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New oscillation results for higher order nonlinear differential equations with a nonlinear neutral terms

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Abstract

The paper deals with the oscillation of higher order nonlinear differential equations with a nonlinear neutral term. The main results are proved via utilizing an integral criterion as well as a comparison theorem with first-order delay differential equation whose oscillatory properties are known. The proposed theorems improve, extend, and simplify existing ones in the literature. The results are associated with four numerical examples.

Keywords: Oscillation, asymptotic behavior, neutral differential equation, comparison method.

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1. Introduction

Consider the higher order nonlinear differential equations with a nonlinear neutral term of the form

$$\left(r(\theta)\left(\mathcal{U}^{(n-1)}(\theta)\right)^{\alpha}\right)' + b(\theta)x^{\gamma}(\tau(\theta))) + q(\theta)x^{\mu}(\omega(\theta)) = 0,\ \theta \geqslant \theta_0, \tag{1.1}$$

where $n > 0$ is an even natural number, and the neutral part $\mathcal{U}(\theta)$ is defined as $\mathcal{U}(\theta) = x(\theta) - p_1(\theta)x^{\beta}(\rho(\theta))$. A solution of (1.1) is a function $x(\theta)$ which is continuous on $[T_x, \infty)$, $T_x \geqslant \theta_0$, and satisfies (1.1) on $[T_x, \infty)$. The solutions which are vanishing identically in some neighborhood of infinity will be excluded from our consideration. Such a solution is said to be *oscillatory* if it is neither eventually positive nor eventually negative, and to be *nonoscillatory* otherwise.

Throughout the remaining part of the paper, we make use of the following assumptions:

$(\mathcal{A}_1)$ α, β, γ, and μ are quotients of positive odd integers, $\alpha \geqslant 1$;

$(\mathcal{A}_2)$ $b(\theta), p_1(\theta)$, and $q(\theta) \in C([\theta_0, \infty), \mathbb{R}_+)$ and $b(\theta) \not\equiv 0$;

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$(\mathcal{A}_3)$ $\rho, \tau, \omega \in C([\theta_0, \infty), \mathbb{R})$ such that $\rho(\theta) \leqslant \theta$, $\tau(\theta) \leqslant \theta$, $\omega(\theta) \geqslant \theta$ and $\lim_{\theta\to\infty} \rho(\theta) = \infty = \lim_{\theta\to\infty} \tau(\theta)$;
$(\mathcal{A}_4)$ $\delta(\theta) = \rho^{-1}(\tau(\theta)) \leqslant \theta$, $\delta^*(\theta) = \rho^{-1}(\omega(\theta)) \geqslant \theta$ and $\delta(\theta) \to \infty$ as $\theta \to \infty$.

Furthermore, we let

$$\mathcal{R}(\nu, u) = \int_u^\nu \frac{ds}{r^{1/\alpha}(s)} \text{ and } \mathcal{R}^*(\theta, \theta_0) = \int_{\theta_0}^\theta (\theta - s)^{n-2}\mathcal{R}(s, \theta_0)ds \to \infty \text{ as } \theta \to \infty. \tag{1.2}$$

Deviating differential equations are thought to be the best tool for modelling a variety of phenomena in science and engineering. In particular, neutral higher order differential equations (NHDEs) arise in the modelling of electric networks containing lossless transmission lines, vibrating masses attached to elastic bars, in some other variational problems, and to mention a few, see [15–19, 21, 22, 31, 34, 36, 37, 56, 60]. Consequently, a great interest has been paid to the study of the qualitative behaviors of NHDEs, such as the existence property, stability property, asymptotic behaviors, oscillatory properties, [3–6, 8, 11, 31–33, 40, 50, 52, 57, 58].

Despite the growing interest in the study of NHDEs, the oscillation and nonoscillation of solutions to these equations remain an ongoing challenge. Starting from the oscillation theory developed by Sturm in 1836, there has been a substantial literature on the oscillation theory of functional differential equations; see [2, 3, 28, 29, 31, 33, 39, 41, 42] and the references cited therein. In this context, one can figure out that many scholars have studied various generalizations of NHDEs and improved the oscillation conditions by using different methods such as the Riccati transformation technique, integral averaging method, comparison method, and inequality technique. The last decades have witnessed the development of the oscillation theory for second-order as well as third-order nonlinear differential equations; the reader is asked to consult the monograph [7] and papers [4–6, 23–27, 30]. However, a little work has been done on the oscillation of higher order neutral differential equations; see the papers [12–14, 20, 38, 46–49, 51, 55]. The motivation behind this paper was initiated upon reading the paper [48], where the authors examined the asymptotic properties and oscillation of the following even-order neutral differential equations of the form

$$\left(r(\theta)\left(u^{(n-1)}(\theta)\right)^\alpha\right)' + p(\theta)\left(u^{(n-1)}(\theta)\right)^\alpha + q(\theta)y^\alpha(\delta(\theta)) = 0, \tag{1.3}$$

where $n \geqslant 4$ is an even number, $u(\theta) = y(\theta) + c(\theta)y(g(\theta))$, $0 \leqslant c(\theta) \leqslant c_0 < 1$, $g(\theta) \leqslant \theta$, $\delta(\theta) \leqslant \theta$ and α is a ratio of odd natural numbers.

It is worth mentioning that the aforementioned equation (1.3) is a particular case of equation (1.1). However, the obtained results in [48] fail to apply to Eq. (1.1). To the best of the authors' knowledge, moreover, there are no existing results for higher order differential equations involving a nonlinear neutral term, especially of the type (1.1). Inspired by this, the objective of this paper is to study the oscillation problem of (1.1) via comparison with the oscillatory behavior of first order differential equations. The obtained results improve and correlate many of the known oscillation criteria existing in the literature, even for the case of (1.1) with $p_1(\theta) = 0$.

For simplicity in what follows, we define:

$$\psi_1(\theta) = \psi(\theta),\ \psi_{k+1}(\theta) = \psi_k(\psi(\theta)),\ J_1(\theta) = \theta - \psi(\theta),\ J_{k+1}(\theta) = \int_{\psi(\theta)}^\theta J_k(s)ds \text{ for } k = 1, 2, \dots, n-1,$$

and

$$\eta_1(\theta) = \eta(\theta),\ \eta_{k+1}(\theta) = \eta_k(\eta(\theta)),\ I_1(\theta) = \eta(\theta) - \theta,\ I_{k+1}(\theta) = \int_\theta^{\eta(\theta)} I_k(s)ds \text{ for } k = 1, 2, \dots, n-1,$$

where $\eta(\theta), \psi(\theta) \in C([\theta_0, \infty))$. We also set

$$\mathcal{Q}(\theta) = \frac{b(\theta)}{\left(p_1\left(\rho^{-1}(\tau(\theta))\right)\right)^{\frac{\gamma}{\beta}}} = \frac{b(\theta)}{\left(p_1\left(\delta(\theta)\right)\right)^{\frac{\gamma}{\beta}}} \text{ and } \mathcal{P}(\theta) = \frac{q(\theta)}{\left(p_1\left(\rho^{-1}(\omega(\theta))\right)\right)^{\frac{\gamma}{\beta}}} = \frac{q(\theta)}{\left(p_1\left(\delta^*(\theta)\right)\right)^{\frac{\mu}{\beta}}}.$$

2. Some essential lemmas

To prove our main results, we need the following lemmas.

