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On bi-topological BCK-algebras

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Abstract

In this paper, we present the concept of bi-topological BCK-algebra. Several characterizations and properties of this concept are obtained. Also, the concept of BCK-ideal of a BCK-algebra is defined and some of its properties are found.

Keywords: BCK-algebra, bi-topological BCK-algebra, BCK-ideal.

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1. introduction

Throughout this paper, we use the standard topological notation and terminology, mainly as in [5]. By A or (A, Ω), we denote a topological space, while (A, Ω_1, Ω_2) denotes a bitopological space [10] (called also bispace), that is a set S equipped with two (in general, unrelated) topologies. The closure and interior of a subset S of a space (A, Ω) are denoted by $\mathrm{Cl}(S)$ and $\mathrm{Int}(S)$, respectively. When (A, Ω_1, Ω_2) is a bitopological space and $S \subseteq A$, then $\mathrm{Cl}_i(S)$ and $\mathrm{Int}_i(S)$, $i = 1, 2$, denote the closure and interior in the space (A, Ω_i). Imai and Iséki [7, 8], introduced the concept of algebras of type $(2, 0)$ called BCK-algebras which generalizes the concept of the algebra of sets with the set subtraction as the only essential and also it is a generalization of implication algebra. Many researchers have combined the concepts of topological spaces with algebras, and they studied the properties of algebras after they are equipped with a specific topology, which they called topological algebras. Alo and Deeba [1] 1996 introduced the concept of topological BCK-algebra and in 1998 Lee and Ryu [11] gave more properties and characterizations of topological BCK-algebras. In 1999 Jun et al. [9] introduced topological BCI-algebras, provided some properties of this structure, and characterized a topological BCI-algebra in terms of neighborhoods. Gonzaga [6] in 2019, introduced the concept of a topological B-algebra which characterized a topological B-algebra concerning open sets.

In 2017, Mehrshad and Golzarpoor [12] presented some properties of uniform topology and topological BE-algebras. In 2019, Satirad and Iampan [14], introduced the concept of topological UP-algebras and they obtained several properties of this concept.

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A recent study on topological B-algebras was investigated by Belleza and Vilela in [2] in 2020, which characterized a topological B-algebra and investigated several properties of B-ideal in a topological B-algebra.

This paper provides a study of a BCK-algebra when it is equipped with two topologies, we call it a bi-topological BCK-algebra. It can be considered as an extension of the concept of topological BCK-algebra which was introduced by Alo and Deeba [1] also, generalization of some of the results in Lee and Ryu [11].

Characterizations and properties of bi-topological BCK-algebra are investigated, and the BCK-ideals in a bi-topological BCK-algebra are studied.

2. Preliminaries

In this section, we give the basic notions of BCK-algebras and investigate the concept of a topological BCK-algebra. For further information, on BCK-algebras we refer to [13].

Definition 2.1. By a BCK-algebra we mean an algebra $(A, \cdot, 0)$ satisfying the following axioms: for every $a, b, c \in A$,

1. $((a \cdot b) \cdot (a \cdot c)) \cdot (c \cdot b) = 0$;
2. $(a \cdot (a \cdot b)) \cdot b = 0$;
3. $a \cdot a = 0$;
4. $a \cdot b = 0$ and $b \cdot a = 0 \Rightarrow a = b$;
5. $0 \cdot a = 0$.

In a BCK-algebra $(A, \cdot, 0)$, we define a partial order relation $(\leqslant)$ by $a \leqslant b$ if and only if $a \cdot b = 0$.

From the definition of BCK-algebras we can get the following properties very easily see [4, Proposition 5.1.3].

Proposition 2.2. *In a BCK-algebra A, the following statements are true for all $a, b, c \in A$:*

1. $a \cdot 0 = a$;
2. $a \cdot b \leqslant a$;
3. $(a \cdot b) \cdot c = (a \cdot c) \cdot b$;
4. $a \leqslant b \Rightarrow a \cdot c \leqslant b \cdot c$ *and* $c \cdot b \leqslant c \cdot a$;
5. $a \cdot (a \cdot (a \cdot b)) = a \cdot b$.

Definition 2.3 ([4, 3])**.** A nonempty subset I of a BCK-algebra $(A, \cdot, 0)$ is called an ideal of A if the following two conditions are satisfied:

1. $0 \in I$.
2. For all $a \in A$ and for all $b \in I$. If $a \cdot b \in I$, then $a \in I$.

If there is an element 1 of A satisfying $x \leqslant 1$, for all $a \in A$, then the element 1 is called unit of A. A BCK-algebra with unit is called a bounded BCK-algebra [4].

Definition 2.4 ([15])**.** A BCK-algebra $(A, \cdot, 0)$ is called negative implicative if $a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$ for all $a, b, c \in A$.

Definition 2.5 ([1])**.** A BCK-algebra A equipped with a topology Ω is called a topological BCK-algebra (for short TBCK-algebra) if $f : A \times A \to A$ defined by $f(x, y) = x \cdot y$ is continuous for all $(x, y) \in A \times A$ where $A \times A$ has the product topology. Equivalently, for each open set O containing $x \cdot y$, there exist open sets U and V containing x and y respectively such that $U \cdot V \subseteq O$.

