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A new fifth-order iterative method for solving non-linear equations using weight function technique and the basins of attraction

M. Q. Khirallah[a,b,*], Asma M. Alkhomsan[b]

[a]Department of Mathematics and Computer Science, Faculty of Science, Ibb University, Yemen.

[b]Department of Mathematics, Faculty of Science and Arts, Najran University, Najran 1988, Saudi Arabia.

Abstract

In this paper, new iterative method is presented of fifth-order for solving non-linear equations $f(x) = 0$ a devoid of the second derivative which requires two derivative functions and evaluations for each step, using both weight functions and synthesis techniques together. This method improves Newton's method and thus the efficiency index has been improved from 1.414 to 1.495. The convergence analysis for the new method is discussed. We provide some numerical examples that illustrate the performance of our proposed method by comparing them with numerical methods of fifth-order also the complex dynamics and basins of attraction is discussed, comparing it with several methods of the same order, thus comparisons show that new method gives the best results.

Keywords: Nonlinear equations, basins of attraction efficiency index, iterative methods, complex dynamics.

2020 MSC: 41A25, 65H05, 65K05.

1. Introduction

Solving nonlinear equations $f(x) = 0$ was and still a considerable issue in numerical analysis. While Newton's method was the first attempt to solve such equation, which converges quadratically, in recent years, researchers have developed Newton's method to achieve higher convergence and more accurate results as, see for example the method shown in [13] has the fourth order and was the first proposed multi-point method of this order, method shown in [8] has fifth order, method shown in [10] has the sixth order and method shown in [1] has eighth order and references therein. In this research, we are interested to find iterative method for solving non-linear equations of order five where methods of this order have been found in several researches as in [7, 8, 10, 11, 14]. These iterative methods have been constructed by using different techniques for solving the non-linear equations such as the variational iteration technique and weight function and etc. We have innovated a new fifth-order iterative method using both weight function and synthesis techniques together, which contains four function evaluations generally. Efficiency in this iterative method is measured with index $I \approx p^{\frac{1}{m}}$, p shows the order of

*Corresponding author
Email address: mqm73@yahoo.com (M. Q. Khirallah)

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convergence and m represents the total number of functional evaluations per iteration. Therefore, this new method has efficiency index defined by $5^{\frac{1}{4}} = 1.495$, it is better than 1.414 the efficiency index to Newton's method.

The rest in this paper arranged as follows. Section 2 displays a new fifth-order iterative method construction. Section 2.1 analyses the convergence order for our supposed method. In Section 3 we discuss and analyze the stability using techniques of complicated dynamics and the using them to compare different methods in terms of attraction basins and the dynamical behavior for iterative method in the complex plane, where compare the new iterative method with several iterative methods of order five. Section 4 illustrates numerical examinations which improve the efficiency and performance of our new supposed method, in the last section we have some conclusion about results.

2. Construction of the new method

We present a new method of two-step where the first step is Newton's method and the second step is a weight function W depending on K included. Thus the iterative expression is as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - W(K_n)\frac{f(y_n)}{f'(x_n)+f'(y_n)}, \tag{2.1}$$

where $W(K) = \dfrac{AK+BK^2}{C+K}$ and $K = \frac{f'(y)}{f'(x)}$. The above method (2.1) has convergence of fifth order and the number of functional evaluations per iteration of four, thus efficiency index is 1.495.

2.1. Convergence analysis

The convergence analysis of the proposed method (2.1) will be discussed in the following theorem where using program mathematica 11 to prove that closeness order is five.

Theorem 2.1. *Consider that a is a simple root of the nonlinear equation and let $f : D \subset R \to R$ be a real sufficiently differentiable function in an open interval D, $a \in D$. If x_0 is close enough to a and the weight function $W(K)$ satisfies $A = \dfrac{4}{3}$, $B = 0$, and $C = -\dfrac{1}{3}$, then iterative method (2.1) converges to a with order of convergence five and the error equation is:*

$$e_{n+1} = -c_2^2 c_3 e_n^5 + O\left(e_n^6\right),$$

where $e_n = x_n - a$ and $c_j = \dfrac{f^{(j)}(a)}{j! f'(a)}, j \geqslant 2$.

Proof. Let $e_n = x_n - a$ be the error at n^{th} iteration. Expanding $f(x_n)$ and $f'(x_n)$ about a, by using Taylor's method we get

$$f(x_n) = f'(a)\left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5\right] + O\left(e_n^6\right) \tag{2.2}$$

and

$$f'(x_n) = f'(a)\left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5\right] + O\left(e_n^6\right). \tag{2.3}$$

Now from (2.2) and (2.3) we get

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2 e_n^2 + \left(2c_2^2 - 2c_3\right)e_n^3 + \left(-4c_2^3 + 7c_2c_3 - 3c_4\right)e_n^4 \\ &\quad + \left(8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5\right)e_n^5 + O\left(e_n^6\right),\end{aligned} \tag{2.4}$$

hence from the first step of (2.1) and (2.4) we have

$$\begin{aligned}y_n - a = e_n - \frac{f(x_n)}{f'(x_n)} &= c_2 e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 + \left(4c_2^3 - 7c_2c_3 + 3c_4\right)e_n^4 \\ &\quad + \left(-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5\right)e_n^5 + O(e_n^6),\end{aligned} \tag{2.5}$$

from (2.5), we obtain

$$f(y_n) = c_2e_n^2 + \left(-2c_2^2 + 2c_3\right)e_n^3 + \left(5c_2^3 - 7c_2c_3 + 3c_4\right)e_n^4 - 2\left(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5\right)e_n^5 + O(e_n^6) \tag{2.6}$$

and

$$f'(y_n) = 1 + 2c_2^2e_n^2 - \left(4c_2^3 - 4c_2c_3\right)e_n^3 + \left(8c_2^4 - 11c_2^2c_3 + 6c_2c_4\right)e_n^4 - 2c_2\left(8c_2^4 - 14c_2^2c_3 + 10c_2c_4 - 4c_5\right)e_n^5 + O(e_n^6). \tag{2.7}$$

Expanding the weight function variable K, we get

$$K = \frac{f'(y_n)}{f'(x_n)} = 1 - 2c_2e_n + \left(6c_2^2 - 3c_3\right)e_n^2 - 4\left(4c_2^3 - 4c_2c_3 + c_4\right)e_n^3 + \left(40c_2^4 - 61c_2^2c_3 + 9c_3^2 + 22c_2c_4 - 5c_5\right)e_n^4 + \left(-96c_2^5 + 198c_2^3c_3 - 66c_2c_3^2 + 24c_3c_4 - 88c_2^2c_4 + 28c_2c_5 - 6c_6\right)e_n^5 + O(e_n^6).$$

