1)a_2a_5a_8a_{11}a_{14}a_{14}a_{17}} \right)}{1 + a_2a_5a_8a_{11}a_{14}a_{17} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_2a_5a_8a_{11}a_{14}a_{14}}{1+(6i+6)a_2a_5a_8a_{11}a_{14}a_{14}a_{17}} \right)}.$$

Hence, we have

$$x_{18n-17} = a_{17} \prod_{i=0}^{n-1} \left(\frac{1+(6i)a_2a_5a_8a_{11}a_{14}a_{17}}{1+(6i+1)\, a_2a_5a_8a_{11}a_{14}a_{17}} \right).$$

Also, it follows from (2.1) and using (2.3) that

$$x_{18n-16} = \frac{x_{18n-34}}{1 + x_{18n-19}x_{18n-22}x_{18n-25}x_{18n-28}x_{18n-31}x_{18n-34}}$$

$$= \frac{a_{16} \prod_{i=0}^{n-2} \left(\frac{1+(6i+M_{16}-1)P_{16}}{1+(6i+M_{16})P_{16}} \right)}{1 + \prod_{j=0}^{5} \left(a_{3j+1} \prod_{i=0}^{n-2} \left(\frac{1+\left(6i+M_{3j+1}-1\right)P_{3j+1}}{1+\left(6i+M_{3j+1}\right)P_{3j+1}} \right) \right)}$$

$$= \frac{a_{16} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_1a_4a_7a_{10}a_{13}a_{16}}{1+(6i+1)a_1a_4a_7a_{10}a_{13}a_{16}} \right)}{1 + a_1a_4a_7a_{10}a_{13}a_{16} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_1a_4a_7a_{10}a_{13}a_{16}}{1+(6i+6)a_1a_4a_7a_{10}a_{13}a_{16}} \right)}.$$

Hence, we have

$$x_{18n-16} = a_{16} \prod_{i=0}^{n-1} \left(\frac{1+(6i)a_1a_4a_7a_{10}a_{13}a_{16}}{1+(6i+1)\, a_1a_4a_7a_{10}a_{13}a_{16}} \right).$$

Also, it follows from (2.1) and using (2.3) that

$$x_{18n-15} = \frac{x_{18n-33}}{1 + x_{18n-18}x_{18n-21}x_{18n-24}x_{18n-27}x_{18n-30}x_{18n-33}}$$

$$= \frac{a_{15} \prod_{i=0}^{n-2} \left(\frac{1+(6i+M_{15}-1)P_{15}}{1+(6i+M_{15})P_{15}} \right)}{1 + \prod_{j=0}^{5} \left(a_{3j} \prod_{i=0}^{n-2} \left(\frac{1+\left(6i+M_{3j}-1\right)P_{3j}}{1+\left(6i+M_{3j}\right)P_{3j}} \right) \right)}$$

$$= \frac{a_{15} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{1+(6i+1)a_0a_3a_6a_9a_{12}a_{15}} \right)}{1 + a_0a_3a_6a_9a_{12}a_{15} \prod_{i=0}^{n-2} \left(\frac{1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{1+(6i+6)a_0a_3a_6a_9a_{12}a_{15}} \right)}.$$

Hence, we have

$$x_{18n-15} = a_{15} \prod_{i=0}^{n-1} \left(\frac{1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{1+(6i+1)\, a_0a_3a_6a_9a_{12}a_{15}} \right).$$

Similarly, one can easily obtain the other relations for (2.2). Hence, the proof is completed. □

Theorem 2.2. *Assume that the initial values of the difference equation* (2.1), $x_{-17}, x_{-16}, \dots, x_0 \in [0, \infty)$, *then every solution of* (2.1) *is bounded.*

Proof. Let $\{x_n\}_{n=-17}^{\infty}$ be a solution of (2.1). It follows from (2.1) that

$$0 \leqslant x_{n+1} = \frac{x_{n-17}}{1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}} \leqslant x_{n-17} \quad \text{for all} \quad n \geqslant 0.$$

Then the sequence $\{x_{18n-i}\}_{n=0}^{\infty}$, $i = 0, 1, \dots, 17$ is decreasing and so is bounded from above by $M = \max\{x_{-17}, x_{-16}, \dots, x_0\}$. □

Theorem 2.3. *The only equilibrium point* $\bar{x}$ *of* (2.1) *is* $\bar{x} = 0$.

