

## Method of lines and Runge-Kutta method for solving delayed one dimensional transport equation



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### Abstract

In this article we consider a delayed one dimensional transport equation. The method of lines with Runge-Kutta method is applied to solve the problem. It is proved that the present method is stable and convergence of order  $O(\Delta t + \bar{h}^4)$ . Numerical examples are presented to illustrate the method presented in this article.

**Keywords:** Stable method, Runge-Kutta method, transport equation, method of lines.

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### 1. Introduction

Delay differential equations frequently arise in a vast variety of scientific problem as relativistic dynamics, nuclear reactor, neural network, electric circuit and control engineering [4, 6, 13, 27]. In general, we must provide the values of unknown functions on some segments in order to solve delay differential equations. When introduced in an explicit way, time delays may change the qualitative behavior of a model; for example, an epidemic model with generalized logistic dynamics can have periodic solutions when the time in the stage of infection is constant [9]. Changing to a delay for the infectious period does destabilize the endemic equilibrium for a small parameter set and leads to periodic solutions in the infectious fraction as the population size approaches extinction [5, 8]. Stein [28] gave a differential-difference equation model incorporating stochastic effects due to neuron excitation and later [29] he generalized the model to deal with the distribution of postsynaptic potential amplitudes. Various other models for neuronal activity have been proposed and many are discussed in [10, 21]. A hyperbolic partial differential equation (HPDE) is one of the types of partial differential equations. There are many examples of HPDEs; for instance, wave equation and telegraph equation. The wave equation is exercised to describe waves, as they occur in classical physics, such as water, sound and seismic waves [3]. For hyperbolic delay differential equations, the authors Sharma and Singh [23, 25] and Karthick and Subburayan [11, 12] discussed the

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Forward Time Backward Space (FTBS) and Backward Time Backward Space (BTBS) numerical techniques. Numerical methods for partial differential equations have been well studied in the literature, to cite a few [14, 15, 18, 26, 30]. Numerical treatments and convergence analysis for ordinary delay differential equations and hyperbolic partial differential equations have been studied in the literature [1, 17, 33, 35]. The application of such a technique necessitates a significant amount of computational techniques discussed in [24]. The maximum principle has been thoroughly examined for hyperbolic, parabolic, and elliptical differential equations [2, 16, 20]. Runge-Kutta techniques are used to solve systems of ordinary differential equations that arise from the discretization of spatial derivatives in hyperbolic equations using the method of lines. In this context unconditionally-stable fully implicit method has been studied in [22, 34]. Implicit Runge-Kutta (IRK) methods are among the more advanced time discretization schemes discussed in [7]. These methods provide high orders of accuracy together with desirable stability properties, error estimators, and other interesting features. These methods exploit the structure arising from carefully chosen time discretization formulae, such as diagonally or singly implicit Runge-Kutta methods. Since the systems arising from spatial discretization of a time-dependent partial differential equation (PDE) can be extremely large, specialized methods that also exploit the structure arising from discretization in space.

The paper is organized as follows: The problem under consideration is given in Section 2. Section 3 present the maximum principle and properties of the solution. In Section 4, we describe time semi-discrete problem using backward Euler scheme in temporal direction. In Section 5, we discretize the spatial domain using fourth order Runge-Kutta method with piecewise cubic Hermite interpolation. Statement of the problem described in Section 2 with discontinuous initial data function presents in Section 6. The Section 7 presents the numerical illustration. The paper is concluded in Section 8.

Throughout the paper it is assumed that,  $C$  is generic positive constant,  $M$  and  $N$  are positive integers.

## 2. Problem statement

Motivated by the works of [23, 25], we consider the following problem: Find  $u \in C(\bar{D}) \cap C^{(1,1)}(D)$  such that

$$\mathcal{L}u := \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + bu(x, t) + cu(x - \delta, t) = f(x, t), \quad (x, t) \in D, \quad (2.1)$$

$$u(x, t) = \phi(x, t), \quad (x, t) \in [-\delta, 0] \times [0, T], \quad (2.2)$$

$$u(x, 0) = u_0(x), \quad x \in [0, x_f], \quad \phi(0, 0) = u_0(0), \quad (2.3)$$

where  $a(x, t) \geq \alpha > 0$ ,  $b(x, t) \geq \beta \geq 0$ ,  $\gamma \leq c(x, t) \leq 0$ ,  $D = (0, x_f] \times (0, T]$ ,  $\delta$  is a delay argument such that  $\delta \leq x_f$  and  $r\delta \leq x_f$  for some positive integer  $r$  such that  $r\delta \leq x_f < (r+1)\delta$ . Further the functions  $a$ ,  $b$ ,  $c$ ,  $f$ ,  $\phi_1$  and  $u_0$  are sufficiently differentiable on their domains. The above equation (2.1) can be written as

$$\mathcal{L}u := \begin{cases} \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + bu = f - c\phi(x - \delta, t), & (x, t) \in [0, \delta] \times (0, T], \\ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + bu = f - cu(x - \delta, t), & (x, t) \in (\delta, x_f] \times (0, T], \end{cases} \quad (2.4)$$

$$u(0, t) = \phi_1(0, t), \quad t \in [0, T], \quad u(x, 0) = u_0(x), \quad x \in [0, x_f]. \quad (2.5)$$

## 3. Stability analysis and propagation of discontinuities

### 3.1. Stability result

In this section we present the maximum principle and the stability results of the above problem (2.4)-(2.5).