Definition 2.6 ([4])**.** Let A be a BCK-algebra and $a \in A$ be a fixed element. The right map $R_a : A \to A$ is a map defined by $R_a(x) = x \star a$ for all $x \in A$.

Definition 2.7 ([11]). A BCK-algebra A equipped with a topology Ω is called a topological BCK-algebra (for short TBCK-algebra) if the operation $\cdot := f : A \times A \to A$ is topologicaly continuous i.e., the inverse image $f^{-1}(O)$ of each open set O containing $x \cdot y$ is open in the product space $A \times A$.

Lemma 2.8 ([1]). *In a TBCK-algebra A,*

1. *If $\{0\}$ is open, then A is discrete.*
2. *$\{0\}$ is closed if and only if A is T_2.*

3. (i, j)-topological BCK-algebras

In this section we introduce the concept of (i, j)-topological BCK-algebras (where $i, j = 1, 2$, $i \neq j$) and establish some of its properties. First, we introduce the following definitions:

Definition 3.1. A function $f : (A, \Omega_1, \Omega_2) \to (A, \Omega_1, \Omega_2)$ is called (i, j)-continuous at an element $x \in A$ if for every Ω_j-open set U containing $f(x)$, there exists a Ω_i-open set V containing x such that $f(V) \subseteq U$. f is said to be (i, j)-open if the image of each Ω_j-open set is Ω_i-open. It is called (i, j)-homoeomorphism if it is a bijection, (i, j)-continuous and (i, j)-open.

Definition 3.2. Let (A, Ω_1, Ω_2) be a bi-topological space. A function $f : A \times A \to A$ defined by $f(x, y) = x \times y$ for all $x, y \in A$ is called (i, j)-continuous if for each Ω_j-open set G containing $x \times y$, there exist two Ω_i-open sets U and V containing x and y, respectively, such that $U \times V \subseteq G$ for $i, j \in \{1, 2\}$. Equivalently, $f : A \times A \to A$ is (i, j)-continuous if and only if $f^{-1}(G)$ is open in the product space $(A, \Omega_i) \times (A, \Omega_i)$ for every Ω_j-open set G.

Definition 3.3. A BCK-algebra A equipped with two topologies Ω_1, Ω_2 is called a (i, j)-topological BCK-algebra (for short (i, j)-BCK-algebra) if $f : A \times A \to A$ defined by $f(x, y) = x \cdot y$ is (i, j)-continuous for all $x, y \in A$ and $i, j \in \{1, 2\}$.

The following example shows that a (i, j)-BCK-algebra may not be either Ω_i-BCK-algebra or Ω_j-BCK-algebra.

Example 3.4. Let $A = \{0, a, b, c\}$ and $\cdot$ be defined as in the following Cayley diagram:

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table 1: A (i, j)-BCK-algebra which is not Ω_i-BCK-algebra.

Then it can be easily checked that $(A, 0, \cdot)$ is a BCK-algebra. Consider the topology

$$\Omega_1 = \{\phi, \{0\}, \{a, b\}, \{c\}, \{0, c\}, \{a, b, c\}, \{0, a, b\}, A\}, \quad \text{and} \quad \Omega_2 = \{\phi, \{0, a, b\}, A\}.$$

Then A is a $(1, 2)$-BCK-algebra which is neither Ω_1-BCK-algebra nor Ω_2-BCK-algebra.

In Example 3.4, suppose that Ω_1=$\{\phi, \{0, a\}, \{b\}, \{c\}, \{b, c\}, \{0, a, b\}, \{0, a, c\}, A\}$ and Ω_2=$\{\phi, \{a\}, \{0, b, c\}, A\}$. Then A is both a Ω_1-BCK-algebra and a Ω_2-BCK-algebra. But it is not a $(1, 2)$-BCK-algebra because $a \cdot c = a$ and $\{0, a\} \cdot \{c\} \not\subseteq \{a\}$. Also, it is not a $(2, 1)$-BCK-algebra because $b \cdot c = b$ and $\{0, b, c\} \cdot \{0, b, c\} \not\subseteq \{b\}$.

The proof of the following theorem follows directly from Definition 3.3.

Theorem 3.5. *A BCK-algebra A is an (i, j)-BCK-algebra if and only if for all $x.y \in A$ and every Ω_j-open set G containing $x \cdot y$, there exist Ω_i-open sets U and V containing x and y respectively such that $U \cdot V \subseteq G$.*

Proposition 3.6. *In any* (i, j)*-BCK-algebra* A*. If* $\{0\}$ *is* Ω_j*-open, then* (A, Ω_i) *is a discrete space.*

Proof. For every $x \in A$, we have $x \cdot x = 0$ and hence, there exist some Ω_i-open set U and V containing x such that $U \cdot V = \{0\}$ because $\{0\}$ is Ω_j-open. Let $O = U \cap V$. Then O is a Ω_i-open sets and $O \cdot O = \{0\}$. This implies that $O = \{x\}$. Therefore, (A, Ω_i) is a discrete space. □

Proposition 3.7. *In any* (i, j)*-BCK-algebra* A*. If* $\{0\}$ *is* Ω_j*-closed, then* (A, Ω_i) *is a Hausdorff space.*