Therefore, weight function $W(K)$ around zero results in

$$\begin{aligned} W(K) = {} & \frac{A+B}{1+C} - \frac{2(B+AC+2BC)}{(1+C)^2}c_2e_n + \frac{1}{(1+C)^3}\Big[\left(2AC(1+3C) + 2B(3+C(9+8C))\right)c_2^2 \\ & - 3(1+C)(B+AC+2BC)c_3\Big]e_n^2 + \frac{1}{(1+C)^4}\big[\big(-8AC^2(1+2C) - 8B\big(2+C\big(8+C\big(12 \\ & +7C\big)\big)\big)\big)C_2^3 + 4(1+C)(AC(1+4C) + B(4+C(12+11C)))c_2c_3 - 4(1+C)^2\left(B+AC \right. \\ & \left. +2BC\right)c_4\big]e_n^3 + \frac{1}{(1+C)^5}\big[AC\big(4(1+2C)\left(-1+5C^2\right)c_2^4 - (1+C)(-3+C(22+61C))c_2^2c_3 \\ & + 2(1+C)^2(3+11C)c_2c_4 + (1+C)^2\left(9Cc_3^2 - 5(1+C)c_5\right)\big) + B\big(4\big(10+C\big(50+C\big(101 \\ & +3C(34+15C)\big)\big)\big)c_2^4 - (1+C)(61+C(244+3C(123+74C)))c_2^2c_3 + 2(1+C)^2\big(11 \\ & +33C+30C^2\big)c_2c_4 + (1+C)^2\left(9(1+3C(1+C))c_3^2 - 5(1+C)(1+2C)c_5\right)\big)\big]e_n^4 + O(e_n^5), \end{aligned} \tag{2.8}$$

from (2.3), (2.6), and (2.7) we have

$$\frac{f(y_n)}{f'(x_n)+f'(y_n)} = \frac{1}{2}c_2e_n^2 - \left(\frac{3}{2}c_2^2 - c_3\right)e_n^3 + \frac{1}{4}\left(14c_2^3 - 21c_2c_3 + 6c_4\right)e_n^4 - \left(7c_2^4 - \frac{35}{2}c_2^2c_3 + \frac{9}{2}c_3^2 + \frac{15}{2}c_2c_4 - 2c_5\right)e_n^5 + O(e_n^6). \tag{2.9}$$

Finally, from (2.8) and (2.9), the error equation of the method of (2.1) is

$$\begin{aligned} e_{n+1} = {} & y_n - a - W(K)\frac{f(y_n)}{f'(x_n)+f'(y_n)} \\ = {} & \left(1 - \frac{(A+B)}{2(1+C)}\right)c_2e_n^2 + \frac{1}{2(1+C)^2}\Big[\left(-4(1+C)^2 + A(3+5C) + B(5+7C)\right)c_2^2 \\ & - 2(1+C)(A+B-2(1+C))c_3\Big]e_n^3 + \frac{1}{4(1+C)^3}\Big[\big(16(1+C)^3 - 2B(19+5C(10+7C)) \\ & - 2A\big(7+C\big(22+19C\big)\big)\big)c_2^3 + 7(1+C)\left(-4(1+C)^2 + A(3+5C) + B(5+7C)\right)c_2c_3 \\ & - 6(1+c)^2(A+B-2(1+C))c_4\Big]e_n^4 + \frac{1}{2(1+c)^4}\Big[\big(-16(1+C)^4 + 2A\big(7+C\big(31 \\ & + C(51+31C)\big)\big) + B\big(31+C\big(117+C(155+73C)\big)\big)\big)c_2^4 - (1+C)\left(-40(1+C)^3\right. \end{aligned} \tag{2.10}$$

$$+A\left(35+3C\left(36+31C\right)\right)+B\left(93+C\left(244+171C\right)\right)\big)c_2^2c_3+5\left(1+C\right)^2\left(-4\left(1+C\right)^2\right.$$
$$\left.+A\left(3+5C\right)+B\left(5+7C\right)\right)c_2c_4+\left(1+C\right)^2\Big(\Big(-12\left(1+C\right)^2+3A\left(3+5C\right)+3B\left(5+7C\right)\Big)c_3^2$$
$$-4\left(1+C\right)\left(A+B-2\left(1+C\right)\right)c_5\Big)\Big]e_n^5+O(e_n^6).$$

Hence from (2.10) the conditions on the weight function W are:

$$\left.\begin{aligned} 2\left(1+C\right)-\left(A+B\right)=0\\ -4\left(1+C\right)^2+A\left(3+5C\right)+B\left(5+7C\right)=0\\ 16\left(1+C\right)^3-2A\left(7+C\left(22+19C\right)\right)-2B\left(19+5C\left(10+7C\right)\right)=0 \end{aligned}\right\}.$$

We solve this system by using maple 15 and we get the values of A, B, and C, where $A=\frac{4}{3}$, $B=0$, and $C=-\frac{1}{3}$. If we put these values into equation (2.10), we get the error equation of any method of (2.1) as

$$e_{n+1}=-c_2^2c_3e_n^5+O\left(e_n^6\right).$$

□

If replacing the weight function $W(K)=\dfrac{4K}{-1+3K}$ in (2.1), we get the iterative scheme

$$y_n=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)},\quad x_{n+1}=y_n+\frac{4f'\left(y_n\right)}{f'\left(x_n\right)-3f'\left(y_n\right)}\cdot\frac{f\left(y_n\right)}{f'\left(x_n\right)+f'\left(y_n\right)}. \tag{2.11}$$

We will denote our new method as AMK1, where it requires only four functional evaluations per step.