**Theorem 3.1** (Maximum principle). *Let  $\psi \in C(\bar{D}) \cap C^{(1,1)}(D)$  be any function satisfying  $\mathcal{L}\psi \geq 0$ ,  $(x, t) \in D$ ,  $\psi(0, t) \geq 0$ ,  $t \in [0, T]$ ,  $\psi(x, 0) \geq 0$ ,  $x \in [0, x_f]$ . Then  $\psi(x, t) \geq 0, \forall (x, t) \in \bar{D}$ .*

A consequence of the above theorem is the following stability result.

**Theorem 3.2** (Stability result). *Let  $\psi \in C(\bar{D}) \cap C^{(1,1)}(D)$  be any function, then*

$$|\psi(x, t)| \leq C \max\{\max_t |\psi(0, t)|, \max_x |\psi(x, 0)|, \sup_{(x,t) \in \bar{D}} |\mathfrak{L}\psi(x, t)|\}, \forall (x, t) \in \bar{D}.$$

### 3.2. Propagation of discontinuities

Let  $t$  be fixed. Then from the equation (2.1)-(2.3):

$$\mathfrak{L}u = \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + bu(x, t) + cu(x - \delta, t) = f(x, t),$$

we have

$$\begin{aligned} au_{xx} &= f_x - u_{xt} - a_x u_x - b_x u - bu_x - c_x u(x - \delta, t) - cu_x(x - \delta, t), \\ \lim_{x \rightarrow \delta^-} au_{xx} &= f_x(\delta^-, t) - u_{xt}(\delta^-, t) - a_x(\delta^-, t)u_x(\delta^-, t) - b_x(\delta^-, t)u(\delta^-, t) - b(\delta^-, t)u_x(\delta^-, t) \\ &\quad - c_x(\delta^-, t)u(0^-, t) - c(\delta^-, t)u_x(0^-, t) \\ &= f_x(\delta^-, t) - u_{xt}(\delta^-, t) - a_x(\delta^-, t)u_x(\delta^-, t) - b_x(\delta^-, t)u(\delta^-, t) - b(\delta^-, t)u_x(\delta^-, t) \\ &\quad - c_x(\delta^-, t)\phi(0^-, t) - c(\delta^-, t)\phi_x(0^-, t) \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow \delta^+} au_{xx} &= f_x(\delta^+, t) - u_{xt}(\delta^+, t) - a_x(\delta^+, t)u_x(\delta^+, t) - b_x(\delta^+, t)u(\delta^+, t) - b(\delta^+, t)u_x(\delta^+, t) \\ &\quad - c_x(\delta^+, t)u(0^+, t) - c(\delta^+, t)u_x(0^+, t) \\ &= f_x(\delta^+, t) - u_{xt}(\delta^+, t) - a_x(\delta^+, t)u_x(\delta^+, t) - b_x(\delta^+, t)u(\delta^+, t) + b(\delta^+, t)u_x(\delta^+, t) \\ &\quad - c_x(\delta^+, t)u(0^+, t) - c(\delta^+, t)u_x(0^+, t). \end{aligned}$$

It is observed that,  $a(\delta^+, t)u_{xx}(\delta^+, t) \neq a(\delta^-, t)u_{xx}(\delta^-, t)$ , since  $\phi_x(0^-, t) \neq u_x(0^+, t)$ . Similarly one can show that,  $u_{xxx}(2\delta^-, t) \neq u_{xxx}(2\delta^+, t)$ . These points  $\delta, 2\delta, 3\delta, \dots$  are primary discontinuities [4]. Hence these point are considered as mesh points while constructing the spatial mesh.

### 3.3. Derivative estimates

From the given differential equation (2.1)-(2.3), one can obtain the following.