Proof. Let x and y be any two distinct points in A. Then either $x \cdot y \neq 0$ or $y \cdot x \neq 0$. Suppose $x \cdot y \neq 0$, Then $x \cdot y \in A \setminus \{0\}$ and by hypothesis $A \setminus \{0\}$ is Ω_j-open, so there exist some Ω_i-open sets U and V containing x and y respectively such that $U \cdot V =\subseteq A \setminus \{0\}$. Obviously, $U \cap V = \phi$ because if there exists some $c \in U \cap V$, then we get $0 = c \cdot c \subseteq A \setminus \{0\}$ which is contradiction. Thus, (A, Ω_i) is a Hausdorff space. □

Corollary 3.8. *If* A *is a finite* (i, j)*-BCK-algebra and* $\{0\} \in \Omega_j \cup (\Omega_j)^c$*, then* (A, Ω_i) *is a discrete space.*

Proof. The proof follows from Propositions 3.6, 3.7 and the fact that a finite Hausdorff space is discrete. □

Proposition 3.9. *Let* A *be a BCK-ideal of an* (i, j)*-BCK-algebra* A*. If* $0 \in \text{Int}_j(A)$*, then* A *is* Ω_i*-open.*

Proof. For every $x \in A$, we have $x \cdot x = 0$ and since $0 \in \text{Int}_j(A)$, so there exists a Ω_j-open set U such that $x \cdot x = 0 \in U \subseteq A$. Since A is a (i, j)-BCK-algebra, there exist Ω_i-open sets V and W of x such that $V \cdot W \subseteq U \subseteq A$. Now for each $y \in V$, we have $x \cdot y \in A$ and since $x \in A$, so $y \in A$ because A is a BCK-ideal implies $x \in V \subseteq A$. Hence, A is Ω_i-open. □

Proposition 3.10. *If* I *is a* Ω_j*-open BCK-ideal of an* (i, j)*- BCK-algebra* A*, then* I *is also* Ω_i*-closed.*

Proof. Suppose I is an Ω_j-open BCK-ideal of a (i, j)- BCK-algebra A. Let $x \in A \setminus I$. Since I is a BCK-ideal of A, so $x \cdot x = 0 \in I$. By Theorem 3.5, there exists a Ω_i-open set $U(x)$ such that $U(x) \cdot U(x) \subseteq I$. We claim that $U(x) \subseteq A \setminus I$. If not, then $U(x) \cap I \neq \phi$. Then there exists $y \in U(x) \cap I$. Hence, for all $z \in U(x)$, we have $z \cdot y \in U(x) \cdot U(x) \subseteq I$. Since $y \in I$ and I is a BCK-ideal, so $z \in I$. Hence, $U(x) \subseteq I$ which implies that $x \in I$, a contradiction. Therefore, $A \setminus I$ is Ω_i-open. Thus, I is Ω_i-closed in A. □

From Proposition 3.9 and Proposition 3.10, we obtain the following result.

Corollary 3.11. *If* I *is a* Ω_j*-open BCK-ideal of a* (i, j)*- BCK-algebra* A*, then* I *is* Ω_i*-clopen.*

Proposition 3.12. *For any subsets* S, R *of an* (i, j)*-BCK-algebra* A*, the following statements are true:*

1. $\text{Cl}_i(S) \cdot \text{Cl}_i(R) \subseteq \text{Cl}_j(S \cdot R)$.
2. *If either* $\text{Cl}_i(S) \cdot \text{Cl}_i(R)$ *or* $S \cdot R$ *is* Ω_j*-closed, then the equality holds.*

Proof. (1) Let $y = a \cdot b \in \text{Cl}_i(S) \cdot \text{Cl}_i(R)$ where $a \in \text{Cl}_i(S)$, $b \in \text{Cl}_i(R)$ and let U be any Ω_j-open set containing y. Since A is a (i, j)-BCK-algebra, so there exist Ω_i-open sets V and W containing a and b respectively such that $V \cdot W \subseteq U$. Also we have $a \in \text{Cl}_i(S)$ implies that $S \cap V \neq \phi$ and $b \in \text{Cl}_i(R)$ implies that $R \cap W \neq \phi$. Suppose that $a_1 \in S \cap V$ and $b_1 \in R \cap W$ implies that $a_1 \cdot b_1 \in V \cdot W$ and hence $a_1 \cdot b_1 \in U$. Also, $a_1 \cdot b_1 \in S \cdot R$ implies that $S \cdot R \cap U \neq \phi$. Thus, $y \in \text{Cl}_j(S \cdot R)$ which implies that $\text{Cl}_i(S) \cdot \text{Cl}_i(R) \subseteq \text{Cl}_j(S \cdot R)$.