Lemma 2.1. *Let* $b \in C([\theta_0,\infty),\mathbb{R}_+)$, $g \in C([\theta_0,\infty),\mathbb{R})$ *such that* $g(\theta) \to \infty$ *as* $\theta \to \infty$ *and* $f \in C(\mathbb{R},\mathbb{R})$ *such that* $f'(\mathcal{U}) > 0$, $\mathcal{U}f(\mathcal{U}) > 0$ *for* $\mathcal{U} \neq 0$.

(I) *The associated delay differential equation (i.e.,* $g(\theta) \leqslant \theta$*)*

$$\mathcal{U}'(\theta) + b(\theta)f(\mathcal{U}(g(\theta)))) = 0$$

has an eventually positive solution if the first-order delay inequality

$$\mathcal{U}'(\theta) + b(\theta)f(\mathcal{U}(g(\theta)))) \leqslant 0$$

does.

(II) *The associated advanced differential equation (i.e.,* $g(\theta) \geqslant \theta$*)*

$$\mathcal{U}'(\theta) - b(\theta)f(\mathcal{U}(g(\theta)))) = 0$$

has an eventually positive solution if the first-order advanced inequality

$$\mathcal{U}'(\theta) - b(\theta)f(\mathcal{U}(g(\theta)))) \geqslant 0$$

does.

Proof. This Lemma is an extension of known results in [39, 45, Corollary 1] and the proof is an immediate consequence. □

In the sequel, we present the following preliminary lemmas to obtain sign properties of possible nonoscillatory solutions of delayed and/or advanced differential equations. We adopt the method of contradiction to prove the following results:

Lemma 2.2. *Let* (A_1)-(A_4) *and* (1.2) *hold. Then the equation*

$$\left(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha}\right)' + b(\theta)\mathcal{U}^{\gamma}(\tau(\theta)) = 0 \tag{2.1}$$

has no eventually positive solution satisfying $\mathcal{U}(\theta) > 0$ *and* $\mathcal{U}^{(n-1)}(\theta) < 0$ *eventually.*

Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.1) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geqslant \theta_2$. Since $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geqslant \theta_2$, then there exists a $\theta_3 > \theta_2$ such that $r(\theta_3)(\mathcal{U}^{(n-1)}(\theta_3))^{\alpha} = c < 0$ and

$$r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha} \leqslant -c < 0 \text{ for } \theta \geqslant \theta_3,$$

or,

$$\mathcal{U}^{(n-1)}(\theta) \leqslant \left(\frac{-c}{r(\theta)}\right)^{1/\alpha} \quad \text{for } \theta \geqslant \theta_3.$$

Integrating the preceding inequality $(n-1)$ times and then applying the condition (1.2) results in the conclusion that $\lim_{\theta\to\infty} \mathcal{U}(\theta) = -\infty$, which is a contradiction. □

Lemma 2.3. *Let* (A_1)-(A_4) *and* (1.2) *hold. If the delay equation*

$$\mathcal{U}'(\theta) + Cb(\theta)\left(\frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))}\right)^{\gamma}\mathcal{U}^{\frac{\gamma}{\alpha}}(\tau(\theta)) = 0 \tag{2.2}$$

is oscillatory for any constant $C > 0$, *then* (2.1) *has no eventually positive solution satisfying* $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) > 0$ *and* $\mathcal{U}^{(n-1)}(\theta) > 0$ *eventually.*

Proof. Suppose that $\mathcal{U}(\theta)$ is a nonoscillatory solution of (2.1) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$ for $\theta \geqslant \theta_2$. Following [29, Lemma 2.2], there exists a constant $\varepsilon \in (0, 1)$ such that

$$\mathcal{U}(\tau(\theta)) \geqslant \frac{\varepsilon}{(n-1)!}\tau^{n-1}(\theta)\mathcal{U}^{(n-1)}(\tau(\theta)) \text{ for } \theta \geqslant \theta_1.$$

Setting $\mathcal{W}(\theta) = r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha}$ in the preceding inequality, we get

$$\mathcal{U}(\tau(\theta)) \geqslant \left(\frac{\varepsilon}{(n-1)!}\right)\left(\frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))}\right)\mathcal{W}^{1/\alpha}(\tau(\theta)). \tag{2.3}$$

Using (2.3) in (2.1), we have

$$\mathcal{W}'(\theta) \leqslant -b(\theta)\left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma}\left(\frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))}\right)^{\gamma}\mathcal{W}^{\frac{\gamma}{\alpha}}(\tau(\theta)) = -Cb(\theta)\left(\frac{\tau^{n-1}(\theta)}{r^{1/\alpha}(\tau(\theta))}\right)^{\gamma}\mathcal{W}^{\frac{\gamma}{\alpha}}(\tau(\theta)), \tag{2.4}$$

where $C = \left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma}$. According to [52, Corollary 1], there is a positive solution $\mathcal{W}(\theta)$ of (2.4) with $\lim_{\theta\to\infty}\mathcal{W}(\theta) = 0$, despite the fact that (2.2) is oscillatory. □

Lemma 2.4. *Let* (A_1)-(A_4) *and* (1.2) *hold. Assume that there exists a nondecreasing function* $\eta(\theta) \in C([\theta_0, \infty))$ *such that*

$$\eta(\theta) > \theta \text{ and } \eta_{n-1}(\tau(\theta)) < \theta. \tag{2.5}$$

If the delay equation

$$\mathcal{X}'(\theta) + b(\theta)I^{\gamma}_{n-1}(\tau(\theta))r^{-\frac{\gamma}{\alpha}}(\eta_{n-1}(\tau(\theta)))\mathcal{X}^{\frac{\gamma}{\alpha}}(\eta_{n-1}(\tau(\theta))) = 0 \tag{2.6}$$

is oscillatory, then (2.1) *has no eventually positive solution satisfying* $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) < 0$ *and* $\mathcal{U}^{(n-1)}(\theta) > 0$ *eventually.*

Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.1) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) < 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$ for $\theta \geqslant \theta_2$. This is the case when n is odd, see [48, Lemma 3]. It is not difficult to see that $\mathcal{U}(\theta)$ satisfies

$$(-1)^m\mathcal{U}^m > 0, \ m = 1, 2, \dots, n-1.$$

Consequently,

$$\begin{aligned}-\mathcal{U}^{(n-2)}(\theta) &\geqslant \mathcal{U}^{(n-2)}(\eta(\theta)) - \mathcal{U}^{(n-2)}(\theta)\\ &= \int_{\theta}^{\eta(\theta)}\mathcal{U}^{(n-1)}(s)ds \geqslant (\eta(\theta)-\theta)\mathcal{U}^{(n-1)}(\eta(\theta)) = I_1(\theta)\mathcal{U}^{(n-1)}(\eta(\theta)).\end{aligned}$$

The repeated integrations of the preceding inequality from θ to $\eta(\theta)$ yields

$$\mathcal{U}(\theta) \geqslant I_{n-1}(\theta)\mathcal{U}^{(n-1)}(\eta_{n-1}(\theta)). \tag{2.7}$$

Using (2.7) in (2.1), we have

$$\begin{aligned}\mathcal{X}'(\theta) = (r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha})' &\leqslant -b(\theta)(I_{n-1}(\tau(\theta))\mathcal{U}^{(n-1)}(\eta_{n-1}(\tau(\theta))))^{\gamma}\\ &\leqslant -b(\theta)I^{\gamma}_{n-1}(\tau(\theta))r^{-\frac{\gamma}{\alpha}}(\eta_{n-1}(\tau(\theta)))\mathcal{X}^{\frac{\gamma}{\alpha}}(\eta_{n-1}(\tau(\theta))),\end{aligned}$$

where $\mathcal{X}(\theta) = r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha}$. The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. □

Next, we consider the equation

$$(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha})' = b(\theta)\mathcal{U}^{\gamma}(\tau(\theta)) + q(\theta)\mathcal{U}^{\mu}(\omega(\theta)). \tag{2.8}$$

Lemma 2.5. *Let* (A_1)-(A_4) *and* (1.2) *hold. Assume that there exists a nondecreasing function* $\xi(\theta) \in C([\theta_0, \infty))$ *such that* $\tau(\theta) \leqslant \xi(\theta)$ *for* $\theta \geqslant \theta_0$. *If the delay first order equation*

$$\mathcal{Y}'(\theta) + \left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma} b(\theta)\tau^{(n-2)\gamma}(\theta)\left(\frac{\xi(\theta)-\tau(\theta)}{r^{\frac{1}{\alpha}}(\xi(\theta))}\right)^{\gamma}(\mathcal{Y}(\xi(\theta)))^{\frac{\gamma}{\alpha}} = 0$$

is oscillatory for $\varepsilon \in (0, 1]$, *then* (2.8) *has no eventually positive solution satisfying* $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) > 0$ *and* $\mathcal{U}^{(n-1)}(\theta) < 0$ *eventually.*

Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.8) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$, and $\mathcal{U}(\omega(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geqslant \theta_2$. Following [29, Lemma 2.2], there exists a constant $\varepsilon \in (0, 1)$ such that

$$\mathcal{U}(\tau(\theta)) \geqslant \frac{\varepsilon}{(n-1)!}\tau^{n-2}(\theta)\mathcal{U}^{(n-2)}(\tau(\theta)) \text{ for } \theta \geqslant \theta_1.$$

Using this inequality in (2.8), we have

$$(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha})' \geqslant b(\theta)\mathcal{U}^{\gamma}(\tau(\theta)) \geqslant b(\theta)\left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma}\tau^{(n-2)\gamma}(\theta)\left(\mathcal{U}^{(n-2)}(\tau(\theta))\right)^{\gamma},$$

or

$$(r(\theta)(\mathcal{Z}'(\theta))^{\alpha})' \geqslant \left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma} b(\theta)\tau^{(n-2)\gamma}(\theta)\mathcal{Z}^{\gamma}(\tau(\theta)), \tag{2.9}$$

where $\mathcal{Z}(\theta) = \mathcal{U}^{(n-2)}(\theta)$. Because $\mathcal{Z}'(\theta) = \mathcal{U}^{(n-1)}(\theta)$ is nondecreasing, then we can see that for $\nu \geqslant u \geqslant \theta_1$,

$$\mathcal{Z}(u) - \mathcal{Z}(\nu) \geqslant (\nu - u)(-\mathcal{Z}'(\nu)).$$

Setting $u = \tau(\theta)$ and $\nu = \xi(\theta)$ in the last inequality yields

$$\mathcal{Z}(\tau(\theta)) \geqslant \mathcal{Z}(\tau(\theta)) - \mathcal{Z}(\xi(\theta)) \geqslant (\xi(\theta) - \tau(\theta))(-\mathcal{Z}'(\xi(\theta))).$$

As a result, the inequality (2.9) can be written as

$$(r(\theta)(\mathcal{Z}'(\theta))^{\alpha})' \geqslant \left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma} b(\theta)\tau^{(n-2)\gamma}(\theta)\left(\xi(\theta) - \tau(\theta)\right)^{\gamma}(-\mathcal{Z}'(\xi(\theta)))^{\gamma},$$

which implies that

$$-\mathcal{Y}'(\theta) \geqslant \left(\frac{\varepsilon}{(n-1)!}\right)^{\gamma} b(\theta)\tau^{(n-2)\gamma}(\theta)\left(\frac{(\xi(\theta)-\tau(\theta))}{r^{\frac{1}{\alpha}}(\xi(\theta))}\right)^{\gamma}\mathcal{Y}^{\frac{\gamma}{\alpha}}(\xi(\theta)),$$

where $\mathcal{Y}(\theta) = -r(\theta)(\mathcal{Z}'(\theta))^{\alpha}$. The rest of the proof is similar to the proof of Lemma 2.4 and hence is omitted. □

Lemma 2.6. *Let* (A_1)-(A_4) *and* (1.2) *hold. Assume that there exists a nondecreasing function* $\eta(\theta) \in C([\theta_0, \infty))$ *such that*

$$\eta(\theta) > \theta \text{ and } \eta_{n-2}(\tau(\theta)) < \theta.$$

If the delay equation

$$\mathcal{X}'(\theta) + r^{-\frac{1}{\alpha}}(\theta)\left(\int_{\theta}^{\infty} b(s)ds\right)^{\frac{1}{\alpha}} I_{n-2}^{\frac{\gamma}{\alpha}}(\tau(\theta))\mathcal{X}^{\frac{\gamma}{\alpha}}(\eta_{n-2}(\tau(\theta))) = 0, \tag{2.10}$$

is oscillatory, then (2.8) *has no eventually positive solution satisfying* $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) < 0$ *and* $\mathcal{U}^{(n-1)}(\theta) < 0$ *eventually.*

Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.8) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$, and $\mathcal{U}(\omega(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) < 0$ and $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geqslant \theta_2$. This is the case when n is even, see [48, Lemma 3]. It is easy to see that $\mathcal{U}(\theta)$ satisfies

$$(-1)^m \mathcal{U}^m > 0, \ m = 1, 2, \ldots, n-1.$$

Proceeding exactly as in the proof of Lemma 2.4, we get

$$\mathcal{U}(\theta) \geqslant I_{n-2}(\theta)\mathcal{U}^{(n-2)}(\eta_{n-2}(\theta)). \tag{2.11}$$

Integrating the inequality $(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha})' \geqslant b(\theta)\mathcal{U}^{\gamma}(\tau(\theta))$ from θ to u and letting $u \to \infty$, we obtain

$$-\mathcal{U}^{(n-1)}(\theta) \geqslant \mathcal{U}^{\frac{\gamma}{\alpha}}(\tau(\theta)) r^{\frac{-1}{\alpha}}(\theta) \left(\int_{\theta}^{\infty} b(s) ds \right)^{\frac{1}{\alpha}}.$$

Using (2.11) in the preceding inequality, we get

$$-\mathcal{U}^{(n-1)}(\theta) \geqslant \left(I_{n-2}(\tau(\theta))\mathcal{U}^{(n-2)}(\eta_{n-2}(\tau(\theta))) \right)^{\frac{\gamma}{\alpha}} r^{\frac{-1}{\alpha}}(\theta) \left(\int_{\theta}^{\infty} b(s) ds \right)^{\frac{1}{\alpha}},$$

or

$$-\mathcal{X}'(\theta) \geqslant r^{\frac{-1}{\alpha}}(\theta) \left(\int_{\theta}^{\infty} b(s) ds \right)^{\frac{1}{\alpha}} I_{n-2}^{\frac{\gamma}{\alpha}}(\tau(\theta)) \mathcal{X}^{\frac{\gamma}{\alpha}}(\eta_{n-2}(\tau(\theta))),$$

where $\mathcal{X}(\theta) = \mathcal{U}^{(n-2)}(\theta)$. The rest of the proof is similar to that of Lemma 2.3 and hence is omitted. □

Lemma 2.7. *Let* (A_1)-(A_4) *and* (1.2) *hold. Assume that there exist nondecreasing function* $\psi(\theta) \in C([\theta_0, \infty))$ *such that*

$$\psi(\theta) < \theta, \ \psi_{n-1}(\omega(\theta)) > \theta \text{ and } \delta(\theta) \leqslant \xi(\theta) \leqslant \psi(\theta) \text{ for } \theta \geqslant \theta_0. \tag{2.12}$$

If the advanced equation

$$\mathcal{W}'(\theta) - q(\theta) J_{n-1}^{\mu}(\omega(\theta)) r^{-\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))) \mathcal{W}^{\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))) = 0, \tag{2.13}$$

is oscillatory, then (2.8) *has no eventually positive solution satisfying* $\mathcal{U}(\theta) > 0$, $\mathcal{U}'(\theta) > 0$ *and* $\mathcal{U}^{(n-1)}(\theta) > 0$ *eventually.*

Proof. Let $\mathcal{U}(\theta)$ be a nonoscillatory solution of (2.8) such that $\mathcal{U}(\theta) > 0$, $\mathcal{U}(\tau(\theta)) > 0$, and $\mathcal{U}(\omega(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$ satisfying $\mathcal{U}'(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$ for $\theta \geqslant \theta_2$. Since $\mathcal{U}(\theta) > 0$ and $\mathcal{U}^{(n-1)}(\theta) > 0$, then we only have the case

$$\mathcal{U}^m(\theta) > 0 \text{ for } m = 0, 1, 2, \ldots, n-1.$$

Consequently,

$$\mathcal{U}^{(n-2)}(\theta) \geqslant \mathcal{U}^{(n-2)}(\theta) - \mathcal{U}^{(n-2)}(\psi(\theta)) = \int_{\psi(\theta)}^{\theta} \mathcal{U}^{(n-1)}(s) ds \geqslant J_1(\theta)\mathcal{U}^{(n-1)}(\psi(\theta)).$$

The repeated integrations of this inequality from $\psi(\theta)$ to θ yields

$$\mathcal{U}(\theta) \geqslant J_{n-1}(\theta)\mathcal{U}^{(n-1)}(\psi_{n-1}(\theta)). \tag{2.14}$$

Using (2.14) in the inequality $(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha})' \geqslant q(\theta)\mathcal{U}^{\mu}(\omega(\theta))$, we have

$$\mathcal{W}'(\theta) \geqslant q(\theta) \left(J_{n-1}(\omega(\theta))\mathcal{U}^{(n-1)}(\psi_{n-1}(\omega(\theta))) \right)^{\mu} \geqslant q(\theta) J_{n-1}^{\mu}(\omega(\theta)) r^{-\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))) \mathcal{W}^{\frac{\mu}{\alpha}}(\psi_{n-1}(\omega(\theta))),$$

where $\mathcal{W}(\theta) = r(\theta)(\mathcal{U}^{(n-1)}(\theta))^{\alpha}$. It follows from Lemma 2.1 (II) that the corresponding differential equation (2.13) also has a positive solution, which is a contradiction. This completes the proof. □

3. Oscillation results

We are ready to present our novel comparison theorem, which reduces the oscillation problem of higher order neutral differential equations to a set of first order delayed and/or advanced differential equations.

Theorem 3.1. *Let (A_1)-(A_4) and (1.2) hold. Assume that there exist nondecreasing functions $\xi(\theta)$, $\eta(\theta)$ and $\psi(\theta) \in C([\theta_0, \infty))$ such that*

$$\psi(\theta) < \theta,\ \psi_{n-1}(\delta^*(\theta)) > \theta \text{ and } \delta(\theta) \leqslant \xi(\theta) \leqslant \psi(\theta) \text{ for } \theta \geqslant \theta_0,$$
$$\eta(\theta) > \theta \text{ and } \eta_{n-1}(\tau(\theta)) < \theta.$$

If the delay equations (2.2), (2.6), and (2.8) with $b(\theta)$ and $\tau(\theta)$ replaced by $\mathcal{Q}(\theta)$ and $\delta(\theta)$, respectively and the advanced equation (2.13) with $q(\theta)$ and $\omega(\theta)$ replaced by $\mathcal{P}(\theta)$ and $\delta^*(\theta)$, respectively and (2.10) is oscillatory, then (1.1) is oscillatory.