Proof. From (2.1), we can write

$$\bar{x} = \frac{\bar{x}}{1 + \bar{x}^6}.$$

Then we have

$$\bar{x} + \bar{x}^7 = \bar{x},$$

or,

$$\bar{x}^7 = 0.$$

Thus, the only equilibrium point of (2.1) is $\bar{x} = 0$. □

Theorem 2.4. *Assume that the initial values of the difference equation* (2.1), $x_{-17}, x_{-16}, \dots, x_0 \in [0, \infty)$, *then the equilibrium point* $\bar{x} = 0$ *of* (2.1) *is locally stable.*

Proof. Let $\epsilon > 0$, and let $\{x_n\}_{n=-17}^{\infty}$ be a solution of (2.1) such that

$$\sum_{j=0}^{17} |x_{-j}| < \epsilon.$$

It suffices to show that $|x_1| < \epsilon$. Now

$$0 < x_1 = \frac{x_{-17}}{1 + x_{-2}x_{-5}x_{-8}x_{-11}x_{-14}x_{-17}} \leqslant x_{-17} < \epsilon,$$

and so the proof is completed. □

Theorem 2.5. *Assume that the initial values of the difference equation* (2.1), $x_{-17}, x_{-16}, \dots, x_0 \in [0, \infty)$, *then the equilibrium point* $\bar{x} = 0$ *of* (2.1) *is globally asymptotically stable.*

Proof. We know by Theorem 2.4 that the equilibrium point $\bar{x} = 0$ of (2.1) is locally stable. So let $\{x_n\}_{n=-17}^{\infty}$ be a positive solution of (2.1). It suffices to show that $\lim_{n\to\infty} x_n = \bar{x} = 0$. From Theorem 2.2 we have $x_{n+1} < x_{n-17}$ for all $n \geqslant 0$, so the sequences $\{x_{18n-i}\}_{n=0}^{\infty}$, $i = 0, 1, \dots, 17$ are decreasing and bounded which implies that the sequences $\{x_{18n-i}\}_{n=0}^{\infty}, i = 0, 1, \dots, 17$ converge to limit (say $L_i > 0$). So

$$L_{17} = \frac{L_{17}}{1 + L_2L_5L_8L_{11}L_{14}L_{17}} = 0, L_{16} = \frac{L_{16}}{1 + L_1L_4L_7L_{10}L_{13}L_{16}} = 0, \dots, L_0 = \frac{L_0}{1 + L_0L_3L_6L_9L_{12}L_{15}} = 0,$$

which implies that $L_0 = L_1 = \cdots = L_{17} = 0$, from which the result follows. □

3. The difference equation $x_{n+1} = \dfrac{x_{n-17}}{1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$

In this section we give a specific form of the solutions of the second equation in the form

$$x_{n+1} = \frac{x_{n-17}}{1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \dots, \tag{3.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \dots, 17$.

Theorem 3.1. *Let $\{x_n\}_{n=-17}^{\infty}$ be a solution of the difference equation (3.1). Then for $n = 0, 1, 2, \ldots,$*

$$x_{18n-k} = a_k \prod_{i=0}^{n-1} \left(\frac{-1 + (6i + M_k - 1) P_k}{-1 + (6i + M_k) P_k} \right), \tag{3.2}$$

where $P_k = \prod_{j=0}^{5} a_{\mathrm{mod}(k,3)+3j}$, $M_k = 6 - \left[\frac{k}{3}\right]$ and $x_{-k} = a_k$, with $rP_k \neq 1$ such that $r \in \{1, 2, 3, \ldots\}$, $k = 0, 1, 2, \ldots, 17$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{18n-18-k} = a_k \prod_{i=0}^{n-2} \left(\frac{-1 + (6i + M_k - 1) P_k}{-1 + (6i + M_k) P_k} \right). \tag{3.3}$$