(2) Suppose that $\text{Cl}_i(S) \cdot \text{Cl}_i(R)$ is Ω_j-closed, then obviously $S \cdot R \subseteq \text{Cl}_i(S) \cdot \text{Cl}_i(R)$ and hence

$$\text{Cl}_j(S \cdot R) \subseteq \text{Cl}_j(\text{Cl}_i(S) \cdot \text{Cl}_i(R)) = \text{Cl}_i(S) \cdot \text{Cl}_i(R).$$

Hence, by (1) we get the result. The other case is obvious. □

Definition 3.13. Let A be a BCK-algebra and let $a \in A$, then we define the subset κ_a of A as follows:

$$\kappa_a = \{x \in A : x = a \cdot (a \cdot x)\}.$$

Example 3.14. let $A = \{0, 1, 2, 3, 4\}$ and let the operation $\cdot$ be given by Table 2. Then $(A, \cdot, 0)$ is a BCK-

$\star$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	1	1	0	0
4	4	3	2	1	0

Table 2: κ_x subsets of a BCK-algebra.

algebra (see [4, Example 5.1.12]). Also, we have $\kappa_0 = \{0\}$, $\kappa_1 = \{0, 1\}$, $\kappa_2 = \{0, 1, 2\}$, $\kappa_3 = \{0, 1, 3\}$ and $\kappa_4 = A$.

Proposition 3.15. *The following statements are true:*

1. $a, 0 \in \kappa_a$ *for all* $a \in A$.
2. $\kappa_0 = \{0\}$ $\kappa_0 \subseteq \kappa_a$ *for all* $a \in A$.
3. $b \in \kappa_a$ *if and only if* $b \cdot (b \cdot a) = a \cdot (a \cdot b) = b$.
4. *If* $b \in \kappa_a$, *then* $\kappa_b \subseteq \kappa_a$.
5. *If* $b \in \kappa_a$ *and* $a \in \kappa_b$, *then* $a = b$.
6. *If* $c \in \kappa_a \cap \kappa_b$, *then* $\kappa_c \subseteq \kappa_a \cap \kappa_b$.

Proof. (1) and (2) are obvious.

(3) Let $b \in \kappa_a$, then by definition $a \cdot (a \cdot b) = b$. Now,

$$b \cdot (b \cdot a) = b \cdot ((a \cdot (a \cdot b)) \cdot a) = b \cdot ((a \cdot a)(a \cdot b)) = b \cdot 0 = b.$$

The converse part is obvious.

(4) Let $b \in \kappa_a$, then by (3), $b \cdot (b \cdot a) = a \cdot (a \cdot b) = b$. Suppose that $c \in \kappa_b$, then $b \cdot (b \cdot c) = c \cdot (c \cdot b) = c$. Now $c = b \cdot (b \cdot c) = [a \cdot (a \cdot b)][(a \cdot (a \cdot b)) \cdot c] = [a \cdot (a \cdot b)][(a \cdot c)(a \cdot b)] \leqslant a \cdot (a \cdot c)$. Hence, $c \leqslant a \cdot (a \cdot c)$. Also, we have $(a \cdot (a \cdot c)) \cdot c = (a \cdot c)(a \cdot c) = 0$, so $a \cdot (a \cdot c)) \leqslant c$. Therefore, $a \cdot (a \cdot c) = c$ implies that $c \in \kappa_a$ and thus $\kappa_b \subseteq \kappa_a$.

(5) If $b \in \kappa_a$ and $a \in \kappa_b$, then we have $a \cdot (a \cdot b) = b$ and $b \cdot (b \cdot a) = a$. From (3), we have $b \cdot (b \cdot a) = a \cdot (a \cdot b)$, so $a = b$.

(6) Follows from (5). □

Proposition 3.16. *Let A be a BCK-algebra and let $\mathcal{B} = \{\kappa_x : x \in A\}$. Then $\mathcal{B}$ forms a base for a topology on A. This topology is denoted by Ω_κ.*

Proof. Since $x \in \kappa_x$ for all $x \in A$, so $A = \bigcup_{x \in A} \kappa_x$ and from Proposition 3.15 (6), we have if $c \in \kappa_a \cap \kappa_b$, then $\kappa_c \subseteq \kappa_a \cap \kappa_b$. Hence, $\mathcal{B}$ forms a base for a topology on A. □

Proposition 3.17. *The space (A, Ω_κ) is a T_0-space.*

Proof. Let a, b be any two distinct points in A. Then by Proposition 3.15 (5), either $a \notin \kappa_b$ or $b \notin \kappa_a$. Therefore, (A, Ω_κ) is T_0. □

Proposition 3.18. *Let (A, Ω_1, Ω_2) be an (i, j)-BCK-algebra and $a \in A$. If κ_a is Ω_j-open, then the following statements are true:*

1. *For each* $x \in \kappa_a$ *there exist* Ω_i*-open sets* U *and* V *containing* a *and* $a \cdot x$ *respectively such that* $U \cdot V \subseteq \kappa_a$.
2. *For each* $x \in A$*, there exists a* Ω_i*-open set* U *containing* x *such that* $U \cdot U \subseteq \kappa_a$.
3. *There exist* Ω_i*-open sets* U *and* V *containing* a *and* 0*, respectively, such that* $U \cdot V \subseteq \kappa_a$.
4. *There exist* Ω_i*-open sets* U *and* V *containing* 0 *and* a*, respectively, such that* $U \cdot V \subseteq \kappa_a$.

Proof. (1) For each $x \in \kappa_a$, we have $a \cdot (a \cdot x) = x \in \kappa_a$. Since κ_a is Ω_j-open and A is (i, j)-BCK-algebra, the result follows.

(2) Follows from the fact that $x \cdot x = 0 \in \kappa_a$.

(3) Follows from the fact that $a \cdot 0 = a \in \kappa_a$.

