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Oscillation of 3^{r}d-order advanced difference equations with a negative middle term

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Abstract

We discuss the situation when all solutions of the half-linear 3rd-order advanced difference equation with a negative middle term are oscillatory. We provided sufficient conditions for all solutions of the studied equation to be oscillatory which are different from the existing results. Examples are presented to illustrate the new results.

Keywords: 3rd-order difference equation, negative middle term, oscillation, advanced argument.

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1. Introduction

Here we investigate the oscillatory behavior of the 3rd-order advanced difference equations

$$\Delta(c(\ell)\Delta(d(\ell)(\Delta x(\ell))^{\alpha})) - a(\ell)(\Delta x(\ell+1))^{\alpha} - b(\ell)g(x(\sigma(\ell))) = 0, \quad \ell \geqslant \ell_0, \tag{1.1}$$

where ℓ_0 is a positive integer and α is a quotient of odd positive integers. We consider the following conditions to provide the results:

(H_1) $\{c(\ell)\}$,$\{d(\ell)\}$ and $\{b(\ell)\}$ are positive real sequences and $\{a(\ell)\}$ is a nonnegative real sequence for all $\ell \geqslant \ell_0$;
(H_2) $\{\sigma(\ell)\}$ is a nondecreasing sequence of integers such that $\sigma(\ell) \geqslant \ell+1$ for all $\ell \geqslant \ell_0$;
(H_3) $mg(m) > 0$, g is nondecreasing for $m \neq 0$, and $-g(-mn) \geqslant g(mn) \geqslant g(m)g(n)$ for $mn > 0$.

By a solution of (1.1), we mean a nontrivial real sequence $\{x(\ell)\}$ which satisfies (1.1) for all $\ell \geqslant \ell_0$. A nontrivial solution $\{x(\ell)\}$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Oscillatory phenomena take part in different models from real world applications; we refer to the papers [15, 23] for models form mathematical biology where oscillation and/or delay actions may be

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formulated by means of cross-diffusion terms. Half-linear equations have numerous applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous medium, and so forth; see, for instance, the papers [9–11, 20] for more details and the papers [9–11, 14, 21, 22] regarding the oscillation of half-linear equations. Recently, the investigation of oscillatory properties of solutions of various classes of difference equations experienced long-term interest to many researches, see for example [1–5, 7, 8, 12, 16–19, 24–26] and the references cited therein. In particular 3rd-order difference equations have very useful applications in hydrodynamics, population dynamics and physics, see for example [1, 6, 19].

In [5, 7, 24, 26] the authors used comparison method and summation averaging technique to establish the oscillation of all solutions of trinomial 3rd-order difference equation

$$\Delta(c(\ell)\Delta(d(\ell)(\Delta x(\ell))^{\alpha}))+a(\ell)(\Delta x(\ell+1))^{\alpha}+b(\ell)g(x(\sigma(\ell)))=0,\quad \ell\geqslant\ell_0, \tag{1.2}$$

where $\{a(\ell)\}$ and $\{b(\ell)\}$ are positive sequences and the auxiliary equation

$$\Delta(c(\ell)\Delta z(\ell))+\frac{a(\ell)}{d(\ell+1)}z(\ell+1)=0,$$

is nonoscillatory.

In [8], the authors studied the asymptotic properties of (1.2) when $\{a(\ell)\}$ is negative and $\{b(\ell)\}$ is positive. They presented several comparison theorems in which the desired properties of solutions are deduced from those of corresponding 1^{st}-order delay difference equations. Their results, however, strongly depend on the knowledge of the auxiliary solution $\{z(\ell)\}$.

In a very recent work [25], the authors discussed the asymptotic and oscillatory behaviour of (1.1) when $d(\ell)\equiv 1$ and $\sigma(\ell)\leqslant\ell-1$. By the conditions (H_1)–(H_3), we derive the results of (1.1), which are new and complement the ones from [7, 8, 24–26].

2. Auxiliary lemmas

Let us define

$$\begin{aligned}
D_0x(\ell)&=x(\ell),\\
D_1x(\ell)&=d(\ell)(\Delta x(\ell))^{\alpha},\\
D_2x(\ell)&=c(\ell)\Delta(D_1x(\ell)),\\
D_3x(\ell)&=\Delta(D_2x(\ell)),
\end{aligned}$$

for all $\ell\geqslant\ell_0$. With this notation, (1.1) can be rewritten as

$$D_3x(\ell)-\frac{a(\ell)}{d(\ell+1)}D_1x(\ell+1)-b(\ell)g(x(\sigma(\ell)))=0.$$