**Lemma 3.3.** *The solution  $u(x, t)$  of (2.1)-(2.3) satisfies the following estimate  $\left| \frac{\partial^{i+j} u}{\partial x^i \partial t^j}(x, t) \right| \leq C, 0 \leq i + j \leq 2$ .*

## 4. Semi-discretization in temporal direction

Let us divide the time domain  $[0, T]$  into equally spaced  $M$  subdomains, then we have the temporal mesh  $\Omega_t^M = \{t_i = i * \Delta t\}_{i=0}^M, \Delta t = \frac{T}{M}$ . On the mesh we discretize the problem (2.1)-(2.3) in temporal direction. Let  $u^0(x) = u_0(x), x \in [0, x_f]$ . Further let,  $u^j(x)$  be the solution of

$$\begin{aligned} \mathfrak{L}^j u^j(x) &:= D_t u^j(x, t_j) + a(x, t_j)u_x^j(x, t_j) + b(x, t_j)u^j(x, t_j) + c(x, t_j)u^j(x - \delta, t_j) = f(x, t_j), \\ u^j(x) &= \phi(x, t_j), x \in [-\delta, 0], j = 1, 2, \dots, M, \end{aligned} \quad (4.1)$$

where  $D_t u^j(x, t_j) = \frac{u(x, t_j) - u(x, t_{j-1})}{\Delta t}$ .

For fixed  $t = t_j$ , the above equation can be written as,

$$\begin{aligned} \Delta t a(x, t_j) \frac{du^j}{dx}(x) + (1 + \Delta t b(x, t_j))u^j(x, t_j) + \Delta t c(x, t_j)u^j(x - \delta, t_j) &= \Delta t f(x, t_j) + u^{j-1}(x, t_{j-1}), \\ j &= 1, 2, \dots, M. \end{aligned} \quad (4.2)$$

**Lemma 4.1.** Let  $u$  be the solution of (2.1)-(2.3) and  $u^j(x)$  be the solution of (4.1) at  $t = t_j$ , then  $\|u - u^j\| \leq C \Delta t$ .

*Proof.* Let  $E_j(x) = u(x, t_j) - u^j(x)$ , and let  $x$  be fixed. Then

$$\begin{aligned} \mathcal{L}^j E_j(x) &= D_t E_j(x) + a(x, t_j) E_j(x) + b(x, t_j) E_j(x) + c(x, t_j) E_j(x - \delta, t_j) \\ &= D_t u(x, t_j) - D_t u^j(x, t_j) + a(x, t_j) (u'_x(x, t_j) - u^{j\prime}(x, t_j)) + b(x, t_j) (u(x, t_j) - u^j(x, t_j)) \\ &\quad + c(x, t_j) (u'_x(x - \delta, t_j) - u^{j\prime}(x - \delta, t_j)), \end{aligned}$$

using [19, Lemma 4.1] we can have  $E_j(x) = (D_t - \frac{\partial}{\partial t})u(x, t_j)$  and  $|E_j(x)| \leq O(\Delta t)$ ,  $\forall j = 1, 2, \dots, M$ ,  $\forall x$  which implies  $\|E_j(x)\| \leq C(\Delta t)$ , therefore  $\|u - u^j(x)\| \leq C(\Delta t)$ .  $\square$

## 5. Fully discretized problem

In this section the semi-discrete problem (4.2) is further discretized in spatial direction using fourth order Runge-Kutta method with piecewise cubic Hermite interpolation on  $[0, x_f]$ .

### 5.1. Spatial mesh points

From the Section (3.2), we observe that,  $\delta, 2\delta, \dots$  are primary discontinuous points. Therefore, we divide the domain  $[0, x_f]$  in the following way:  $[0, \delta], [\delta, 2\delta], \dots, [(r-1)\delta, r\delta]$  and  $[r\delta, x_f]$ . Divide each sub-domain with  $\frac{N}{r+1}$  sub-domain. Hence  $\bar{\Omega}_x^N = \{x_i\}_{i=0}^N$ ,  $x_i = x_{i-1} + h_i$ , where  $h_i = x_{i-1} + h_i$ ,  $i = 1, 2, \dots, N$ .

The problem (4.2) can be written in the following way:

$$\begin{aligned} f^*(x, u^j, u^{j-1}, t_j) &= \frac{1}{\Delta t} [\Delta t f(x, t_j) - (1 + b(x, t_j) \Delta t) u^j(x, t_j) \\ &\quad + u^{j-1}(x, t_{j-1}) - c(x, t_j) \Delta t u^{j,I}(x)]. \end{aligned} \quad (5.1)$$

One can refer [31, 32] for numerical scheme of piecewise cubic Hermite interpolation for interpolating the solution  $u(x_i - \delta, t_j)$  in the interval  $[\delta, x_f]$ . We apply fourth order Runge-Kutta method with piecewise cubic Hermite interpolation in space direction on  $[0, x_f]$ , we get

$$U_{i+1}^j = U_i^j + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4], \quad i = 0, 1, \dots, N-1, \quad j = 1, 2, \dots, M, \quad (5.2)$$