Now, it follows from (3.1) and using (3.3) that

$$\begin{aligned}
x_{18n-17} &= \frac{x_{18n-35}}{1 - x_{18n-20}x_{18n-23}x_{18n-26}x_{18n-29}x_{18n-32}x_{18n-35}} \\
&= \frac{a_{17} \prod_{i=0}^{n-2} \left(\frac{-1+(6i+M_{17}-1)P_{17}}{-1+(6i+M_{17})P_{17}} \right)}{1 - \prod_{j=0}^{5} \left(a_{3j+2} \prod_{i=0}^{n-2} \left(\frac{-1+\left(6i+M_{3j+2}-1\right)P_{3j+2}}{-1+\left(6i+M_{3j+2}\right)P_{3j+2}} \right) \right)} \\
&= \frac{a_{17} \prod_{i=0}^{n-2} \left(\frac{-1+(6i)a_2a_5a_8a_{11}a_{14}a_{17}}{-1+(6i+1)a_2a_5a_8a_{11}a_{14}a_{17}} \right)}{1 - a_2a_5a_8a_{11}a_{14}a_{17} \prod_{i=0}^{n-2} \left(\frac{-1+(6i)a_2a_5a_8a_{11}a_{14}a_{17}}{-1+(6i+6)a_2a_5a_8a_{11}a_{14}a_{17}} \right)}.
\end{aligned}$$

Hence, we have

$$x_{18n-17} = a_{17} \prod_{i=0}^{n-1} \left(\frac{-1 + (6i)a_2a_5a_8a_{11}a_{14}a_{17}}{-1 + (6i+1)\, a_2a_5a_8a_{11}a_{14}a_{17}} \right).$$

Also, it follows from (3.1) and using (3.3) that

$$\begin{aligned}
x_{18n-16} &= \frac{x_{18n-34}}{1 - x_{18n-19}x_{18n-22}x_{18n-25}x_{18n-28}x_{18n-31}x_{18n-34}} \\
&= \frac{a_{16} \prod_{i=0}^{n-2} \left(\frac{-1+(6i+M_{16}-1)P_{16}}{-1+(6i+M_{16})P_{16}} \right)}{1 - \prod_{j=0}^{5} \left(a_{3j+1} \prod_{i=0}^{n-2} \left(\frac{-1+\left(6i+M_{3j+1}-1\right)P_{3j+1}}{-1+\left(6i+M_{3j+1}\right)P_{3j+1}} \right) \right)} \\
&= \frac{a_{16} \prod_{i=0}^{n-2} \left(\frac{-1+(6i)a_1a_4a_7a_{10}a_{13}a_{16}}{-1+(6i+1)a_1a_4a_7a_{10}a_{13}a_{16}} \right)}{1 - a_1a_4a_7a_{10}a_{13}a_{16} \prod_{i=0}^{n-2} \left(\frac{-1+(6i)a_1a_4a_7a_{10}a_{13}a_{16}}{-1+(6i+6)a_1a_4a_7a_{10}a_{13}a_{16}} \right)}.
\end{aligned}$$

Hence, we have

$$x_{18n-16} = a_{16} \prod_{i=0}^{n-1} \left(\frac{-1 + (6i)a_1a_4a_7a_{10}a_{13}a_{16}}{-1 + (6i+1)\, a_1a_4a_7a_{10}a_{13}a_{16}} \right).$$

Also, it follows from (3.1) and using (3.3) that

$$x_{18n-15} = \frac{x_{18n-33}}{1 - x_{18n-18}x_{18n-21}x_{18n-24}x_{18n-27}x_{18n-30}x_{18n-33}}$$

$$= \frac{a_{15}\prod_{i=0}^{n-2}\left(\frac{-1+(6i+M_{15}-1)P_{15}}{-1+(6i+M_{15})P_{15}}\right)}{1-\prod_{j=0}^{5}\left(a_{3j}\prod_{i=0}^{n-2}\left(\frac{-1+\left(6i+M_{3j}-1\right)P_{3j}}{-1+\left(6i+M_{3j}\right)P_{3j}}\right)\right)}$$

$$= \frac{a_{15}\prod_{i=0}^{n-2}\left(\frac{-1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{-1+(6i+1)a_0a_3a_6a_9a_{12}a_{15}}\right)}{1-a_0a_3a_6a_9a_{12}a_{15}\prod_{i=0}^{n-2}\left(\frac{-1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{-1+(6i+6)a_0a_3a_6a_9a_{12}a_{15}}\right)}.$$