The functions given below will be used for the easy understanding in this paper:

$$\begin{aligned}
&E_1(\ell,\ell_1)=\sum_{s=\ell_1}^{\ell-1}d^{\frac{-1}{\alpha}}(s),\quad E_2(\ell,\ell_1)=\sum_{s=\ell_1}^{\ell-1}\frac{1}{c(s)},\\
&E_{12}(\ell,\ell_1)=\sum_{s=\ell_1}^{\ell-1}d^{\frac{-1}{\alpha}}(s)E_2^{\frac{1}{\alpha}}(s,\ell_1),\quad A(\ell)=\prod_{s=\ell_1}^{\ell-1}\left(1-\frac{1}{c(s)}\sum_{t=s}^{\infty}\frac{Q(t)}{d(t+1)}\right),\\
&B(\ell)=\sum_{s=\ell}^{\infty}\frac{1}{c(s)}\sum_{t=s}^{\infty}b(t),\quad Q(\ell)=\frac{a(\ell)}{d(\ell+1)}B(\ell+1)+b(\ell),\\
&\bar{Q}(\ell)=\frac{1}{c(\ell)}\sum_{s=\ell}^{\infty}Q(s),
\end{aligned}$$

$$\bar{b}(s,\ell)=\sum_{t=\ell}^{s}\frac{1}{c(t)}\sum_{j=\ell}^{t-1}b(j)g(E_{12}(\sigma(j),\ell_1)),$$
$$E(s,\ell)=\frac{a(s)}{d(s+1)}\bar{b}(s,\ell)+b(s)g(E_{12}(\sigma(s),\ell_1)),$$

for all $s \geqslant \ell \geqslant \ell_1 \geqslant \ell_0$.

Throughout, we assume without further mention that

(H_4) $E_1(\ell,\ell_0)\to\infty$ and $E_2(\ell,\ell_0)\to\infty$ as $\ell\to\infty$,

which implies that the operator $D_3x(\ell)$ is so-called canonical form.

To make the sense of the definitions of $A(\ell)$ and $B(\ell)$, here we consider

(H_5) $\sum_{\ell=\ell_0}^{\infty}\frac{a(\ell)}{d(\ell+1)}<\infty$ and $\sum_{\ell=\ell_0}^{\infty}b(\ell)<\infty$.

In what follows, all difference inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all sufficiently large ℓ. Moreover, in the sequel and without loss of generality, we look over only the positive solutions of (1.1).

The solutions of (1.1) and auxiliary second-order linear ordinary difference equations are closely relevant

$$\Delta(c(\ell)\Delta z(\ell))=\frac{a(\ell)}{d(\ell+1)}z(\ell+1), \tag{2.1}$$

as the following lemma says.

Lemma 2.1. *Let the positive solution of (2.1) be $\{z(\ell)\}$ for all $\ell\geqslant\ell_0$. Then (1.1) can be written in the form*

$$\Delta\left(c(\ell)z(\ell)z(\ell+1)\Delta\left(\frac{d(\ell)}{z(\ell)}(\Delta x(\ell))^{\alpha}\right)\right)=b(\ell)z(\ell+1)g(x(\sigma(\ell))),\quad \ell\geqslant\ell_0.$$

Proof. From a straightforward calculation, we see that

$$\Delta(c(\ell)\Delta(d(\ell)(\Delta x(\ell))^{\alpha}))-a(\ell)(\Delta x(\ell+1))^{\alpha}=\frac{1}{z(\ell+1)}\Delta\left(c(\ell)z(\ell)z(\ell+1)\Delta\left(\frac{d(\ell)}{z(\ell)}(\Delta x(\ell))^{\alpha}\right)\right).$$

The proof is obtained by [25, Lemma 2.1] and so it is omitted. □

In view of [1, Theorem 6.3.4] Equation (2.1) always has a pair of nonoscillatory solutions $\{z(\ell)\}$ such that either

$$z(\ell)\Delta z(\ell)>0,$$

or

$$z(\ell)\Delta z(\ell)<0, \tag{2.2}$$

for all $\ell\geqslant\ell_0$.

The following property of a nonoscillatory solution $\{z(\ell)\}$ satisfying (2.2) to disclose the structure of possible nonoscillatory solutions of (1.1).

Lemma 2.2. *If*

$$\sum_{\ell=\ell_1}^{\infty}\left(\frac{A(\ell)}{d(\ell)}\right)^{\frac{1}{\alpha}}=\infty, \tag{2.3}$$

then (2.1) has a positive solution $\{z(\ell)\}$ satisfying

$$\sum_{\ell=\ell_1}^{\infty}\frac{1}{c(\ell)z(\ell)z(\ell+1)}=\sum_{\ell=\ell_1}^{\infty}\left(\frac{z(\ell)}{d(\ell)}\right)^{\frac{1}{\alpha}}=\infty.$$