3. Basins of attraction on the complex plane

In this section dynamical behavior of iterative methods of fifth-order are studied for solving nonlinear equation $f\left(z\right)=0$, where the function $f: C \longrightarrow C$ in a complicated plane. In 1879, the complex method of Newton's for the basins of attraction was first started by Cayley [12], where it was solved when $f\left(z\right)$ is a quadratic polynomial. The Newton's method has been outspread to other iterative methods, with convergence order higher than two (see, for example [3–6, 9]). The basins of attraction for methods are used to make a visual comparison, this method had used first by Stewart in [12], where he compared Newton's method and several other methods of different order such as Popovski method, Halley's method, and Leguerre method by presenting the attraction basins of the zeros that had established by the iterative methods. He noticed that the basins of attraction is a manner to visually understand how an iterative method work as a function of the various beginning points thus the iterative method is better when it has a bigger area of convergence. For comparisons, we will use four fifth-order known methods as described below:

method (FLM) in [7]:

$$y_n=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)},\quad x_{n+1}=y_n-\frac{5f'^2\left(x_n\right)+3f'^2\left(y_n\right)}{f'^2\left(x_n\right)+7f'^2\left(y_n\right)}\cdot\frac{f\left(y_n\right)}{f';\left(x_n\right)} \tag{3.1}$$

method (M2) in [10]:

$$y_n=x_n-\frac{2}{3}\frac{f\left(x_n\right)}{f'\left(x_n\right)},\quad z_n=x_n-\frac{1}{2}\frac{f\left(x_n\right)}{f'\left(x_n\right)}-\frac{f\left(x_n\right)}{3f'\left(y_n\right)-f'\left(x_n\right)},\quad x_{n+1}=z_n-\frac{f\left(z_n\right)}{f'\left(x_n\right)}; \tag{3.2}$$

method (SH3) in [14]:

$$y_n=x_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)},\quad z_n=y_n-\frac{f\left(x_n\right)}{f'\left(x_n\right)}\cdot\left[\frac{f\left(y_n\right)}{f\left(x_n\right)-2f\left(y_n\right)}\right],\quad x_{n+1}=z_n-\frac{f\left(z_n\right)}{f'\left(x_n\right)}; \tag{3.3}$$

method (PJ) in [11]:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = y_n - \left(\frac{5}{4} - \frac{f'(y_n)}{2f'(x_n)} + \frac{f'^2(y_n)}{4f'^2(x_n)}\right) \cdot \frac{f(y_n)}{f'(y_n)}. \tag{3.4}$$

In the following we will give four different polynomial examples, with different degree and we draw the attraction basins by using five methods (2.11), (3.1), (3.2), (3.3), and (3.4). We will construct them using this strategy: we take square $[-2, 2] \times [-2, 2] \subset \mathbb{C}$ and in Examples 3.4 and 3.6 we will compute in the square $[-3, 3] \times [-3, 3] \subset \mathbb{C}$ from 256×256, in which it contains all roots of nonlinear equation $f(z) = 0$, we will give different colors for those different roots, we put a color to point for every $z_0 \in D$ and $D \subset \mathbb{C}$ according to the root in which has convergence the iterative method and a black colored one to indicate non convergence for any of those roots, which started of z_0 with conditions $\varepsilon = 10^{-3}$ and 20 iterative in maximum. We notice the black colored dots in the picture means that the method failed to find the solution under the conditions specified for convergence, such as the number of steps and the fixed stopping criterion of error allowed in the solution. For that we will introduce comparison of Examples 3.1-3.6 in Tables 1-6 for these iterative methods. The column I/P displays the average of iterations per point until the method decides that a root has been reached or the point is not convergent, where measured from iterations/point. The column NB displays the number of black points and it is clear that nonconvergent points have a significant impact on I/P values, as these points always contribute with the maximum number of iterations allowed. The column BI displays brightness indicator where the higher the brightness the better. Finally, the column T displays the time it took to get to the solution. Note that the graph with higher brightness will take less iteration. Thus, the results in those Tables 1-6 show that our suggested method AMK1 is the best. We have used Mathematica 11 to solve all examples in complex plane.

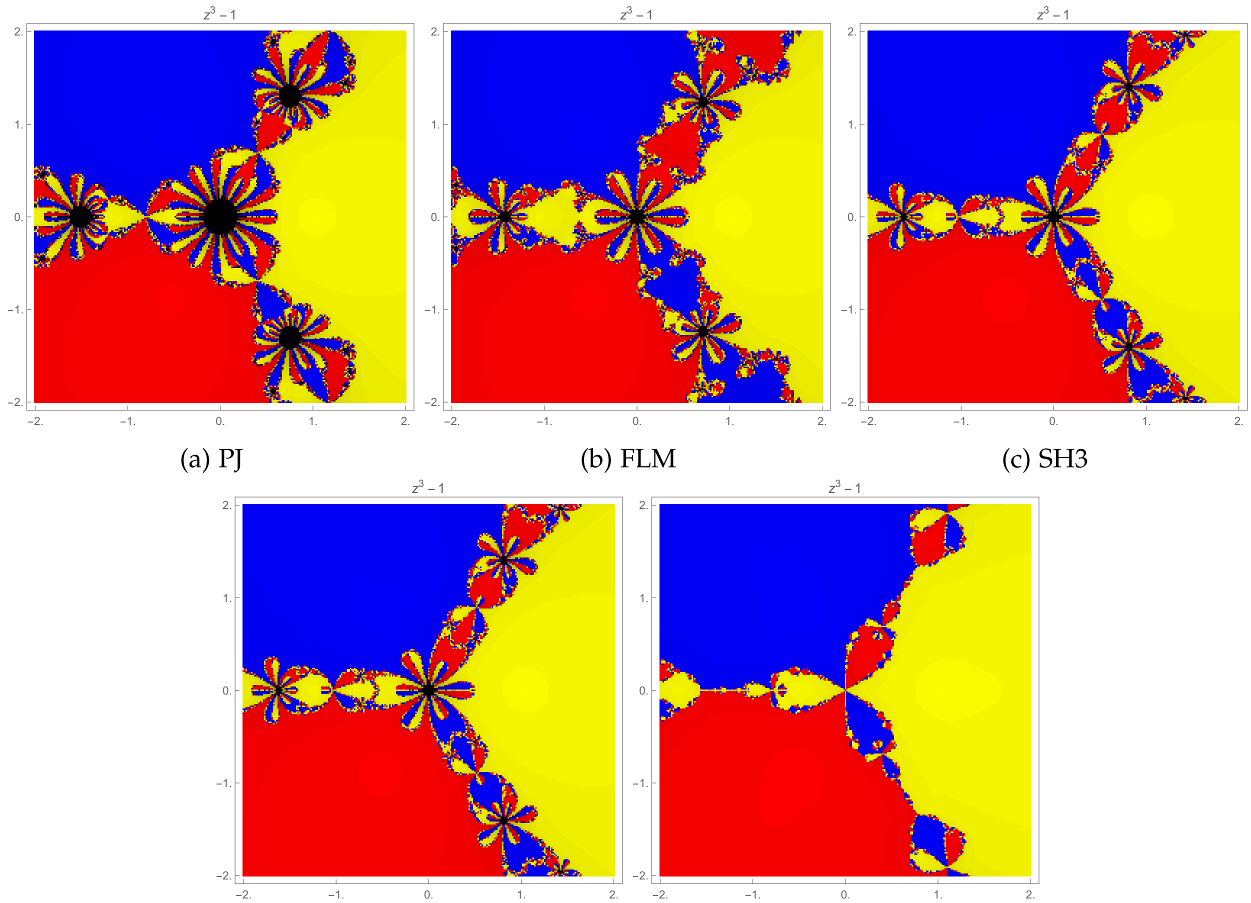
Example 3.1. We consider the nonlinear equation