Proof. On the contrary, assume that $x(\theta)$ is a nonoscillatory solution of (1.1) such that $x(\theta)$, $x(\rho(\theta))$, $x(\tau(\theta))$, and $x(\omega(\theta))$ positive eventually for $\theta \geqslant \theta_1 > \theta_0$. It follows from (1.1) that

$$(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha)' = -b(\theta)x^\gamma(\tau(\theta)) - q(\theta)x^\mu(\omega(\theta)) \leqslant 0.$$

Hence $r(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha$ is nonincreasing and of one sign. That is, there exists a $\theta_2 \geqslant \theta_1$ such that $\mathcal{U}^{(n-1)}(\theta) > 0$ or, $\mathcal{U}^{(n-1)}(\theta) < 0$ for $\theta \geqslant \theta_2$. We shall distinguish the following four cases:

1. $\mathcal{U}(\theta) > 0$, $\mathcal{U}^{(n-1)}(\theta) < 0$;
2. $\mathcal{U}(\theta) > 0$, $\mathcal{U}^{(n-1)}(\theta) > 0$;
3. $\mathcal{U}(\theta) < 0$, $\mathcal{U}^{(n-1)}(\theta) > 0$;
4. $\mathcal{U}(\theta) < 0$, $\mathcal{U}^{(n-1)}(\theta) < 0$.

Case 1. Since $\mathcal{U}^{(n-1)}(\theta)$ is nonincreasing and negative, we apply Lemma 2.2 and by condition (1.2), we conclude that $\lim_{\theta\to\infty} \mathcal{U}(\theta) = -\infty$, which is a contradiction.

Case 2. Clearly, we see that either (i) $\mathcal{U}'(\theta) > 0$ or (ii) $\mathcal{U}'(\theta) < 0$ for $\theta \geqslant \theta_2$.

Subcase (i). From the definition of $\mathcal{U}(\theta)$, we have

$$\mathcal{U}(\theta) = x(\theta) - p_1(\theta)x^\beta(\rho(\theta)) \leqslant x(\theta).$$

Thus, from (1.1) we have

$$(r(\theta)(\mathcal{U}^{(n-1)}(\theta))^\alpha)' + b(\theta)x^\gamma(\tau(\theta)) \leqslant 0.$$

Applying Lemmas 2.1 and 2.3, we see that Eq. (2.2) is nonoscillatory, a contradiction.

Subcase (ii). By applying Lemma 2.4, we see that Eq. (2.6) is nonoscillatory, a contradiction.

After that, we'll have a look at the situation $\mathcal{U}(\theta) < 0$ for $\theta \geqslant \theta_2$. Let

$$z(\theta) = -\mathcal{U}(\theta) = -x(\theta) + p_1(\theta)x^\beta(\rho(\theta)) \leqslant p_1(\theta)x^\beta(\rho(\theta)),$$

which implies that

$$x(\rho(\theta)) \geqslant \left(\frac{z(\theta)}{p_1(\theta)}\right)^{1/\beta} \quad \text{or} \quad x(\theta) \geqslant \left(\frac{z(\rho^{-1}(\theta))}{p_1(\rho^{-1}(\theta))}\right)^{1/\beta}$$

and so,

$$\begin{aligned}(r(\theta)(z^{(n-1)}(\theta))^\alpha)' &= b(\theta)x^\gamma(\tau(\theta)) + q(\theta)x^\mu(\omega(\theta)) \\ &\geqslant \frac{b(\theta)}{(p_1(\rho^{-1}(\tau(\theta))))^{\frac{\gamma}{\beta}}} z^{\frac{\gamma}{\beta}}(\rho^{-1}(\tau(\theta))) + \frac{q(\theta)}{(p_1(\rho^{-1}(\omega(\theta))))^{\frac{\mu}{\beta}}} z^{\frac{\mu}{\beta}}(\rho^{-1}(\omega(\theta))).\end{aligned}$$

Therefore,

$$(r(\theta)(z^{(n-1)}(\theta))^\alpha)' \geqslant \mathcal{Q}(\theta)z^{\frac{\gamma}{\beta}}(\delta(\theta)) + \mathcal{P}(\theta)z^{\frac{\mu}{\beta}}(\delta^*(\theta)). \tag{3.1}$$

Case 3. Clearly, we see that $z^{(n-1)}(\theta) < 0$ and either (i) $z'(\theta) > 0$ or (ii) $z'(\theta) < 0$ for $\theta \geqslant \theta_2$.

Subcase (i). From (3.1), we have

$$(r(\theta)(z^{(n-1)}(\theta))^\alpha)' \geqslant \mathcal{Q}(\theta)z^{\frac{\gamma}{\beta}}(\delta(\theta)).$$

Since $z^{(n-1)}(\theta) < 0$ and $z'(\theta) > 0$, then following the line of proof of Lemma 2.5, we obtain the desired conclusion.

Subcase (ii). In this case, we apply Lemma 2.6 to Eq. (3.1) and get the desired conclusion.

Case 4. It is not difficult to see that $z^{(n-1)}(\theta) > 0$ and $z(\theta) > 0$. Following the line of proof of Lemma 2.7, we obtain the desired conclusion. This completes the proof of the theorem. □

The oscillatory and asymptotic behavior of Eq. (1.1) is addressed in the following theorem.

Theorem 3.2. *In Theorem 3.1, if we exclude the equations (2.6) and (2.10), then every solution $x(\theta)$ of (1.1) is either oscillatory or converges to zero as θ tends to ∞.*

Proof. Let $x(\theta)$ be a nonoscillatory solution of (1.1) with $\lim_{\theta\to\infty} x(\theta) \neq 0$, $x(\theta) > 0$, $x(\rho(\theta)) > 0$, $x(\tau(\theta)) > 0$, and $x(\omega(\theta)) > 0$ for $\theta \geqslant \theta_1 > \theta_0$. In this case we exclude Subcase (ii) of Case 2 and Subcase (ii) of Case 3, that is, following the line of proof of Theorem 3.1, we proof all possible cases expect Subcase (ii) of Case 2 and Subcase (ii) of Case 3. □

Remark 3.3. We may exclude the part of advanced argument of Case 4 of Theorem 3.2 by imposing the condition

$$\int_{\theta_0}^{\infty} \mathcal{Q}(s)(\mathcal{R}^*(\delta(s), \theta_0))^{\frac{\gamma}{\beta}} ds = \infty \tag{3.2}$$

and by replacing the proof of Case 4 of Theorem 3.1 by the following proof.

Clearly, we have

$$r(\theta)(z^{(n-1)}(\theta))^\alpha > 0$$

and so there exists a constant $c > 0$ such that $r(\theta)(z^{(n-1)}(\theta))^\alpha \geqslant c$, or $z^{(n-1)}(\theta) \geqslant \left(\frac{c}{r(\theta)}\right)^{\frac{1}{\alpha}}$. Thus,

$$z(\theta) \geqslant c^{1/\alpha}\mathcal{R}^*(\theta, \theta_2) \text{ for } \theta \geqslant \theta_3 > \theta_2. \tag{3.3}$$

Using (3.3) in (3.1), we have

$$(r(\theta)(z^{(n-1)}(\theta))^\alpha)' \geqslant \mathcal{Q}(\theta)z^{\frac{\gamma}{\beta}}(\delta(\theta)) \geqslant \mathcal{Q}(\theta)\left(c^{1/\alpha}\mathcal{R}^*(\theta, \theta_2)\right)^{\frac{\gamma}{\beta}}.$$

The rest of the proof is easy and hence is omitted.