where,

$$\begin{aligned} K_1 &= \frac{1}{a(x_i, t_j) \Delta t} \{ \Delta t f(x_i, t_j) + U^{j-1}(x_i, t_{j-1}) - (1 + b(x_i, t_j) \Delta t) U^j(x_i, t_j) - c(x_i, t_j) \Delta t U_i^{j,I}(x_i) \}, \\ K_2 &= \frac{1}{a(x_i + \frac{h_i}{2}, t_j) \Delta t} \{ \Delta t f(x_i + \frac{h_i}{2}, t_j) + (U_i^{j-1} + \frac{K_1}{2}) - (1 + b(x_i + \frac{h_i}{2}, t_j) \Delta t) (U_i^j + \frac{K_1}{2}) \\ &\quad - c(x_i + \frac{h_i}{2}, t_j) \Delta t (U_i^{j,I}(x_i + \frac{h_i}{2})) \}, \\ K_3 &= \frac{1}{a(x_i + \frac{h_i}{2}, t_j) \Delta t} \{ \Delta t f(x_i + \frac{h_i}{2}, t_j) + (U_i^{j-1} + \frac{K_2}{2}) - (1 + b(x_i + \frac{h_i}{2}, t_j) \Delta t) (U_i^j + \frac{K_2}{2}) \\ &\quad - c(x_i + \frac{h_i}{2}, t_j) \Delta t (U_i^{j,I}(x_i + \frac{h_i}{2})) \}, \\ K_4 &= \frac{1}{a(x_i + h_i, t_j) \Delta t} \{ \Delta t f(x_i + h_i, t_j) + (U_i^{j-1} + K_3) - (1 + b(x_i + h_i, t_j) \Delta t) (U_i^j + K_3) \\ &\quad - c(x_i + h_i, t_j) \Delta t (U_i^{j,I}(x_i + h_i)) \}, \\ U^{j,I}(x) &= \begin{cases} \phi(x_i - \delta, t_j), & \text{if } (x_i - \delta) \leq 0, \\ U_k^j A_k(x) + U_{k+1}^j A_{k+1}(x) \\ \quad + B_k(x) f^*(x_k, U_k^j, U_k^{j-1}, t_j) + B_{k+1}(x) f^*(x_{k+1}, U_{k+1}^j, U_{k+1}^{j-1}, t_j), & \text{if } (x_i - \delta) > 0, \end{cases} \end{aligned}$$

and  $k$  is an integer such that  $x_i - \delta \in (x_k, x_{k+1})$ .

$$A_k(x) = \left[ 1 - \frac{2(x - x_k)}{x_k - x_{k+1}} \right] \left[ \frac{x - x_{k+1}}{x_k - x_{k+1}} \right]^2, \quad A_{k+1}(x) = \left[ 1 - \frac{2(x - x_{k+1})}{x_{k+1} - x_k} \right] \left[ \frac{x - x_k}{x_{k+1} - x_k} \right]^2,$$

$$B_k(x) = \frac{(x - x_k)(x - x_{k+1})^2}{(x_k - x_{k+1})^2}, \quad B_{k+1}(x) = \frac{(x - x_{k+1})(x - x_k)^2}{(x_{k+1} - x_k)^2}, \quad k = i - N.$$

To prove the stability of the solution, equation (5.1) can be written as

$$\frac{du^j}{dx} = f^*(x, u^j, u^{j-1}, t_j), \quad j = 1, 2, \dots, M.$$

Expanding the right hand side of the above equation about  $(x_i, t_j)$ , we get

$$\frac{du^j}{dx} = \frac{1}{\Delta t a(x_i, t_j)} [\Delta t f(x_i, t_j) - (1 + b(x_i, t_j) \Delta t) u^j(x_i, t_j) + u^{j-1}(x_i, t_{j-1}) - c(x_i, t_j) \Delta t u^{j,1}(x_i)].$$

Since  $u_i^{j-1}$  and  $u_i^{j,1}$  are known functions, can be considered as  $\mu(x_i)$  and  $\gamma(x_i)$ . Therefore the above equation becomes,

$$K_1 = \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) - \frac{h}{a} (\lambda + b) u_i^j,$$

$$K_2 = \frac{h}{a} [f + \lambda \mu(x) - c \gamma(x) - (\lambda + b)(u_i^j + \ell_{21} K_1)]$$

$$= \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) \left[ 1 - (\lambda + b) \ell_{21} \frac{h}{a} \right] - \frac{h}{a} (\lambda + b) u_i^j \left[ 1 - \ell_{21} (\lambda + b) \frac{h}{a} \right],$$

$$K_3 = \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) - (\lambda + b) \frac{h}{a} (u_i^j + \ell_{31} K_1 + \ell_{32} K_2)$$

$$= \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) \left[ 1 - (\lambda + b) \frac{h}{a} \ell_{31} - (\lambda + b) \frac{h}{a} \ell_{32} - (\lambda + b)^2 \frac{h^2}{a^2} \ell_{32} \ell_{21} \right]$$