$$f_1(z) = z^3 - 1.$$

This polynomial has three roots as $\{1, -0.5 - 0.866025i, -0.5 + 0.866025i\}$. In Figure 1 we plotted the fifth order methods, as PJ in [11], FLM in [7], M2 in [10], SH3 in [14], and the new method AMK1. We compare from where the number of black points NB, the brightness index BI, the time taken to reach the solution T, the average of iterations per point until the method decides that a root has been reached I/P. In Figure 1 each basin was given a different color to indicate whether the method had converged to the closest root. We have also used lighter shade when the number of iterations is lower and it means if the graph has higher brightness it will take less iterations, thus we will rule that the method is the best. The methods are listed in the Table 1 from worst to best.

Table 1: Comparison of different methods in complex plane in Example 3.1.

Method	NB	BI	T	I/P
PJ	1907	0.401867	19.64	3.84
FLM	588	0.404924	17.11	3.80
M2	254	0.417466	16.03	3.24
SH3	254	0.417466	15.76	3.24
AMK1	1	0.428942	11.56	2.43



(a) PJ (b) FLM (c) SH3

(d) M2 (e) AMK1

Figure 1: Basins of attraction to iterative methods for $f_1(z) = z^3 - 1$.

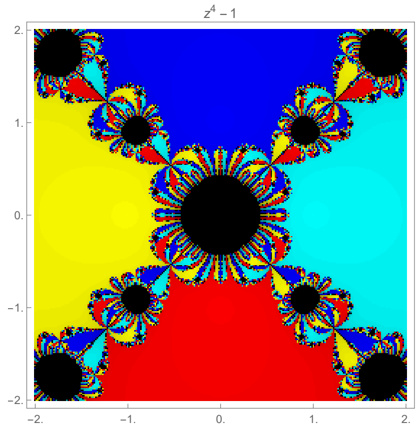
Example 3.2. We consider the nonlinear equation

$$f_2(z) = z^4 - 1.$$

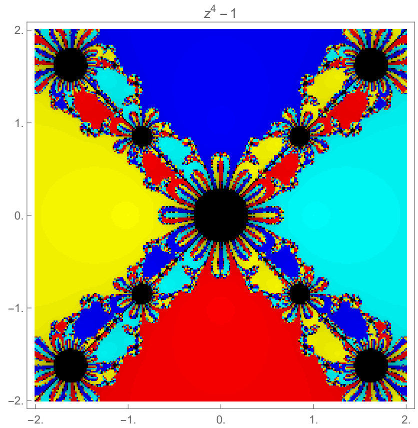
It has four roots which are $\{-1, +1, -i, +i\}$ through the results shown in Table 2 the new method AMK1 is comparable with all of the methods and gives the best results. Note that the order of the methods from worst to best are listed in the Table 2. Figure 2 illustrates the basins of attraction for each root in this example for all methods.

Table 2: Comparison of different methods in complex plane in Example 3.2.

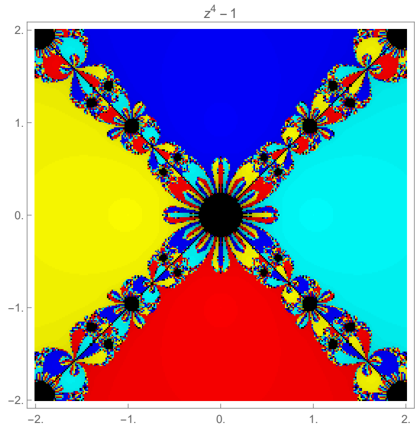
Method	NB	BI	T	I/P
PJ	9561	0.387804	31.84	3.98
FLM	5833	0.410141	27.34	4.57
M2	3317	0.436208	23.89	3.87
SH3	3477	0.434614	23.13	3.91
AMK1	513	0.465024	14.95	3.11



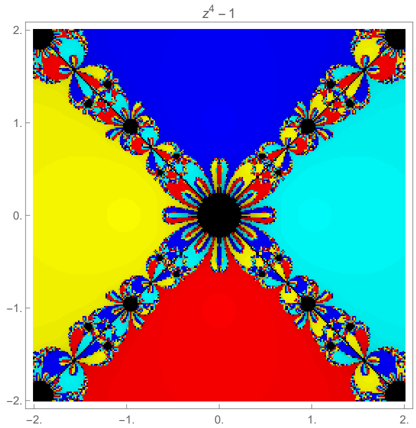
(a) PJ



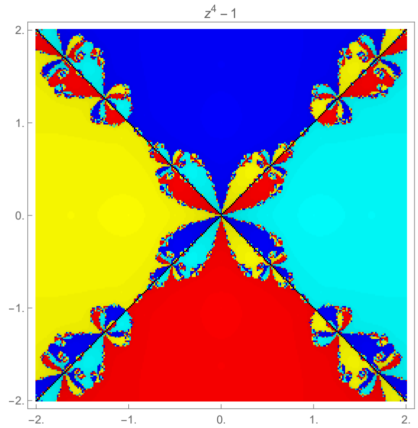
(b) FLM



(c) SH3



(d) M2



(e) AMK1

Figure 2: Basins of attraction to iterative methods for $f_1(z) = z^4 - 1$.