From Theorem 3.2 and Remark 3.3, the following result follows immediately.

Theorem 3.4. *Assume that all the hypotheses of Theorem 3.2 and condition (3.2) hold, then every solution of (1.1) is oscillatory, converges to zero or diverges to infinity as θ tends to ∞.*

It is possible to generate sufficient conditions for oscillation of (1.1) by applying existing oscillation criteria to first order differential equations. The following corollary follows immediately from [41, Theorem 2], [42], and Theorem 3.2.

Corollary 3.5. *Let* (A_1)-(A_4) *and* (1.2) *hold. Assume that there exists a function* $a : [\theta_0, \infty) \to (0, \infty)$ *such that* (2.3) *is satisfied. Furthermore, assume that there exist nondecreasing functions* $\eta(\theta)$, $\xi(\theta)$, $\psi(\theta) \in C([\theta_0, \infty))$ *such that* (2.5) *and* (2.12) *hold. If*

$$\liminf_{\theta\to\infty} \int_{\tau(\theta)}^{\theta} b(s) \left(\frac{\tau^{n-1}(s)}{r^{\frac{1}{\alpha}}(\tau(s))} \right)^{\gamma} ds = \infty, \quad when \ \ \gamma \leqslant \alpha,$$

$$\liminf_{\theta\to\infty} \int_{\xi(\theta)}^{\theta} \mathcal{Q}(s)\delta^{\frac{(n-2)\gamma}{\beta}}(s) \left(\frac{\xi(s)-\delta(s)}{r^{\frac{1}{\alpha}}(\xi(s))} \right)^{\frac{\gamma}{\beta}} ds = \infty, \quad when \ \ \gamma \leqslant \alpha\beta,$$

and

$$\liminf_{\theta\to\infty} \int_{\theta}^{\psi_{n-1}(\delta^*(\theta))} \mathcal{P}(s) J_{n-1}^{\frac{\mu}{\beta}}(\delta^*(s)) r^{-\frac{\mu}{\alpha\beta}}(\psi_{n-1}(\delta^*(s)))ds \begin{cases} > \frac{1}{e}, & when \ \mu = \alpha\beta, \\ = \infty, & when \ \mu < \alpha\beta, \end{cases}$$

hold, then every solution of (1.1) *is oscillatory, or converges to zero as* θ *tends to* ∞.

4. Examples

For the sake of completeness, the following examples are presented.

Example 4.1. *Consider the higher order differential equation*

$$\left(\theta^3 \left(\left(x(\theta) - x^{1/3}(\frac{\theta}{2}) \right)^{(n-1)} \right)^3 \right)' + b(\theta)x^{1/3}(\frac{\theta}{2})) + q(\theta)x^3(2\theta) = 0, \ \theta \geqslant \theta_0, \tag{4.1}$$

where $n \in \mathbb{N}$, $\alpha = 3$, $\beta = \frac{1}{3}$, $r(\theta) = \theta^3$, $\gamma = \frac{1}{3}$, $\mu = 3$, $\rho(\theta) = \frac{\theta}{2}$, $\tau(\theta) = \frac{\theta}{2}$, $\omega(\theta) = 2\theta$, *and* b *and* q *are appropriate positive continuous functions. It is easy to check that all conditions of Theorem* 3.1 *are satisfied and conclude that* (4.1) *is oscillatory.*

Example 4.2. *Consider the higher order differential equation*

$$\left(e^{-\theta} \left(x(\theta) - x^3(\frac{\theta}{2}) \right)' \right)' + \frac{3}{4}x(\frac{\theta}{2}) = 0, \ \theta \geqslant \theta_0,$$

where $\alpha = 1$, $\beta = 3$, $r(\theta) = e^{-\theta}$, $\gamma = 1$, $\mu = 3$, $\rho(\theta) = \frac{\theta}{2}$, $\tau(\theta) = \frac{\theta}{2}$, $p_1(\theta) = 1$, $b(\theta) = \frac{3}{4}$, *and* $q(\theta) = 0$. *It is not difficult to check that all conditions of Theorem* 3.4 *are satisfied. One such solution is* $x(\theta) = e^{\theta}$.

Example 4.3. *Consider the higher order differential equation*

$$\left(e^{-\theta} \left(x(\theta) - \frac{1}{\theta^2}x^3(\frac{\theta}{2}) \right)' \right)' + b(\theta)x(\frac{\theta}{2}) + q(\theta)x(2\theta) = 0, \ \theta \geqslant \theta_0 = 4,$$

where $\alpha = 1$, $\beta = 3$, $\gamma = 1$, $\mu = 1$, $\rho(\theta) = \frac{\theta}{2}$, $\tau(\theta) = \frac{\theta}{2}$, $\omega(\theta) = 2\theta$, $r(\theta) = e^{-\theta}$, $p_1(\theta) = \frac{1}{\theta^2}$, $b(\theta) = \frac{4e^{\theta}-24}{96e^{\theta}}$, *and* $q(\theta) = \frac{20e^{\theta}-72}{96\theta}$. *It is not difficult to check that all conditions of Theorem* 3.4 *are satisfied. One such solution is* $x(\theta) = \theta e^{2\theta}$.

Example 4.4. *Consider a second order differential equation*

$$\left(e^{-t} \left((x(t) - e^{-2\pi}x(t-2\pi))' \right)^{\alpha} \right)' + e^{3t-6\pi}x^3(\tau(t))) + e^{3t+9\pi}x^3(\omega(t)) = 0, \ t \geqslant t_0 = 3\pi, \tag{4.2}$$

where $\alpha = 5$, $\beta = 1$, $\gamma = \mu = 3$, $\rho(\theta) = \theta - 2\pi$, $\tau(\theta) = \theta - 3\pi$, $\omega(\theta) = \theta + 2\pi$, $r(t) = e^{-t}$, $p_1(t) = e^{-2\pi}$, $b(t) = e^{3t-6\pi}$, $q(\theta) = e^{3\theta+9\pi}$. *Clearly,* $\delta(\theta) = \rho^{-1}(\tau(\theta)) = \theta - \pi \leqslant \theta$ *and* $\delta^*(\theta) = \rho^{-1}(\omega(\theta)) = \theta + 4\pi \geqslant t$. *A straight forward verification shows that all the conditions of Corollary* 3.5 *hold. Therefore, every solution of* (4.2) *is oscillatory, or converge to zero. Indeed,* $x(\theta) = e^{-\theta}\sin\theta$ *is such a solution of* (4.2).