Hence, we have

$$x_{18n-15} = a_{15}\prod_{i=0}^{n-1}\left(\frac{-1+(6i)a_0a_3a_6a_9a_{12}a_{15}}{-1+(6i+1)\,a_0a_3a_6a_9a_{12}a_{15}}\right).$$

Similarly, one can easily obtain the other relations for (3.2). Hence, the proof is completed. □

Theorem 3.2. (3.1) *has a unique equilibrium point* $\bar{x} = 0$, *which is a non hyperbolic equilibrium point.*

Proof. From (3.1), we can write

$$\bar{x} = \frac{\bar{x}}{1-\bar{x}^6}.$$

Then we have

$$\bar{x} - \bar{x}^7 = \bar{x},$$

or,

$$\bar{x}^7 = 0.$$

Thus the only equilibrium point of (3.1) is $\bar{x} = 0$.

Define the function $f\,(x,y,z,u,v,t) = \frac{x}{1-xyzuvt}$ on I^6 where I is a subset of R such that $0 \in I$ and $f(I^6) \subseteq I$. Clearly, f is continuously differentiable on I^6 and we have

$$f\,(x,y,z,u,v,t) = \frac{x}{1-xyzuvt}.$$

Then we have

$$f_x(x,y,z,u,v,t) = \frac{1}{(1-xyzuvt)^2}, \qquad f_y(x,y,z,u,v,t) = \frac{x^2zuvt}{(1-xyzuvt)^2},$$
$$f_z(x,y,z,u,v,t) = \frac{x^2yuvt}{(1-xyzuvt)^2}, \qquad f_u(x,y,z,u,v,t) = \frac{x^2yzvt}{(1-xyzuvt)^2},$$
$$f_v(x,y,z,u,v,t) = \frac{x^2yzut}{(1-xyzuvt)^2}, \qquad f_t(x,y,z,u,v,t) = \frac{x^2yzuv}{(1-xyzuvt)^2},$$

which implies that

$$f_x(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 1,$$
$$f_y(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = f_z(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = f_u(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = f_v(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = f_t(\bar{x},\bar{x},\bar{x},\bar{x},\bar{x},\bar{x}) = 0.$$

So, the linearized equation of (3.1) about the equilibrium point $\bar{x} = 0$ is

$$z_{n+1} = z_{n-17}, \tag{3.4}$$

and the characteristic equation of (3.4) about the equilibrium point $\bar{x} = 0$ is

$$\lambda^{18} - 1 = 0,$$

which implies that

$$|\lambda_i| = 1\,;i = 1,2,\ldots,18,$$

so, $\bar{x}$ is a non hyperbolic equilibrium point. □

4. The difference equation $x_{n+1} = \dfrac{x_{n-17}}{-1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$

In this section we give a specific form of the solutions of the third equation in the form

$$x_{n+1} = \frac{x_{n-17}}{-1 + x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \ldots, \tag{4.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 17$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 4.1. *Let $\{x_n\}_{n=-17}^{\infty}$ be a solution of the difference equation (4.1). Then for $n = 0, 1, 2, \ldots$,*

$$x_{18n-k} = \frac{a_k}{(-1 + P_k)^{nq_k}}, \tag{4.2}$$

where $P_k = \prod_{j=0}^{5} a_{\mathrm{mod}(k,3)+3j}$, $q_k = (-1)^{\left[\frac{k}{3}\right]+1}$, and $x_{-k} = a_k$, with $P_k \neq 1$, $k = 0, 1, 2, \ldots, 17$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{18n-18-k} = \frac{a_k}{(-1 + P_k)^{(n-1)q_k}}, \tag{4.3}$$