$$+ u_i^j \left[ (\lambda + b)^2 \frac{h^2}{a^2} \ell_{31} - (\lambda + b) \frac{h}{a} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{32} - (\lambda + b)^3 \frac{h^3}{a^3} \ell_{21} \ell_{32} \right],$$

$$K_4 = \frac{h}{a} [f + \lambda \mu(x) - c \gamma(x) - (\lambda + b)(u_i^j + \ell_{41} K_1 + \ell_{42} K_2 + \ell_{43} K_3)]$$

$$= \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) \{ 1 - (\lambda + b) \frac{h}{a} \ell_{41} - (\lambda + b) \frac{h}{a} \ell_{42} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{21} \ell_{42}$$

$$- (\lambda + b) \frac{h}{a} \ell_{43} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{31} \ell_{43} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{32} \ell_{43} - (\lambda + b)^3 \frac{h^3}{a^3} \ell_{21} \ell_{32} \ell_{43} \}$$

$$+ u_i^j \{ -(\lambda + b)^3 \frac{h^3}{a^3} \ell_{31} \ell_{43} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{43} - (\lambda + b)^3 \frac{h^3}{a^3} \ell_{32} \ell_{43} + (\lambda + b)^4 \frac{h^4}{a^4} \ell_{21} \ell_{32} \ell_{43}$$

$$- (\lambda + b) \frac{h}{a} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{41} + (\lambda + b)^2 \frac{h^2}{a^2} \ell_{42} - (\lambda + b)^3 \frac{h^3}{a^3} \ell_{21} \ell_{42} \},$$

$$u_{i+1}^j = u_i^j + W_1 K_1 + W_2 K_2 + W_3 K_3 + W_4 K_4,$$

$$u_{i+1}^j = u_i^j \left[ 1 - (\lambda + b) \frac{h}{a} + (\lambda + b)^2 \frac{h^2}{2a^2} - (\lambda + b)^3 \frac{h^3}{6a^3} + (\lambda + b)^4 \frac{h^4}{24a^4} \right]$$

$$+ \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) \left[ 1 - (\lambda + b) \frac{h}{a} + (\lambda + b)^2 \frac{h^2}{2a^2} - (\lambda + b)^3 \frac{h^3}{6a^3} \right],$$

$$u_{i+1}^j = u_i^j \left[ 1 - \bar{h} + \frac{\bar{h}^2}{2} - \frac{\bar{h}^3}{6} + \frac{\bar{h}^4}{12} \right] + \frac{h}{a} (f + \lambda \mu(x) - c \gamma(x)) \left[ 1 - \bar{h} + \frac{\bar{h}^2}{2} - \frac{\bar{h}^3}{6} \right],$$

where,

$$\bar{h} = (\lambda + b) \frac{h}{a}, \quad \lambda = \frac{1}{\Delta t}, \quad c_2 = \ell_{21}, \quad c_3 = \ell_{31} + \ell_{32}, \quad c_4 = \ell_{41} + \ell_{42} + \ell_{43}, \quad W_1 + W_2 + W_3 + W_4 = 1,$$

$$W_2 c_2 + W_3 c_3 + W_4 c_4 = \frac{1}{2}, \quad W_3 c_2 \ell_{32} + W_4 (c_2 \ell_{42} + c_3 \ell_{43}) = \frac{1}{6}, \quad W_4 c_2 \ell_{32} \ell_{43} = \frac{1}{24}.$$

The above method is stable if  $|\bar{h}| < 1$ . That is  $\frac{h}{a\Delta t} + \frac{b}{a}h < 1$ . Let us assume that  $\frac{b}{a}h < \frac{1}{2}$  and  $\frac{h}{a\Delta t} < \frac{1}{2}$ . Then the above scheme is stable and convergence.

**Theorem 5.1** ([4]). Let  $u(x_i, t_j)$  be the solution of the problem (4.2) and  $U_i^j$  be the solution of the problem (5.2), then  $\|u(x_i, t_j) - U_i^j\| \leq C(\bar{h}^4)$ .

The following theorem gives an error estimate for the above method.

**Theorem 5.2.** Let  $u_i^j$  be the exact solution of (2.1) at the point  $(x_i, t_j)$  and  $U_i^j$  be the numerical solution of (5.2), then  $\|u(x_i, t_j) - U_i^j\| \leq C(\Delta t + \bar{h}^4)$ .