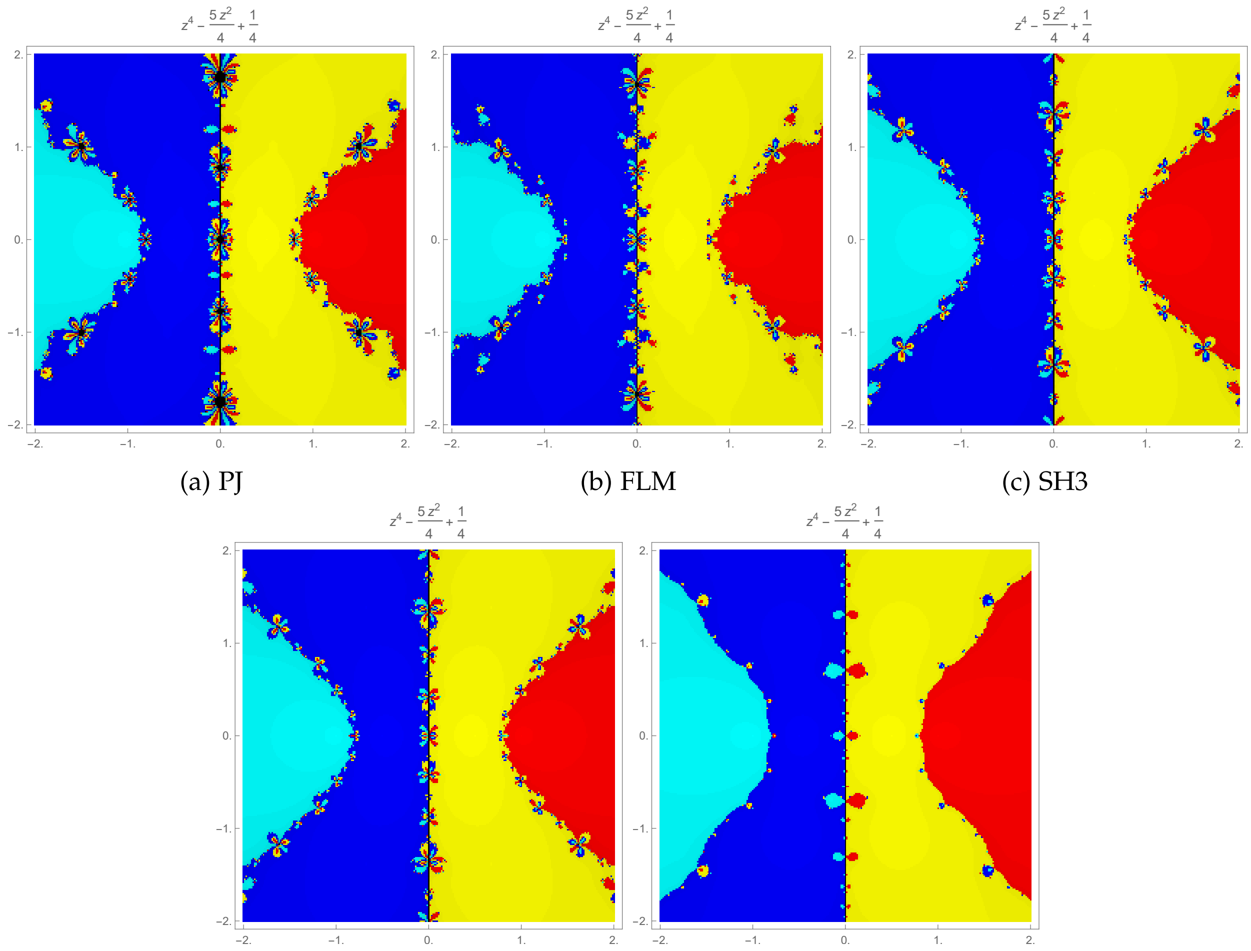
Example 3.3. We consider the nonlinear equation

$$f_3(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4}.$$

The roots are $\{-1, +1, -0.5, +0.5\}$. We have the basins in Figure 3. Compared the methods and results in Table 3, the best methods are AMK1 then M2 then SH3 then FLM then PJ. Thus the fewer black points and it was of high brightness and has less time to obtain the basins of attraction of the roots of the example considered the methods, the better the method.

Table 3: Comparison of different methods in complex plane in Example 3.3.

Method	NB	BI	T	I/P
PJ	557	0.459471	31.64	3.63
FLM	301	0.461623	26.86	3.61
SH3	265	0.464767	26.50	3.32
M2	265	0.464759	30.03	3.32
AMK1	257	0.469977	21.79	2.81



(a) PJ (b) FLM (c) SH3

(d) M2 (e) AMK1

Figure 3: Basins of attraction to iterative methods for $f_3(z) = z^4 - \frac{5}{4}z^2 + \frac{1}{4}$.

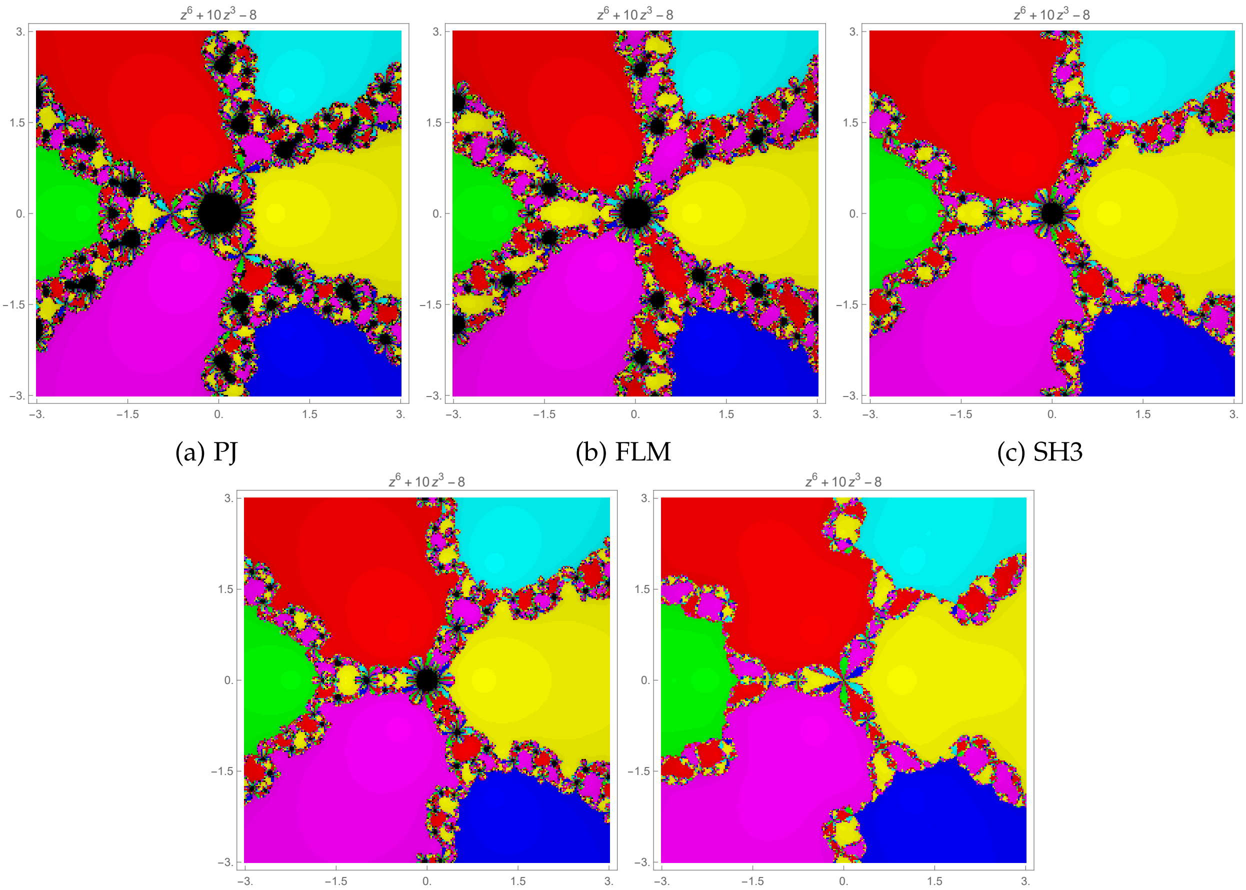
Example 3.4. We consider the nonlinear equation