Now, it follows from (4.1) and using (4.3) that

$$\begin{aligned}
x_{18n-17} &= \frac{x_{18n-35}}{-1 + x_{18n-20}x_{18n-23}x_{18n-26}x_{18n-29}x_{18n-32}x_{18n-35}} \\
&= \frac{\frac{a_{17}}{(-1+P_{17})^{n-1}}}{-1 + a_2 (-1 + P_2)^{n-1} \frac{a_5}{(-1+P_5)^{n-1}} a_8 (-1 + P_8)^{n-1} \frac{a_{11}}{(-1+P_{11})^{n-1}} a_{14} (-1 + P_{14})^{n-1} \frac{a_{17}}{(-1+P_{17})^{n-1}}} \\
&= \frac{a_{17}}{(-1 + a_2a_5a_8a_{11}a_{14}a_{17})^{n-1} (-1 + a_2a_5a_8a_{11}a_{14}a_{17})}.
\end{aligned}$$

Hence, we have

$$x_{18n-17} = \frac{a_{17}}{(-1 + a_2a_5a_8a_{11}a_{14}a_{17})^{n}}.$$

Also, it follows from (4.1) and using (4.3) that

$$\begin{aligned}
x_{18n-16} &= \frac{x_{18n-34}}{-1 + x_{18n-19}x_{18n-22}x_{18n-25}x_{18n-28}x_{18n-31}x_{18n-34}} \\
&= \frac{\frac{a_{16}}{(-1+P_{16})^{n-1}}}{-1 + a_1 (-1 + P_1)^{n-1} \frac{a_4}{(-1+P_4)^{n-1}} a_7 (-1 + P_7)^{n-1} \frac{a_{10}}{(-1+P_{10})^{n-1}} a_{13} (-1 + P_{13})^{n-1} \frac{a_{16}}{(-1+P_{16})^{n-1}}} \\
&= \frac{a_{16}}{(-1 + a_1a_4a_7a_{10}a_{13}a_{16})^{n-1} (-1 + a_1a_4a_7a_{10}a_{13}a_{16})}.
\end{aligned}$$

Hence, we have

$$x_{18n-16} = \frac{a_{16}}{(-1 + a_1a_4a_7a_{10}a_{13}a_{16})^{n}}.$$

Also, it follows from (4.1) and using (4.3) that

$$\begin{aligned}
x_{18n-15} &= \frac{x_{18n-33}}{-1 + x_{18n-18}x_{18n-21}x_{18n-24}x_{18n-27}x_{18n-30}x_{18n-33}} \\
&= \frac{\frac{a_{15}}{(-1+P_{15})^{n-1}}}{-1 + a_0 (-1 + P_0)^{n-1} \frac{a_3}{(-1+P_3)^{n-1}} a_6 (-1 + P_6)^{n-1} \frac{a_9}{(-1+P_9)^{n-1}} a_{12} (-1 + P_{12})^{n-1} \frac{a_{15}}{(-1+P_{15})^{n-1}}}
\end{aligned}$$

$$= \frac{a_{15}}{(-1 + a_0 a_3 a_6 a_9 a_{12} a_{15})^{n-1} (-1 + a_0 a_3 a_6 a_9 a_{12} a_{15})}.$$

Hence, we have

$$x_{18n-15} = \frac{a_{15}}{(-1 + a_0 a_3 a_6 a_9 a_{12} a_{15})^{n}}.$$

Similarly, one can easily obtain the other relations for (4.2). Hence, the proof is completed. □

Theorem 4.2. (4.1) *has three equilibrium points* 0 *and* $\pm\sqrt[6]{2}$, *which are nonhyperbolic equilibrium points.*

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted. □

Theorem 4.3. (4.1) *is periodic of period* 18 *iff* $P_k = 2$; $k = 0, 1, \ldots, 17$ *and will take the form*

$$x_{18n-k} = a_k; \quad k = 0, 1, \ldots, 17, \text{ and } n = 0, 1, 2, \ldots.$$

Proof. The proof follows immediately from Theorem 4.1. □

Theorem 4.4. *Assume that* $a_0, a_1, \ldots, a_{17} \in (0, 1)$. *Then the solution* $\{x_n\}_{n=-17}^{\infty}$ *oscillates about the equilibrium point* $\bar{x} = 0$, *with positive semicycles of length* 18, *and negative semicycles of length* 18.