*Proof.* Using the Lemma 4.1 and Theorem 5.1, one can prove that,

$$\|u - U_i^j\| = \|u - u_i^j + u_i^j - U_i^j\| \leq \|u - u_i^j\| + \|u_i^j - U_i^j\| \leq C(\Delta t + \bar{h}^4).$$

□

## 6. Discontinuous initial data

Let us assume that  $x^*$  be a point at which the function  $u_0$  has a jump discontinuity. Then the method discussed in the above section can be applied to the problem,

$$\mathcal{L}_1 u := \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} + bu(x, t) + cu(x - \delta, t) = f(x, t),$$

$$u(x, t) = \phi(x, t), \quad (x, t) \in [-\delta, 0] \times [0, T],$$

$$u(x, 0) = u_0(x), \quad x \in (0, x_f], \quad \phi(0, 0) = u_0(0),$$

*Remark 6.1.* For  $j = 1$ , then the differential equation

$$\Delta t a \frac{du^j}{dx} + b u^j + c u^j(x - \delta) = \Delta t f(x, t_j) + u^0(x).$$

It is noted that the right hand side function is discontinuous at  $x = x^*$ , so that the function  $u^j(x, t_j)$  is not smooth at  $x = x^*$ . Further, the function  $u^j(x)$  is not smooth enough at  $x^* + \delta, x^* + 2\delta, \dots$ . So that while dividing the domain  $[x_0, x_f]$ , these points  $x^*, x^* + \delta, x^* + 2\delta, \dots$  are to be considered as mesh points.

Similar to the above Theorem 5.2, we have the estimate for the difference between the exact and numerical solution as follows. Due to the presence of the delay argument, the initial discontinuity propagates in the forward direction.

**Theorem 6.2.** Let  $u_i^j$  be the exact solution of (2.1) and  $u(x_i, t_j)$  be the numerical solution of (5.2), then  $\|u(x_i, t_j) - u_i^j\| \leq C(\Delta t + \bar{h}^4)$ .

## 7. Numerical examples

Two examples are given in this section to illustrate the numerical methods presented in this paper. We use the half mesh principle to estimate the maximum error. For this we put

$$E^{N,M} = \max_{i,j} |U_i^j(h, \Delta t) - U_i^j(h/2, \Delta t/2)|, \quad 0 \leq i \leq N, \quad 0 \leq j \leq M,$$

where  $U_i^j(h, \Delta t)$  and  $U_i^j(\frac{h}{2}, \Delta t/2)$  are the numerical solution at the node  $(x_i, t_j)$  with mesh sizes  $(h, \Delta t)$

and  $(\frac{h}{2}, \Delta t/2)$ , respectively. Graph of the numerical solutions, numerical solution at different time level and the maximum point-wise error plot are drawn.

**Example 7.1.** Consider the following first order hyperbolic delay differential equation.

$$\begin{aligned} \frac{\partial u}{\partial t} + a(x, t) \frac{\partial u}{\partial x} + b(x, t)u(x, t) + c(x, t)u(x - \delta, t) &= 0, \quad (x, t) \in (0, 4] \times (0, 4], \\ u(x, t) &= 0, \quad (x, t) \in [-\delta, 0] \times [0, 4], \\ u(x, 0) &= x \exp(-(4x - 1)^2/2) \times (2 - x), \quad x \in [0, 2], \\ u(x, 0) &= (x - 2) \times (4 - x) \times \exp(-(4x - 1)^2/10), \quad x \in [2, 4], \\ a(x, t) &= \frac{3 + x^2 + t^2}{1 + 2tx + 2x^2}, \quad b(x, t) = 1, \quad c(x, t) = -2. \end{aligned}$$

In this problem, it is assumed that,  $\delta = 1$ . Numerical solution is plotted in the Figure 1 and for different time levels the solution curves are plotted in Figure 2. The maximum point-wise error is given in Table 1. The maximum error is plotted in Figure 3.

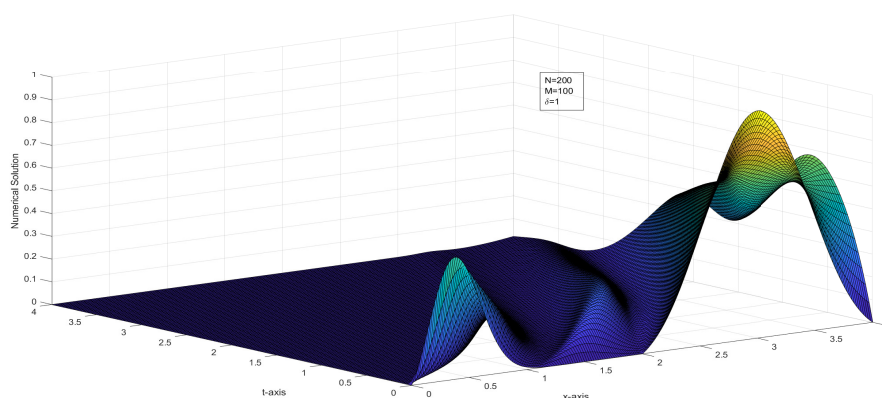


Figure 1: The surface plot of the U-numerical solution of Example 7.1.