5. Concluding remarks and open problems

In the present work, with the help of a comparison utility with the behavior of first order delay and/or advanced differential equations as well as an integral criterion, several results for the oscillation and asymptotic behavior of solutions of equation (1.1) are presented. The main results are formulated by the Theorems 3.1, 3.2, and 3.4. For this purpose, five auxiliary lemmas are given. As an application of the main results, a Corollary 3.5 as well as some examples are then presented. The examples are illustrative and support the theoretical results. Article [48] is concerned with the asymptotic behavior and oscillation of solutions to even order neutral differential equations, which is a topic very close to our investigations. Our obtained theorems not only generalize the existing results in the literature but also can be used to plan future research papers in a variety of directions. For example:

(Q1) One can consider equation (1.1) with

$$\mathcal{U}(\theta) = x(\theta) - p_1(\theta)x^{\beta_1}(\rho_1(\theta)) + p_2(\theta)x^{\beta_2}(\rho_2(\theta))$$

with $\rho_1(\theta) \leqslant \theta$ and $\rho_2(\theta) \leqslant \theta$.

(Q2) One can consider equation (1.1) under the case

$$\lim_{\theta\to\infty} \mathcal{R}(\theta, \theta_0) = \lim_{\theta\to\infty} \int_{\theta_0}^{\theta} \frac{ds}{r^{1/\alpha}(s)} < \infty.$$

(Q3) It would be of interest to extend the results of this paper for higher order equations of type (1.1), where $n \geqslant 3$ is an odd natural number.

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References

[1] H. A. Abd El-Hamid, H. M. Rezk, A. M. Ahmed, G. AlNemer, M. Zakarya, H. A. El Saify, *Some dynamic Hilbert-type inequalities for two variables on time scales*, J. Inequal. Appl., **2021** 2021, 21 pages.

[2] R. P. Agarwal, M. Bohner, T. X. Li, C. H. Zhang, *Oscillation of second-order differential equations with a sublinear neutral term*, Carpathian J. Math., **30** (2014), 1–6. 1

[3] R. P. Agarwal, S. R. Grace, *The oscillation of higher-order differential equations with deviating arguments*, Comput. Math. Appl., **38** (1999), 185–199. 1

[4] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation criteria for certain nth order differential equations with deviating arguments*, J. Math. Anal. Appl., **262** (2001), 601–622. 1

[5] R. P. Agarwal, S. R. Grace, D. O'Regan, *The oscillation of certain higher-order functional differential equations*, Math. Comput. Modelling, **37** (2003), 705–728.

[6] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation of certain fourth order functional differential equations*, Ukrainian Math. J., **59** (2007), 315–342. 1

[7] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation Theory for Difference and Functional Difference Equations*, Springer Science & Business Media, Berlin/Heidelberg, (2013). 1

[8] P. Agarwal, A.-A. Hyder, M. Zakarya, *Well-posedness of Stochastic modified Kawahara equation*, Adv. Difference Equ., **2020** (2020), 10 pages. 1

[9] G. AlNemer, M. Zakarya, H. A. Abd El-Hamid, M. R. Kenawy, H. M. Rezk, *Dynamic Hardy-type inequalities with non-conjugate parameters*, Alex. Eng. J., **59** (2020), 4523–4532.

[10] B. Baculiková, J. Džurina, T. X. Li, *Oscillation results for even-order quasilinear neutral functional differential equations*, Electron. J. Differ. Equ., **2011** (2011), 9 pages.

[11] M. Bartušek, M. Cecchi, Z. Došlá, M. Marini, *Asymptotics for higher order differential equations with a middle term*, J. Math. Anal. Appl., **388** (2012), 1130–1140. 1

[12] O. Bazighifan, R. A. El-Nabulsi, *Different techniques for studying oscillatory behavior of solution of differential equations*, Rocky Mountain J. Math., **51** (2021), 77–86. 1