(4) Follows from the fact that $0 \cdot a = 0 \in \kappa_a$. □

Theorem 3.19. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra satisfying the condition that* $y = x \cdot (x \cdot y)$ *for all distinct points* $x, y \in A$ *and* $x \neq 0$*, then for any* $0 \neq a \in S$ *and* $S, R \subseteq A$ *the following statements are true:*

1. $a \cdot Cl_i(R) \subseteq Cl_j(a \cdot R)$.
2. $a \cdot Cl_j(R) \supseteq Cl_i(a \cdot R)$.
3. $a \cdot Int_j(R) \subseteq Int_i(a \cdot R)$.
4. $a \cdot Int_i(R) \supseteq Int_j(a \cdot R)$.
5. $S \cdot Int_j(R) \subseteq Int_i(S \cdot R)$.

Proof.

1. Let $y \in a \cdot Cl_i(R)$, then $y = a \cdot b$ where $b \in Cl_i(R)$ and let $U \in \Omega_j$ with $y = a \cdot b \in U$. Since (A, Ω_1, Ω_2) is a (i, j)-BCK-algebra, then there exists $V \in \Omega_i$ with $b \in V$ and $a \cdot V \subseteq U$. Since $b \in Cl_i(R)$, so there is $c \in R \cap V$, thus $a \cdot c \in a \cdot V \subseteq U$. Therefore, $a \cdot c \in (a \cdot R) \cap U$ which implies $y = a \cdot b \in Cl_j(a \cdot R)$. Hence, $a \cdot Cl_i(R) \subseteq Cl_j(a \cdot R)$.
2. Let $c \in Cl_i(a \cdot R)$ we have to show that $c \in a \cdot Cl_j(R)$. Let $U \in \Omega_j$ containing $a \cdot c$. As A is (i, j)-BCK-algebra, so there exists a Ω_i-open set H containing c such that $a \cdot H \subseteq U$. Since $c \in Cl_i(a \cdot R)$, so $(a \cdot R) \cap H \neq \phi$. Let $a \cdot z \in (a \cdot R) \cap H$, then $(a \cdot z) \in H$ implies that $z = a \cdot (a \cdot z) \in a \cdot H \subseteq U$. Therefore, we obtain that $R \cap U \neq \phi$. Hence $a \cdot c \in Cl_j(R)$ which implies that $c \in a \cdot Cl_j(R)$. Therefore, $Cl_i(a \cdot R) \subseteq a \cdot Cl_j(R)$.
3. Let $a \cdot b \in a \cdot Int_j(R)$, then there is an Ω_j-open set O such that $b \in O \subseteq R$. Since $b = a \cdot (a \cdot b) \in O$ and O is Ω_j-open, there is a Ω_i-open set V with $a \cdot b \in V$ and $a \cdot V \subseteq O$. By hypothesis, we have $V = a \cdot (a \cdot V)$, so $V \subseteq a \cdot O$. Thus, $a \cdot b \in V \subseteq a \cdot O \subseteq a \cdot R$. Therefore, $a \cdot b \in Int_i(a \cdot R)$ and hence, $a \cdot Int_j(R) \subseteq Int_i(a \cdot R)$.
4. Let $c \in Int_j(a \cdot R)$, then there is a Ω_j-open set O such that $c \in O \subseteq a \cdot R$, so we can write $c = a \cdot x \in O \subseteq a \cdot R$. Since A is (i, j)-BCK-algebra, then there is a Ω_i-open set V containing x such that $a \cdot V \subseteq O$. Therefore, $x \in V \subseteq a \cdot O \subseteq R$. Hence, $x \in Int_i(R)$ implies that $c \in a \cdot Int_i(R)$. Thus, $a \cdot Int_i(R) \supseteq Int_j(a \cdot R)$.
5. Let $0 \neq a \in S$, then by (2), $a \cdot Int_j(R) \subseteq Int_i(a \cdot R) \subseteq Int_i(S \cdot R)$. Hence, $S \cdot Int_j(R) = \bigcup_{a \in S}(a \cdot Int_j(R)) \subseteq \bigcup_{a \in S} Int_i(a \cdot R) \subseteq Int_i(S \cdot R)$.

□

Corollary 3.20. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra* $a \in A$*, then the following statements are true:*

1. $a \cdot Cl_i(\kappa_a) \subseteq Cl_j(a \cdot \kappa_a)$.
2. $a \cdot Cl_j(\kappa_a) \supseteq Cl_i(a \cdot \kappa_a)$.
3. $a \cdot Int_j(\kappa_a) \subseteq Int_i(a \cdot \kappa_a)$.
4. $a \cdot Int_i(\kappa_a) \supseteq Int_j(a \cdot \kappa_a)$.

Proof. Follows from that fact that $a \cdot (a \cdot x) = x$ for all $x \in \kappa_a$. Thus in Theorem 3.19, if we replace κ_a instead of R the result follows. □

Theorem 3.21. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra and let* $a \in A$*. If* $R \subseteq A$ *satisfying the condition* $x = (x \cdot a) \cdot a$ *for all* $x \in R$*, then the following statements are true:*

1. $Cl_i(R) \cdot a \subseteq Cl_j(R \cdot a)$.
2. $Cl_i(R \cdot a) \supseteq Cl_j(R) \cdot a$.
3. $Int_j(R) \cdot a \subseteq Int_i(R \cdot a)$.
4. $Int_i(R) \cdot a \supseteq Int_j(R \cdot a)$.
5. *If the condition is true for all* $a \in A$*, then* $Int_j(R) \cdot A \subseteq Int_i(R \cdot A)$.