$$f_4(z) = z^6 + 10z^3 - 8.$$

The roots are $\{0.906359, -0.45318 + 0.78493I, -0.45318 - 0.78493I, -2.20663, 1.10332 + 1.911I, 1.10332 - 1.911I\}$. We illustrate the efficiency of the new method in complex function, comparing between the new method and some of the known methods of fifth order for solving nonlinear equations from where the number of black points, brightness indicator, and time to obtain the basins of attraction for these the methods. In Table 4 AMK1 is the best methods, also the order of the methods from this example from worst to best and the attractions basins for these roads are Illustrated in Figure 4.

Table 4: Comparison of different methods in complex plane in Example 3.4.

Method	NB	BI	T	I/P
PJ	5385	0.415969	36.89	4.15
FLM	3515	0.430532	34.39	4.30
M2	1186	0.456193	30.14	3.72
SH3	1139	0.457666	28.58	3.66
AMK1	3	0.476154	19.50	2.93



(a) PJ (b) FLM (c) SH3

(d) M2 (e) AMK1

Figure 4: Basins of attraction to iterative methods for $f_4(z) = z^6 + 10z^3 - 8$.

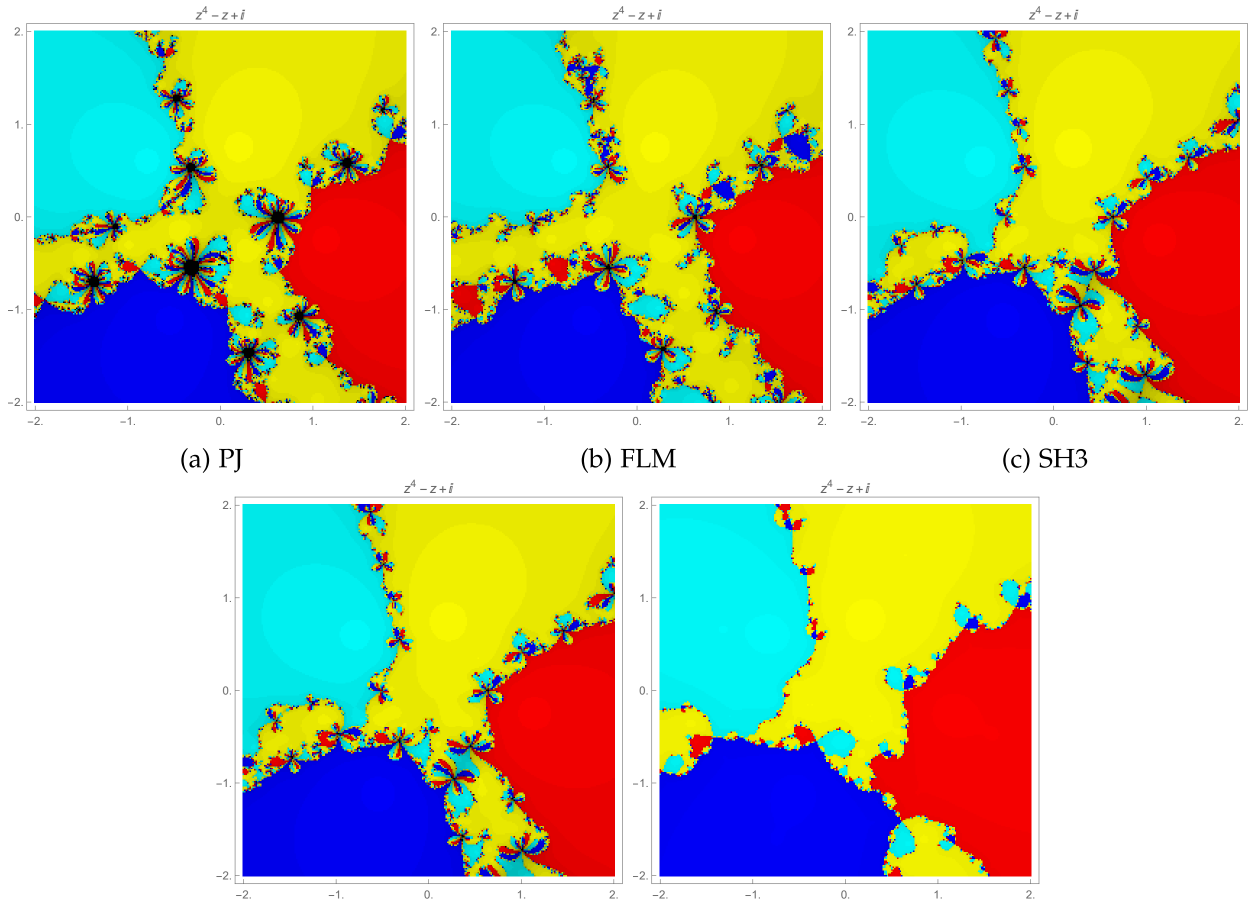
Example 3.5. We consider the nonlinear equation

$$f_5(z) = z^4 - z + I.$$

Its roots are $\{-0.759845 + 0.592595i, -0.532605 - 1.08829i, 0.181924 + 0.732098i, 1.11052 - 0.236405i\}$. We compared the new iterative method AMK1 with four different methods such as PJ in [11], FLM in [7], SH3 in [14], and M2 in [10], the new method AMK1 has less number of black points and higher brightness. Results are given in Table 5, from worst to best. The basins of attraction for this methods are given in Figure 5.