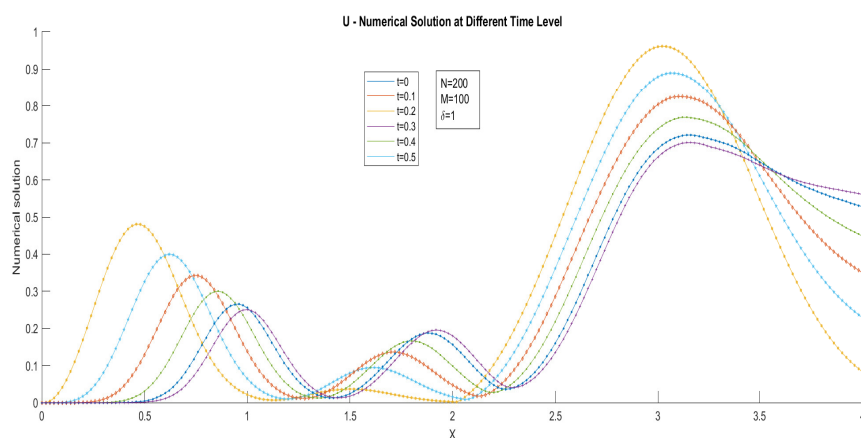


Figure 2: U-numerical solution of Example 7.1 at different time level.

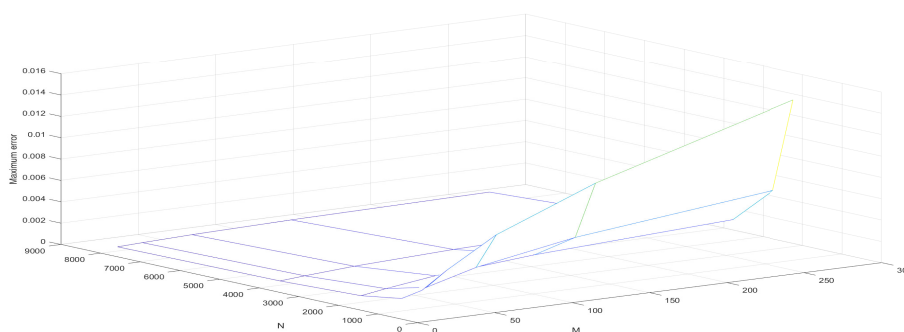


Figure 3: Maximum point-wise error of the Example 7.1.

Table 1: Maximum error for the Example 7.1 using conditional method.

N and $\delta = 1$							
M ↓	64	128	256	512	1024	2048	4096
64	2.3175e-03	1.1365e-03	5.6285e-04	2.8009e-04	1.3971e-04	6.9773e-05	3.4866e-05
128	3.8978e-03	1.8843e-03	9.2652e-04	4.5942e-04	2.2876e-04	1.1414e-04	5.7013e-05
256	6.6230e-03	3.1471e-03	1.5356e-03	7.5853e-04	3.7698e-04	1.8792e-04	9.3821e-05
512	1.0256e-02	4.7577e-03	2.2978e-03	1.1299e-03	5.6030e-04	2.7901e-04	1.3922e-04
1024	1.5652e-02	6.7670e-03	3.1902e-03	1.5526e-03	7.6626e-04	3.8069e-04	1.8974e-04

**Example 7.2.** Consider the following first order hyperbolic delay differential equation,

$$\frac{\partial u}{\partial t} + a(x, t) \frac{\partial u}{\partial x} + b(x, t)u(x, t) + c(x, t)u(x - \delta, t) = 0, \quad (x, t) \in (0, 4] \times (0, 4],$$

$$u(x, t) = 0, \quad (x, t) \in [-\delta, 0] \times [0, 4],$$

$$u(x, 0) = x \exp(-(4x - 1)^2/2) \times (2 - x), \quad x \in [0, 2],$$

$$u(x, 0) = 0.1, \quad x \in (2, 4],$$

$$a(x, t) = \frac{3 + x^2 + t^2}{1 + 2tx + 2x^2}, \quad b(x, t) = 1, \quad c(x, t) = -2.$$

In this problem, it is assumed that,  $\delta = 1$ . Numerical solution is plotted in the Figure 4 and for different time levels the solution curves are plotted in Figure 5. The maximum point-wise error is given in Table 2. The maximum error is plotted in Figure 6.

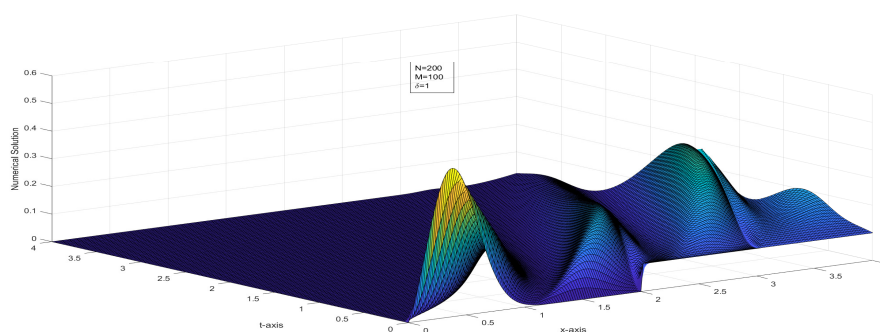


Figure 4: The surface plot of the U-numerical solution of Example 7.2.