Proof.
1. Let $y \in Cl_i(R) \cdot a$, then $y = b \cdot a$ where $b \in Cl_i(R)$ and let $U \in \Omega_j$ with $y = b \cdot a \in U$. Since (A, Ω_1, Ω_2) is a (i, j)-BCK-algebra, then there exists $V \in \Omega_i$ with $b \in V$ and $V \cdot a \subseteq U$. Since $b \in Cl_i(R)$, so there is $c \in R \cap V$, thus $c \cdot a \in V \cdot a \subseteq U$. Therefore, $c \cdot a \in (R \cdot a) \cap U$ which implies $y = b \cdot a \in Cl_j(R \cdot a)$. Hence, $Cl_i(R) \cdot a \subseteq Cl_j(R \cdot a)$.
2. Let $c \in Cl_i(R \cdot a)$ we have to show that $c \cdot a \in Cl_j(R)$. Let $U \in \Omega_j$ containing $c \cdot a$. As A is (i, j)-BCK-algebra, so there exists a Ω_i-open set H containing c such that $H \cdot a \subseteq U$. Since $c \in Cl_i(R \cdot a)$, so $(R \cdot a) \cap H \neq \phi$. Let $z \cdot a \in (R \cdot a) \cap H$, then $(z \cdot a) \in H$ implies that $z = (z \cdot a) \cdot a \in H \cdot a \subseteq U$. Therefore, we obtain that $R \cap U \neq \phi$. Hence $c \cdot a \in Cl_j(R)$ which implies that $c \in Cl_j(R) \cdot a$. Therefore, $Cl_i(R \cdot a) \subseteq Cl_j(R) \cdot a$.
3. Let $b \cdot a \in Int_j(R) \cdot a$, then there is an Ω_j-open set O such that $b \in O \subseteq R$. Since $b = (b \cdot a) \cdot a \in O$ and O is Ω_j-open, there is a Ω_i-open set V with $a \cdot b \in V$ and $a \cdot V \subseteq O$. By hypothesis, we have $V = (V \cdot a) \cdot a$, so $V \subseteq O \cdot a$. Thus, $b \cdot a \in V \subseteq O \cdot a \subseteq R \cdot a$. Therefore, $b \cdot a \in Int_i(R \cdot a)$ and hence, $Int_j(R) \cdot a \subseteq Int_i(R \cdot a)$.
4. Let $c \in Int_j(R \cdot a)$, then there is a Ω_j-open set O such that $c \in O \subseteq R \cdot a$, so we can write $c = x \cdot a \in O \subseteq R \cdot a$ where $x \in R$. Since A is (i, j)-BCK-algebra, then there is a Ω_i-open set V containing x such that $V \cdot a \subseteq O$. Therefore, $x \in V \subseteq O \cdot a \subseteq R$. Hence, $x \in Int_i(R)$ implies that $c \in Int_i(R) \cdot a$. Thus, $Int_i(R) \cdot a \supseteq Int_j(R \cdot a)$.
5. Let $a \in A$, then by (2), $Int_j(R) \cdot a \subseteq Int_i(R \cdot a) \subseteq Int_i(R \cdot A)$. Hence,

$$Int_j(R) \cdot A = \bigcup_{a \in A} (Int_j(R) \cdot a) \subseteq \bigcup_{a \in A} Int_i(R \cdot a) \subseteq Int_i(R \cdot A).$$

□

Corollary 3.22. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra satisfying the condition that* $y = x \cdot (x \cdot y)$ *for all distinct points* $x, y \in A$ *and* $x \neq 0$*, then for any* $0 \neq a \in A$ *and* $A, R \subseteq A$ *the following statements are true:*

1. *If* R *is* Ω_j*-closed, then* $a \cdot R$ *is* Ω_i*-closed.*
2. *If* R *is* Ω_j*-open, then* $A \cdot R$ *is* Ω_i*-open.*

Proof. The proof follows from Theorem 3.19. □

Corollary 3.23. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra and let* $a \in A$*.If* $R \subseteq A$ *satisfying the condition* $x = (x \cdot a) \cdot a$ *for all* $x \in R$*, then the following statements are true:*

1. *If* R *is* Ω_j*-closed, then* $R \cdot a$ *is* Ω_i*-closed.*
2. *If If* R *is* Ω_j*-open, then* $R \cdot A$ *is* Ω_i*-open.*

Proof. The proof follows from Theorem 3.21. □

Theorem 3.24. *Let* (A, Ω_1, Ω_2) *be a* (i, j)*-BCK-algebra and let* $a \in A$*. If* $R \subseteq A$ *satisfying the condition that* $x = a \cdot (a \cdot x)$ *for all* $x \in R$*, then the following statements are true:*