Table 5: Comparison of different methods in complex plane in Example 3.5.

Method	NB	BI	T	I/P
PJ	807	0.476169	33.34	3.84
FLM	176	0.48627	29.67	3.73
SH3	71	0.483391	30.86	3.29
M2	61	0.484638	27.67	3.28
AMK1	0	0.503473	15.97	2.50



(a) PJ (b) FLM (c) SH3

(d) M2 (e) AMK1

Figure 5: Basins of attraction to iterative methods for $f_5(z) = z^4 - z + I$.

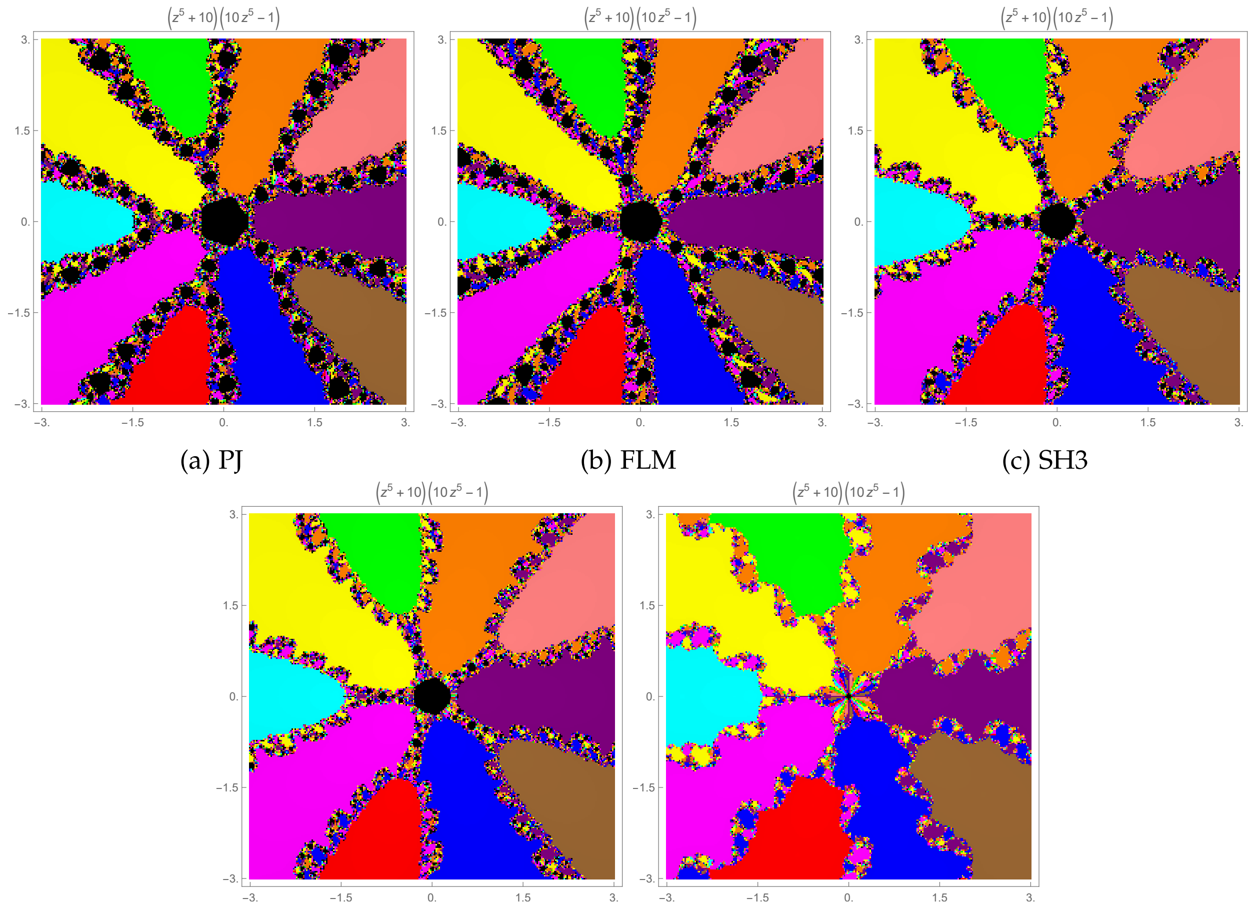
Example 3.6. We consider the nonlinear equation

$$f_6(z) = \left(z^5 + 10\right)\left(10z^5 - 1\right).$$

It has roots as $\{-1.58489, -0.51046 - 0.37087i, -0.51046 + 0.37087i, -0.48976 - 1.50732i, -0.48976 + 1.50732i, 0.19498 - 0.60008i, 0.19498 + 0.60008i, 0.63096, 1.28221 - 0.93158i, 1.28221 + 0.93158i\}$. We compared the new iterative method AMK1 with four different methods, where the new method AMK1 has less number of black points and higher brightness. Results are given in Table 6, from worst to best. The basins of attraction for this methods are given in Figure 6.

Table 6: Comparison of different methods in complex plane in Example 3.6.

Method	NB	BI	T	I/P
PJ	10208	0.413022	97.52	5.09
FLM	9145	0.419687	91.75	5.31
SH3	3912	0.460927	79.69	4.81
M2	3433	0.464424	78.58	4.93
AMK1	223	0.488489	41.68	3.85



(a) PJ (b) FLM (c) SH3

(d) M2 (e) AMK1

Figure 6: Basins of attraction to iterative methods for $f_6(z) = \left(z^5 + 10\right)\left(10z^5 - 1\right)$.

4. Numerical testing

In this section, we compared the new iterative method with some fifth-order iterative methods, we will compared it by using two different ways: the first way comparing using some numerical examples and the second way comparing using the asymptotic constant.