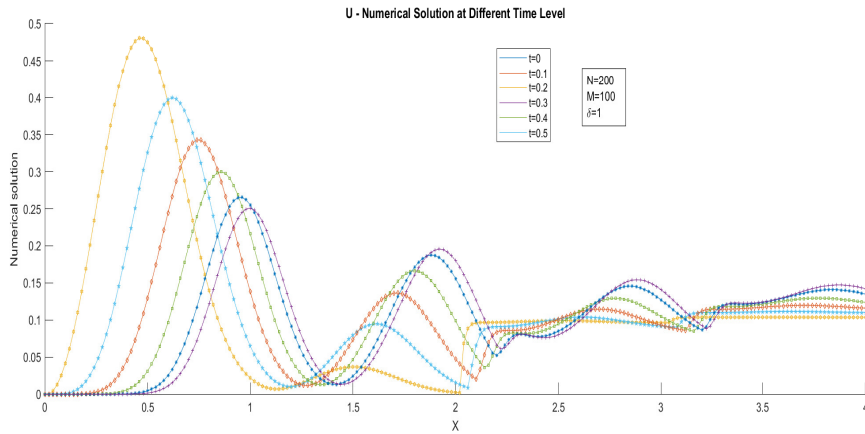


Figure 5: U-numerical solution of Example 7.2 at different time level.

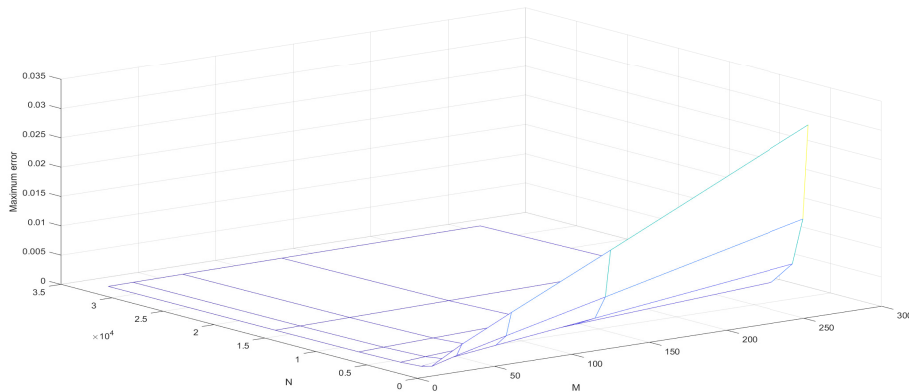


Figure 6: Maximum point-wise error of the Example 7.2.

Table 2: Maximum error for the Example 7.2 using conditional method.

		N and $\delta = 1$					
M ↓	64	128	256	512	1024	2048	4096
64	1.8966e-03	9.4441e-04	4.7126e-04	2.3540e-04	1.1764e-04	5.8806e-05	2.9400e-05
128	4.2727e-03	2.1272e-03	1.0613e-03	5.3009e-04	2.6490e-04	1.3242e-04	6.6199e-05
256	8.3897e-03	4.1851e-03	2.0901e-03	1.0444e-03	5.2206e-04	2.6099e-04	1.3049e-04
512	1.6388e-02	8.1765e-03	4.0839e-03	2.0409e-03	1.0202e-03	5.1001e-04	2.5499e-04
1024	3.2511e-02	1.6220e-02	8.1012e-03	4.0484e-03	2.0236e-03	1.0117e-03	5.0581e-04

## 8. Conclusions

In this article, we considered the first order hyperbolic delay differential equation with space delay, which serves the model for more scientific applications. For the numerical solution of this problem, we apply semi-discretization in temporal direction on uniform mesh using backward finite difference scheme and the truncation error of this method produces first order convergence for fixed  $\chi$ , that is  $O(\Delta t)$ . The semi-discretized problem is, then further discretized using fourth order Runge-Kutta method with piecewise cubic Hermite interpolation in spatial direction and the method produces  $O(\Delta t + \bar{h}^4)$ . The problem (2.1) with smooth and non-smooth data functions and its properties of the solutions are also discussed. The numerical examples are given to validate the theoretical results in the form of Figures 1-6 and Tables 1-2. It is observed that, for fixed integer M and for increasing the size of N, the maximum error decreases, whereas for fixed N and for increasing M the maximum error increases. Since the method is

conditionally stable. That is if  $\bar{h} < 1$ , then only the method is stable. From Figures 3 and 6, it is observed the same, that is the method is stable only if  $h \leq C\Delta t$ .

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