1. *The left map* $l_a : A \to A$ *defined by* $l_a(x) = a \cdot x$*, is an* (i, j)*-homeomorphism of* A *onto* A*.*
2. *For any elements* x, y *in* A *such that* $y = a \cdot x$*, there exists an* (i, j)*-homeomorphism* f *of* A *onto itself such that* $f(y) = x$*.*

Proof. 1. Let $x, y \in A$ and $l_a(x) = l_a(y)$, then $a \cdot x = a \cdot y$ implies that $a \cdot (a \cdot x) = a \cdot (a \cdot y)$ and hence $x = y$. Therefore, l_a is one-to-one. For every $x \in A$, $a \cdot x \in A$, thus $l_a(a \cdot x) = a \cdot (a \cdot x) = x$. Hence l_a is onto. Let O be a Ω_j-open set, then $l_a(O) = a \cdot O$. By Theorem 3.19 (3), $a \cdot O$ is Ω_i-open. Hence l_a is (i, j)-open. Let $x \in A$ and O be any Ω_j-open set containing $l_a(x)$, then by Theorem 3.19 (3), $a \cdot O$ is Ω_i-open. Since $l_a(x) = a \cdot x \in O$. Hence, $x \in a \cdot O$ and $l_a(a \cdot O) \subseteq O$. Hence, l_a is (i, j)-continuous. Thus, l_a is an (i, j)-homeomorphism.

2. Let $x, y \in A$, then the function $f = l_a : A \to A$ is (i, j)-homeomorphism, and

$$l_a(y) = (a \cdot y) = a \cdot (a \cdot x) = x.$$

□

Theorem 3.25. *Let (A, Ω_1, Ω_2) be a (i, j)-BCK-algebra and let $a \in A$.If $R \subseteq A$ satisfying the condition $x = (x \cdot a) \cdot a$ for all $x \in R$, then the following statements are true:*

1. *The right map $r_a : A \to A$ defined by $r_a(x) = x \cdot a$, is an (i, j)-homeomorphism of A onto A.*
2. *For every element $x \in A$, there exists a (i, j)-homeomorphism f of A onto itself such that $f(a) = x$.*

Proof. 1. Let $x, y \in A$ and $r_a(x) = r_a(y)$, then $x \cdot a = y \cdot a$ implies that $(x \cdot a) \cdot a = (y \cdot a) \cdot a\cdot$ and hence $x = y$. Therefore, r_a is one-to-one. For every $x \in A$, $x \cdot a \in A$, thus $r_a(x \cdot a) = (x \cdot a) \cdot a = x$. Hence r_a is onto. To prove that r_a is (i, j)-open, let O be a Ω_j-open set in A, then $r_a(O) = O \cdot a$. By Theorem 3.21 (3), $O \cdot a$ is Ω_i-open. Hence r_a is (i, j)-open. Let $x \in A$ and let O be any Ω_j-open set containing $l_a(x)$, then by Theorem 3.21 (3), $O \cdot a$ is Ω_i-open. Since $l_a(x) = x \cdot a \in O$. Hence, $x \in O \cdot a$ and $r_a(a \cdot O) \subseteq O$. Hence, r_a is (i, j)-continuous. Thus, r_a is an (i, j)-homeomorphism.

2. Let $x \in A$, we define $f = r_{x \cdot a} : A \to A$ as above, then $r_{x \cdot a}$ is (i, j)-homeomorphism, and $r_{x \cdot a}(a) = (x \cdot a) \cdot a = x$.

□

Theorem 3.26. *Let (A, Ω_1, Ω_2) be an (i, j)-BCK-algebra and let S be a BCK-subalgebra satisfying the condition $y = x \cdot (x \cdot y)$ for all $x, y \in S$. If S is Ω_j-open, then $\mathrm{Int}_i(S)$ is also a BCK-subalgebra.*

Proof. Let $x, y \in \mathrm{Int}_i(S)$, then by hypothesis, S is a Ω_j-open set containing y. Since S is closed under the operation $(\cdot)$, so $(x) \cdot S \subseteq S$. Hence, by Theorem 3.19 (3), $x \cdot S$ is Ω_i-open and $x \cdot y \in x \cdot S \subseteq S$. Hence, $x \cdot y \in \mathrm{Int}_i(S)$. Therefore, $\mathrm{Int}_i(S)$ is closed under the operation $(\cdot)$. □

Proposition 3.27. *If (A, Ω_1, Ω_2) is a negative implicative BCK-algebra. If S is a BCK-subalgebra, then $x \cdot S$ is also a BCK-subalgebra.*

Proof. Let $x \cdot a, x \cdot b \in (x \cdot S)$, then obviously, $a, b \in S$ and since S is closed, so $a \cdot b \in S$. Therefore, $x \cdot (a \cdot b) \in x \cdot S$. Hence, by hypothesis, $x \cdot (a \cdot b) = (x \cdot a) \cdot (x \cdot b) \in x \cdot S$. Thus $x \cdot S$ is also a BCK-subalgebra. □

Proposition 3.28. *Let A be an (i, j)BCK-algebra and $\phi \neq W \in \Omega_j$, then the following statements are true:*

1. *If $x \in W$, then there exists a Ω_i-open set U containing 0 such that $x \cdot U \subseteq W$.*
2. *If $0 \in W$, then there exists a Ω_i-open set U containing x such that $U \cdot U \subseteq W$.*
3. *If $0 \in W$, then there exist two Ω_i-open sets U and V containing 0 and x respectively such that $(U \cdot V) \subseteq W$.*
4. *If $0 \in W$, then for each $x, y \in A$ there exist two a Ω_i-open sets U and V containing $y \cdot x$ and y respectively such that $U \cdot V \subseteq W$.*

Proof. 1. Obvious.