[13] O. Bazighifan, O. Moaaz, R. A. El-Nabulsi, A. Muhib, *Some new oscillation results for fourth-order neutral differential equations with delay argument*, Symmetry, **12** (2020), 10 pages.
[14] C. Cesarano, O. Moaaz, B. Qaraad, N. A. Alshehri, S. K. Elagan, M. Zakarya, *New results for oscillation of solutions of odd-order neutral differential equations*, Symmetry, **13** (2021), 12 pages. 1
[15] R. A. El-Nabulsi, *A generalized nonlinear oscillator from non-standard degenerate lagrangians and its consequent hamiltonian formalism*, Proc. Nat. Acad. Sci. India Sect. A, **84** (2014), 563–569. 1
[16] R. A. El-Nabulsi, *Non-standard power-law Lagrangians in classical and quantum dynamics*, Appl. Math. Lett., **43** (2015), 120–127.
[17] R. A. El-Nabulsi, *Complex backward-forward derivative operator in non-local-in-time Lagrangians mechanics*, Qual. Theory Dyn. Syst., **16** (2017), 223–234.
[18] R. A. El-Nabulsi, *Dynamics of pulsatile flows through microtubes from nonlocality*, Mech. Res. Commun., **86** (2017), 18–26.
[19] R. A. El-Nabulsi, *Nonlocal-in-time kinetic energy in nonconservative fractional systems, disordered dynamics, jerk and snap and oscillatory motions in the rotating fluid tube*, Int. J. Non-Linear Mechanics, **93** (2017), 65–81. 1
[20] R. A. El-Nabulsi, O. Moaaz, O. Bazighifan, *New results for oscillatory behavior of fourth-order differential equations*, Symmetry, **12** (2020), 12 pages. 1
[21] H. A. Ghany, Abd-Allah Hyder, M. Zakarya, *Exact solutions of stochastic fractional Korteweg de-Vries equation with conformable derivatives*, Chin. Phys. B, **29** (2020), 1–8. 1
[22] H. A. Ghany, M. Zakarya, *Exact solitary wave solutions for wick-type stochastic $(2+1)$-dimensional coupled KdV equations*, J. Comput. Anal. Appl., **29** (2020), 617–633. 1
[23] S. R. Grace, J. Džurina, I. Jadlovská, T. X. Li, *An improved approach for studying oscillation of second-order neutral delay differential equations*, J. Inequal. Appl., **2018** (2018), 13 pages. 1
[24] S. R. Grace, J. R. Graef, *Oscillatory behavior of second order nonlinear differential equations with a sublinear neutral term*, Math. Model. Anal., **23** (2018), 217–226.
[25] S. R. Grace, J. R. Graef, M. A. El-Beltagy, *On the oscillation of third order delay dynamic equations on time scales*, Comput. Math. Appl., **63** (2012), 775–782.
[26] S. R. Grace, J. R. Graef, I. Jadlovská, *Oscillation criteria for second-order half-linear delay differential equations with mixed neutral terms*, Math. Slovaca, **69** (2019), 1117–1126.
[27] S. R. Grace, J. R. Graef, E. Tunc, *Oscillatory behaviour of second order damped neutral differential equations with distributed deviating arguments*, Miskolc Math. Notes, **18** (2017), 759–769. 1
[28] J. Graef, S. Grace, E. Tunc, *Oscillation criteria for even-order differential equations with unbounded neutral coefficients and distributed deviating arguments*, Funct. Differ. Equ., **25** (2018), 143–153. 1
[29] J. R. Graef, S. R. Grace, E. Tunc, *Oscillatory behaviour of even-order nonlinear differential equations with a sublinear neutral term*, Opuscula Math., **39** (2019), 39–47. 1, 2, 2
[30] S. R. Grace, I. Jadlovska, *Oscillatory behavior of odd order nonlinear differential equations with a nonpositive neutral term*, Dyn. Syst. Appl., **27** (2018), 125–136. 1
[31] J. R. Graef, M. K. Grammatikopoulos, P. W. Spikes, *On the asymptotic behavior of solutions of a second order nonlinear neutral delay differential equation*, J. Math. Anal. Appl., **156** (1991), 23–39. 1
[32] J. R. Graef, C. Qian, B. Yang, *A three point boundary value problem for nonlinear fourth order differential equations*, J. Math. Anal. Appl., **287** (2003), 217–233.
[33] J. R. Graef, P. W. Spikes, *On the oscillation of an nth-order nonlinear neutral delay differential equation*, J. Comput. Appl. Math., **41** (1992), 35–40. 1
[34] G. Gupta, M. Massoudi, *Flow of a generalized second grade fluid between heated plates*, Acta Mechanica, **99** (1993), 21–33. 1
[35] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, Cambridge, (1934).
[36] T. Hayat, Y. Wang, A. M. Siddiqui, K. Hutter, S. Asghar, *Peristaltic transport of a third-order fluid in a circular cylindrical tube*, Math. Models Methods Appl. Sci., **12** (2002), 1691–1706. 1
[37] D. B. Ingham, B. Yan, *Fluid flow induced by a small amplitude harmonically oscillating cascade*, Acta Mechanica, **91** (1992), 27–46. 1
[38] K. Kamo, H. Usami, *Nonlinear oscillations of fourth Order quasilinear ordinary Differential equations*, Acta Math. Hungar., **132** (2011), 207–222. 1
[39] I. T. Kiguradze, *On the oscillatory character of solutions of the equation* $dmu/dtm + a(t)|u|n \text{ sign } u = 0$, (Russian) Mat. Sb. (N.S.), **65** (1964), 172–187. 1, 2
[40] M. M. Kipnis, D. A. Komissarova, *A note on explicit stability conditions for autonomous higher order difference equations*, J. Difference Equ. Appl., **13** (2007), 457–461. 1
[41] Y. Kitamura, T. Kusano, *Oscillation of first order nonlinear differential equations with deviating arguments*, Proc. Amer. Math. Soc., **78** (1980), 64–68. 1, 3
[42] R. G. Koplatadze, T. A. Chanturiya, *Oscillating and monotone solutions of first-order differential equations with deviating argument*, (Russian) Differentsial'nye Uravneniya, **18** (1982), 1463–1465. 1, 3
[43] W. N. Li, *Oscillation of higher order delay differential equations of neutral type*, Georgian Math. J., **7** (2000), 347–353.
[44] T.-X. Li, Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Appl. Math. Lett., **61** (2016), 35–41.

[45] T. X. Li, Y. V. Rogovchenko, *Oscillation criteria for second–order superlinear Emden–Fowler neutral differential equations*, Monatsh. Math., **184** (2017), 489–500. 2

[46] O. Moaaz, R. A. El-Nabulsi, O. Bazighifan, *Behavior of non-oscillatory solutions of fourth-order neutral differential equations*, Symmetry, **12** (2020), 12 pages. 1

[47] O. Moaaz, R. A. El-Nabulsi, O. Bazighifan, *Oscillatory behavior of fourth-order differential equations with neutral delay*, Symmetry, **12** (2020), 8 pages.

[48] O. Moaaz, R. A. El-Nabulsi, O. Bazighifan, A. Muhib, *New comparison theorems for the even-order neutral delay differential equation*, Symmetry, **12** (2020), 11 pages. 1, 1, 2, 2, 5

[49] O. Moaaz, R. A. El-Nabulsi, A. Muhib, S. K. Elagan, M. Zakarya, *New improved results for oscillation of fourth-order neutral differential equations*, Mathematics, **9** (2021), 12 pages. 1

[50] M. Naito, F. Wu, *A note on the existence and asymptotic behavior of nonoscillatory solutions of fourth order quasilinear differential equations*, Acta Math. Hungar., **102** (2004), 177–202. 1

[51] N. Parhi, A. K. Tripathy, *On oscillatory fourth order nonlinear neutral differential equations-I*, Math. Slovaca, **54** (2004), 389–410. 1

[52] C. G. Philos, *On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays*, Arch. Math. (Basel), **36** (1981), 168–178. 1, 2

[53] H. M. Rezk, G. AlNemer, H. A. Abd El-Hamid, A. H Abdel-Aty, K. S. Nisar, M. Zakarya, *Hilbert-type inequalities for time scale nabla calculus*, Adv. Difference Equ., **2020** (2020), 21 pages.

[54] Y. Sun, Z. Han, S. Sun, C. Zhang, *Oscillation criteria for even order nonlinear neutral differential equations*, Electron. J. Qual. Theory Differ. Equ., **2012** (2012), 12 pages.

[55] P. G. Wang, W. Y. Shi, *Oscillatory theorems of a class of even-order neutral equations*, Appl. Math. Lett., **16** (2003), 1011–1018. 1

[56] Y. Wang, W. Wu, *Unsteady flow of a fourth-grade fluid due to an oscillating plate*, Int. J. Non-Linear Mechanics, **42** (2007), 432–441. 1

[57] W. M. Xiong, G. G. Yue, *Almost periodic solutions for a class of fourth-order nonlinear differential equations with a deviating argument*, Comput. Math. Appl., **60** (2010), 1184–1190. 1

[58] J. L. Yao, F. W. Meng, *Asymptotic behavior of solutions of certain higher order nonlinear difference equation*, J. Comput. Appl. Math., **205** (2007), 640–650. 1

[59] A. Zafer, *Oscillation criteria for even order neutral differential equations*, Appl. Math. Lett., **11** (1998), 21–25.

[60] M. Zakarya, *Hypercomplex Systems and Non-Gaussian Stochastic solutions of χ- wick type $(3+1)$-dimensional modified Benjamin-Bona-Mahony Equation*, Thermal Science, **24** (2020), 209–223. 1

[61] Q. Zhang, J. Yan, L. Gao, *Oscillation behaviour of even-order nonlinear neutral differential equations with variable coefficients*, Comput. Math. Appl., **59** (2010), 426–430.