4.1. Comparing using some the numerical examples

We present five numerical examples in real domain to illustrate efficiency of the new fifth-order iterative method described as AMK1, comparing it with some fifth-order iterative methods such as FLM method in [7], M2 method in [10], PJ method in [11], and SH3 method in [14]. We performed all the calculations in Table 7 by using the Mathematica 11 with 128 significant digits, where 10 digits are displayed for x_n, we use the number of iteration $n = 4$ and with precision $\varepsilon = 10^{-50}$. The stopping criterion is used in computer program as $|f(x_{n+1})| < \varepsilon$. In Table 7, we calculated computation of the iterative process error $|x_4 - x_3|$, we calculated $|f(x_4)|$ and (COC) the computational order of convergence, as it is approximated using the following formula:

$$\mathrm{COC} \approx \frac{\ln |(x_{n+1} - x_n)\,/\,(x_n - x_{n-1})|}{\ln |(x_n - x_{n-1})\,/\,(x_{n-1} - x_{n-2})|}.$$

Now we will give the numerical examples for numerical testing and comparing between the methods.

$$f_1(x) = x^3 + 4x^2 - 10, \quad x_0 = 1, \qquad\qquad f_2(x) = \sin^2 x - x^2 + 1, \quad x_0 = 2,$$

$f_3(x) = x^2 - e^{-x} - 3x + 2, \quad x_0 = 2.5, \qquad f_4(x) = \cos x - x, \; x_0 = 1.7,$
$f_5(x) = x^3 - 10, \quad x_0 = 2.5.$

Table 7: Comparing of the iterative methods over some the examples in real domain.

Method	x_0	x_4	COC	$\|x_4 - x_3\|$	$\|f(x_4)\|$	Time
f_1	1					
PJ		1.36523001341409	5.00	2.695×10^{-62}	$0. \times 10^{-125}$	0.0016
FLM		1.36523001341409	5.00	1.502×10^{-67}	$0. \times 10^{-125}$	0.0009
SH3		1.36523001341409	5.00	7.118×10^{-79}	$0. \times 10^{-124}$	0.0014
M2		1.36523001341409	5.00	7.118×10^{-79}	$0. \times 10^{-125}$	0.0008
AMK1		1.36523001341409	4.99	2.739×10^{-99}	$0. \times 10^{-125}$	0.0013
f_2	2					
PJ		1.40449164821534	4.99	1.143×10^{-51}	$0. \times 10^{-125}$	0.0023
FLM		1.40449164821534	4.99	9.483×10^{-52}	$0. \times 10^{-125}$	0.0021
SH3		1.40449164821534	4.99	7.941×10^{-56}	$0. \times 10^{-124}$	0.0023
M2		1.40449164821534	4.99	1.867×10^{-56}	$0. \times 10^{-126}$	0.0022
AMK1		1.40449164821534	4.99	4.490×10^{-79}	$0. \times 10^{-125}$	0.0019
f_3	2.5					
PJ		2.10935699557101	4.99	3.658×10^{-70}	$0. \times 10^{-122}$	0.0015
FLM		2.10935699557101	4.99	1.764×10^{-70}	$0. \times 10^{-122}$	0.0020
SH3		2.10935699557101	4.99	7.429×10^{-77}	$0. \times 10^{-118}$	0.0015
M2		2.10935699557101	4.99	7.107×10^{-77}	$0. \times 10^{-122}$	0.0061
AMK1		2.10935699557101	4.99	2.892×10^{-111}	$0. \times 10^{-122}$	0.0014
f_4	1.7					
PJ		0.73908513321516	4.99	1.108×10^{-106}	$0. \times 10^{-126}$	0.0013
FLM		0.73908513321516	4.99	2.798×10^{-107}	$0. \times 10^{-126}$	0.0017
SH3		0.73908513321516	4.99	3.890×10^{-94}	$0. \times 10^{-125}$	0.0017
M2		0.73908513321516	4.99	2.418×10^{-96}	$0. \times 10^{-126}$	0.0026
AMK1		0.73908513321516	4.99	8.927×10^{-118}	0×10^{-126}	0.0014
f_5	2.5					
PJ		2.15443469003188	4.99	3.716×10^{-91}	$0. \times 10^{-125}$	0.0008
FLM		2.15443469003188	4.99	7.574×10^{-92}	$0. \times 10^{-125}$	0.0012
SH3		2.15443469003188	4.99	2.966×10^{-102}	$0. \times 10^{-124}$	0.0006
M2		2.15443469003188	4.99	2.966×10^{-102}	$0. \times 10^{-125}$	0.0011
AMK1		2.15443469003188	5.00	1.4×10^{-124}	$0. \times 10^{-125}$	0.0006

From the results that's in Table 7 we notice that new method was comparable with all methods, where it gives the best result in the examples that mentioned above.

4.2. *Comparing using asymptotic constant*

We will compare the methods by using asymptotic constant c_2^4, where The asymptotic constant affects the speed of convergence but not to the extent of the order in [2]. In the Table 8, we compare all the methods that mentioned above with our suggested method in terms of the asymptotic error constant c_2^4, where the error equations are given to all methods and we will arrange the methods from the worst to the best.

Table 8: Comparison between asymptotic error constant c_2^4.

Method	approximation	Co c_2^4
PJ	$\left(4c_2^4 - c_2^2c_3\right)e_n^5 + O\left(e_n^6\right)$	4
FLM	$\left(\frac{7}{2}c_2^4 - c_2^2c_3\right)e_n^5 + O\left(e_n^6\right)$	3.5
SH3	$\left(2c_2^4 - 2c_2^2c_3\right)e_n^5 + O\left(e_n^6\right)$	2
M2	$\left(2c_2^4 - 2c_2^2c_3 + \frac{2}{9}c_2c_4\right)e_n^5 + O\left(e_n^6\right)$	2
AMK1	$-c_2^2c_3e_n^5 + O\left(e_n^6\right)$	0

In the Table 8 the results indicate that whenever the asymptotic error constant is lower, the method will be the better because it affects the velocity of convergence, that is converges faster to zero, so the new method is the better.

5. Conclusions

In the present article, we have given a two-step method of fifth-order for solving nonlinear equations $f(x) = 0$, which contains two evaluations of the function and two evaluations of the first derivative for each step. We compared these methods with the examples in real domain also compared it from the asymptotic error constant c_2^4 in complex domain and also have been illustrated and discussed the basins of attracting in methods. From the comparisons, it has been shown that our supposed method gives the best results.

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