2. Let $0 \in W$ and $x \in A$. Since $x \cdot x = 0 \in W$ and A is (i, j)-BCK-algebra, then there exist two Ω_i-open sets G and H containing x such that $G \cdot H \subseteq W$. Suppose that $U = G \cap H$, then U is a Ω_i-open set containing x. Hence, $U \cdot U \subseteq W$.

3. Let $0 \in W$ and $x \in A$. Since $0 \cdot x = 0$, and A is (i,j)-BCK-algebra, then there exist Ω_i-open sets G, H containing 0 and A such that $G \cdot H \subseteq W$.
4. Let $0 \in W$ and $x, y \in A$. Since $x \cdot y \leqslant x$ and A is (i,j)-BCK-algebra, then there exist a Ω_i-open set U containing $x \cdot y$, and a Ω_i-open set G containing x such that $U \cdot G \subseteq W$.

□

Proposition 3.29. *Let A be an (i,j)-BCK-algebra and U_0 be the least Ω_i and Ω_j-open set containing 0. If $x \in U_0$, then U_0 is the least Ω_j-open set containing x.*

Proof. Let $x \in U_0$ and N be any Ω_j-open in A which contains x. By Definition 2.1, we have $x \cdot 0 = x \in N$. By Theorem 3.5, there exist Ω_i-open sets N_x and N_0 such that $N_x \cdot N_0 \subseteq N$. Since N_0 is a Ω_i-open set containing 0, it follows from assumption and Proposition 2.2 that $0 = x \cdot x \in N_x \cdot U_0 \subseteq N_x \cdot N_0 \subseteq N$. Therefore, N is a Ω_j-open set containing 0. By assumption, we have $U_0 \subseteq N$. Hence, U_0 is the least Ω_j-open set containing A. □

Proposition 3.30. *In every (i,j)-BCK-algebra (A, Ω_1, Ω_2), the following statements are true:*

1. *If (A, Ω_j) is T_0, then (A, Ω_i) is T_1.*
2. *If (A, Ω_j) is T_1, then (A, Ω_i) is T_2.*
3. *If (A, Ω_j) is T_2, then (A, Ω_i) is T_0.*

Proof. (1): Suppose that (A, Ω_j) is T_0 and let $x, y \in A$ such that $x \neq y$. Thus we have either $x \cdot y \neq 0$ or $y \cdot x \neq 0$ without loss of generality, assume that $x \cdot y \neq 0$, so we have two cases:

Case 1: There exists a Ω_j-open set W containing $x \cdot y$ but not 0. Since A is (i,j)-BCK-algebra, then there exist two Ω_i-open sets U and V containing x and y respectively such that $U \cdot V \subseteq W$. Since $0 \notin W$ so that $0 \notin U \cdot V$. Hence, $y \notin U$ and $x \notin V$.

Case 2: There exists Ω_j-open W containing 0 but not $x \cdot y$. Since $x \cdot x = 0$, $y \cdot y = 0$ and A is (i,j)-BCK-algebra, then by Proposition 3.28 there exists Ω_i-open set U containing x such that $U \cdot U \subseteq W$. Also, there exists Ω_i-open set V containing y such that $V \cdot V \subseteq W$. Obviously, $y \notin U$ and $x \notin V$. Therefore, (A, Ω_i) is T_1.

(2): Suppose that (A, Ω_j) is T_1 then $\{0\}$ is Ω_j- closed. Therefore, by Proposition 3.7, (A, Ω_i) is T_2. (3): Obvious. □

The converse of the above proposition is not true in general, for this if we take Ω_i is a discrete space and Ω_j is any space which is not T_0 on a BCK-algebra A, then (A, Ω_1, Ω_2) is an (i,j)-BCK-algebra.

Proposition 3.31. *Let A be an (i,j)-BCK-algebra and U_0 be the least Ω_i and Ω_j-open set containing 0. If $x \in U_0$, then U_0 is the least Ω_j-open set containing x.*

Proof. Let $x \in U_0$ and N be any Ω_j-open in A which contains x. By Definition 2.1, we have $x \cdot 0 = x \in N$. By Theorem 3.5, there exist Ω_i-open sets N_x and N_0 such that $N_x \cdot N_0 \subseteq N$. Since N_0 is a Ω_i-open set containing 0, it follows from assumption and Proposition 2.2 that $0 = x \cdot x \in N_x \cdot U_0 \subseteq N_x \cdot N_0 \subseteq N$. Therefore, N is a Ω_j-open set containing 0. By assumption, we have $U_0 \subseteq N$. Hence, U_0 is the least Ω_j-open set containing x. □

4. Conclusion

In this paper we extended the concept of topological BCK-algebra to a bitopological BCK-algebra. We proved some properties of this concept and gave illustrative examples when they are needed. Some relations linked with bitopological BCK-algebras to separation axioms and homeomorphisms are investigated.

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