Proof. From Theorem 4.1, we have $x_1, x_2, \ldots, x_{18} < 0$ and $x_{19}, x_{20}, \ldots, x_{36} > 0$, and the result follows by induction. □

5. The difference equation $x_{n+1} = \frac{x_{n-17}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}$

In this section, we give a specific form of the solutions of the fourth equation in the form

$$x_{n+1} = \frac{x_{n-17}}{-1 - x_{n-2}x_{n-5}x_{n-8}x_{n-11}x_{n-14}x_{n-17}}, \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

with conditions posed on the initial values x_{-j}, $j = 0, 1, 2, \ldots, 17$. Also, we investigate the oscillation and periodicity of these solutions.

Theorem 5.1. *Let* $\{x_n\}_{n=-17}^{\infty}$ *be a solution of the difference* (5.1). *Then for* $n = 0, 1, 2, \ldots$,

$$x_{18n-k} = \frac{a_k}{(-1 - P_k)^{n q_k}}, \tag{5.2}$$

where $P_k = \prod_{j=0}^{5} a_{\text{mod}(k,3)+3j}$, $q_k = (-1)^{[\frac{k}{3}]+1}$ *and* $x_{-k} = a_k$, *with* $P_k \neq -1$, $k = 0, 1, 2, \ldots, 17$.

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is

$$x_{18n-18-k} = \frac{a_k}{(-1 - P_k)^{(n-1) q_k}}. \tag{5.3}$$

Now, it follows from (5.1) and using (5.3) that

$$\begin{aligned} x_{18n-17} &= \frac{x_{18n-35}}{-1 - x_{18n-20}x_{18n-23}x_{18n-26}x_{18n-29}x_{18n-32}x_{18n-35}} \\ &= \frac{\frac{a_{17}}{(-1-P_{17})^{n-1}}}{-1 - a_2 (-1 - P_2)^{n-1} \frac{a_5}{(-1-P_5)^{n-1}} a_8 (-1 - P_8)^{n-1} \frac{a_{11}}{(-1-P_{11})^{n-1}} a_{14} (-1 - P_{14})^{n-1} \frac{a_{17}}{(-1-P_{17})^{n-1}}} \\ &= \frac{a_{17}}{(-1 - a_2 a_5 a_8 a_{11} a_{14} a_{17})^{n-1} (-1 - a_2 a_5 a_8 a_{11} a_{14} a_{17})}. \end{aligned}$$

Hence, we have

$$x_{18n-17} = \frac{a_{17}}{(-1 - a_2 a_5 a_8 a_{11} a_{14} a_{17})^n}.$$

Also, it follows from (5.1) and using (5.3) that

$$\begin{aligned} x_{18n-16} &= \frac{x_{18n-34}}{-1 - x_{18n-19} x_{18n-22} x_{18n-25} x_{18n-28} x_{18n-31} x_{18n-34}} \\ &= \frac{\frac{a_{16}}{(-1-P_{16})^{n-1}}}{-1 - a_1 (-1 - P_1)^{n-1} \frac{a_4}{(-1-P_4)^{n-1}} a_7 (-1 - P_7)^{n-1} \frac{a_{10}}{(-1-P_{10})^{n-1}} a_{13} (-1 - P_{13})^{n-1} \frac{a_{16}}{(-1-P_{16})^{n-1}}} \\ &= \frac{a_{16}}{(-1 - a_1 a_4 a_7 a_{10} a_{13} a_{16})^{n-1} (-1 - a_1 a_4 a_7 a_{10} a_{13} a_{16})}. \end{aligned}$$

Hence, we have

$$x_{18n-16} = \frac{a_{16}}{(-1 - a_1 a_4 a_7 a_{10} a_{13} a_{16})^n}.$$

Also, it follows from (5.1) and using (5.3) that

$$\begin{aligned} x_{18n-15} &= \frac{x_{18n-33}}{-1 - x_{18n-18} x_{18n-21} x_{18n-24} x_{18n-27} x_{18n-30} x_{18n-33}} \\ &= \frac{\frac{a_{15}}{(-1-P_{15})^{n-1}}}{-1 - a_0 (-1 - P_0)^{n-1} \frac{a_3}{(-1-P_3)^{n-1}} a_6 (-1 - P_6)^{n-1} \frac{a_9}{(-1-P_9)^{n-1}} a_{12} (-1 - P_{12})^{n-1} \frac{a_{15}}{(-1-P_{15})^{n-1}}} \\ &= \frac{a_{15}}{(-1 - a_0 a_3 a_6 a_9 a_{12} a_{15})^{n-1} (-1 - a_0 a_3 a_6 a_9 a_{12} a_{15})}. \end{aligned}$$

Hence, we have

$$x_{18n-15} = \frac{a_{15}}{(-1 - a_0 a_3 a_6 a_9 a_{12} a_{15})^n}.$$

Similarly, one can easily obtain the other relations for (5.2). Hence, the proof is completed. □

Theorem 5.2. (5.1) *has a unique equilibrium point* $\overline{x} = 0$*, which is a nonhyperbolic equilibrium point.*

Proof. The proof is similar to the proof of Theorem 3.2, and will be omitted. □

Theorem 5.3. (5.1) *is periodic of period* 18 *iff* $P_k = 2$; $k = 0, 1, \ldots, 17$ *and will take the form*

$$x_{18n-k} = a_k; \quad k = 0, 1, \ldots, 17, \text{ and } n = 0, 1, 2, \ldots.$$

Proof. The proof follows immediately from Theorem 5.1. □

Theorem 5.4. *Assume that* $a_0, a_1, \ldots, a_{17} \in (0, 1)$*. Then the solution* $\{x_n\}_{n=-17}^{\infty}$ *oscillates about the equilibrium point* $\overline{x} = 0$*, with positive semicycles of length* 18*, and negative semicycles of length* 18.

Proof. From Theorem 5.1, we have $x_1, x_2, \ldots, x_{18} < 0$ and $x_{19}, x_{20}, \ldots, x_{36} > 0$, and the result follows by induction. □

6. Numerical examples

To verify the results of this paper, we consider some numerical examples as follows.

Example 6.1. The graph of the difference equation (2.1) and the case when $x_{-17} = 9$, $x_{-16} = 8$, $x_{-15} = 7$, $x_{-14} = 6$, $x_{-13} = 5$, $x_{-12} = 4$, $x_{-11} = 3$, $x_{-10} = 2$, $x_{-9} = 1$, $x_{-8} = 0.9$, $x_{-7} = 0.8$, $x_{-6} = 0.7$, $x_{-5} = 0.6$, $x_{-4} = 0.5$, $x_{-3} = 0.4$, $x_{-2} = 0.3$, $x_{-1} = 0.2$, and $x_0 = 0.1$ is shown in Figure 1.

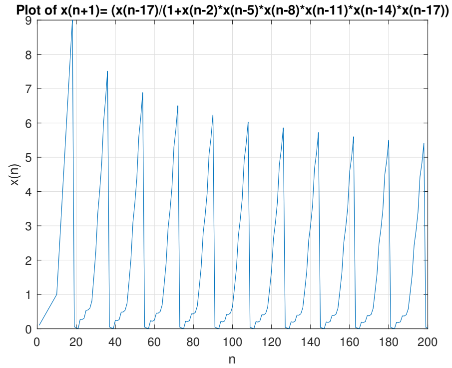

Figure 1: Graph of the difference equation (2.1).

Example 6.2. The graph of the difference equation (3.1) and the case when $x_{-17} = 23$, $x_{-16} = 21$, $x_{-15} = 28$, $x_{-14} = 2.5$, $x_{-13} = 9$, $x_{-12} = 3$, $x_{-11} = 1.5$, $x_{-10} = 10$, $x_{-9} = 0.1$, $x_{-8} = 7$, $x_{-7} = 6$, $x_{-6} = 5$, $x_{-5} = 1$, $x_{-4} = 6$, $x_{-3} = 2.5$, $x_{-2} = 2.5$, $x_{-1} = 1.5$, and $x_0 = 0.2$ is shown in Figure 2.

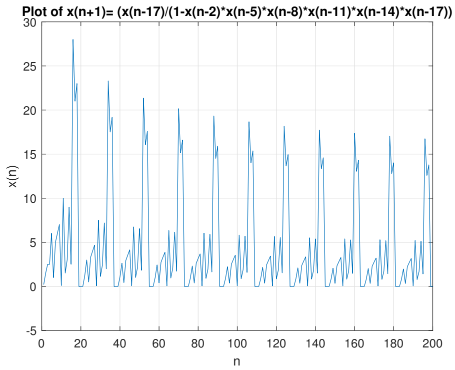

Figure 2: Graph of the difference equation (3.1).

Example 6.3. The graph of the difference equation (4.1) and the case when $x_{-17} = 0.75$, $x_{-16} = 0.22$, $x_{-15} = 0.2$, $x_{-14} = 0.15$, $x_{-13} = 0.30$, $x_{-12} = 0.6$, $x_{-11} = 0.6$, $x_{-10} = 0.22$, $x_{-9} = 0.7$, $x_{-8} = 0.9$, $x_{-7} = 0.8$, $x_{-6} = 0.7$, $x_{-5} = 0.8$, $x_{-4} = 0.5$, $x_{-3} = 0.4$, $x_{-2} = 0.9$, $x_{-1} = 0.8$, and $x_0 = 0.1$ is shown in Figure 3.

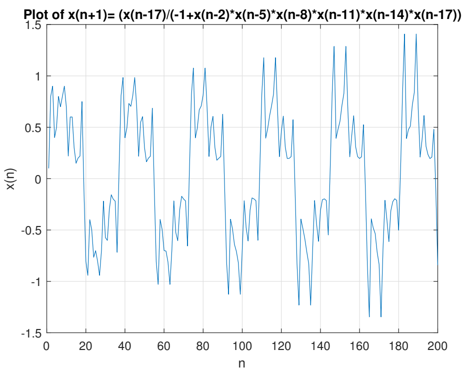

Figure 3: Graph of the difference equation (4.1).

Example 6.4. The graph of the difference equation (5.1) and the case when $x_{-17} = 2.75$, $x_{-16} = 0.22$, $x_{-15} = 0.13$, $x_{-14} = 5.15$, $x_{-13} = 0.30$, $x_{-12} = 2.2$, $x_{-11} = 0.6$, $x_{-10} = 0.22$, $x_{-9} = 0.7$, $x_{-8} = 0.12$, $x_{-7} = 0.10$, $x_{-6} = 0.8$, $x_{-5} = 0.20$, $x_{-4} = 0.6$, $x_{-3} = 0.4$, $x_{-2} = 0.2$, $x_{-1} = 15$, and $x_0 = 0.10$ is shown in Figure 4.

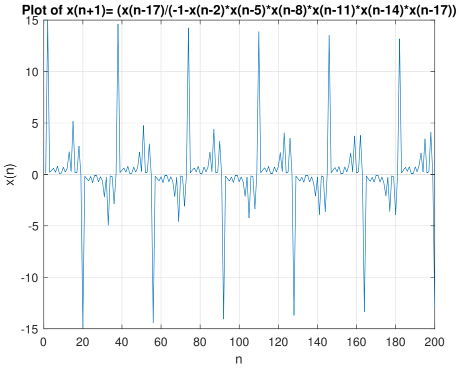

Figure 4: Graph of the difference equation (5.1).

7. Conclusion

To summarize, this paper has investigated four main rational difference equations with Eighteenth-Order. We have introduced the solutions of the considered equations using modulus operator. In Theorem 2.1, we have presented and proved the solutions of (2.1), while Theorem 2.2 has shown the boundedness of the solutions of (2.1). It has been proved that the fixed point of (2.1) is globally asymptotically stable. Theorem 4.3 has presented that (4.1) is periodic of period 18 if and only if $P_k = 2$. Furthermore, in Theorem 5.3, we have explored the solutions of (5.1) which are periodic of period 18 if and only if $P_k = -2$. We have also plotted the periodicity of (4.1) and (5.1) in Figures 3 and 4, respectively. Finally, the used approaches can be simply applied for other nonlinear equations.

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