Proof. The proof is obtained by [1, Lemma 2.1] and so the details are omitted. Now, we introduce the following classes of nonoscillatory (let us say positive) solutions:

$$x(\ell) \in W_1 \implies x(\ell) > 0,\ D_1x(\ell) > 0,\ D_2x(\ell) < 0,\ D_3x(\ell) > 0,$$

$$x(\ell) \in W_3 \implies x(\ell) > 0,\ D_1x(\ell) > 0,\ D_2x(\ell) > 0,\ D_3x(\ell) > 0,$$

for all $\ell \geqslant \ell_1 \geqslant \ell_0$. □

Lemma 2.3. *Assume that* (2.3) *holds. If* $\{x(\ell)\}$ *is a positive solution of* (1.1) *for all* $\ell \geqslant \ell_0$*, then there exists an integer* ℓ_1 *such that either* $x(\ell) \in W_1$ *or* $x(\ell) \in W_3$ *for all* $\ell \geqslant \ell_1 \geqslant \ell_0$.

Proof. The proof is obtained by [25, Lemma 2.3] and hence the details are omitted. If we assume (2.3), the set W of all positive solutions of (1.1) has the following decomposition:

$$W = W_1 \cup W_3.$$

The results in [1, 2] of oscillation criteria have been commonly established by introducing property A and / or B. Equation (1.1) is said to have property B if $W = W_3$. □

We provided few useful estimates for our main results.

Lemma 2.4. *Let* $x(\ell) \in W_1$ *be a positive solution of* (1.1) *for all* $\ell \geqslant \ell_1$*. Then*

$$\frac{x(\ell)}{E_1(\ell,\ell_1)} \text{ is nonincreasing,}$$

and there exists $\ell_2 > \ell_1$ *such that*

$$D_1x(\ell) \geqslant B(\ell)g(x(\sigma(\ell))), \tag{2.4}$$

for all $\ell \geqslant \ell_2$.

Proof. Assume that $x(\ell) \in W_1$ is a positive solution of (1.1) for all $\ell \geqslant \ell_1$. It follows from the monotonicity of $D_1x(\ell)$ that

$$x(\ell) \geqslant x(\ell) - x(\ell_1) = \sum_{s=\ell_1}^{\ell-1} d^{\frac{-1}{\alpha}}(s)D_1^{\frac{1}{\alpha}}x(s) \geqslant E_1(\ell,\ell_1)D^{\frac{1}{\alpha}}x(\ell).$$

Therefore,

$$\Delta\left(\frac{x(\ell)}{E_1(\ell,\ell_1)}\right) = \frac{E_1(\ell,\ell_1)D_1^{\frac{1}{\alpha}}x(\ell) - x(\ell)}{d^{\frac{1}{\alpha}}(\ell)E_1(\ell,\ell_1)E_1(\ell+1,\ell_1)} \leqslant 0,$$

and so $\frac{x(\ell)}{E_1(\ell,\ell_1)}$ is nonincreasing.

Next, summing (1.1) form ℓ to ∞, yields

$$\begin{aligned} -D_2x(\ell) &\geqslant \sum_{s=\ell}^{\infty} \frac{a(s)}{d(s+1)}D_1x(s+1) + \sum_{s=\ell}^{\infty} b(s)g(x(\sigma(s))) \\ &\geqslant g(x(\sigma(n)))\sum_{s=\ell}^{\infty} b(s). \end{aligned}$$

By repeated summation, we obtain

$$D_1x(\ell) \geqslant \sum_{s=\ell}^{\infty} \frac{g(x(\sigma(s)))}{c(s)} \sum_{t=s}^{\infty} b(t) \geqslant B(\ell)g(x(\sigma(\ell))).$$

The proof is complete. □

Lemma 2.5. *Let $x(\ell) \in W_3$ be a positive solution of (1.1). For all $\ell \geqslant \ell_1 \geqslant \ell_0$. If*

$$\sum_{s=\ell}^{\infty} \frac{a(s)}{d(s)} E_2(s,\ell_1) + b(s)g(E_{12}(\sigma(s),\ell_1)) = \infty, \tag{2.5}$$

then there is an integer $\ell_2 \geqslant \ell_1$ such that

$$\frac{x(\ell)}{E_{12}(\ell,\ell_1)} \text{ is nondecreasing for all } \ell \geqslant \ell_2.$$

Proof. The proof is obtained by [25, Lemma 2.5] and hence the details are skipped. □

We conclude this section by recalling a use full result found in [2].

Lemma 2.6. *If the inequality*

$$\Delta x(\ell) - b(\ell)g(x(\sigma(\ell))) \geqslant 0,$$

has an eventually positive solution, then so does equation

$$\Delta x(\ell) - b(\ell)g(x(\sigma(\ell))) = 0.$$

3. Main results

In this section, we first present sufficient conditions under which (1.1) enjoys property B. Then using these results, we obtain conditions for the oscillation of all solutions of (1.1).

Theorem 3.1. *Let (2.3) hold. If the 1st-order advanced difference equation*

$$\Delta x(\ell) - c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(x(\sigma(\ell+1)) = 0, \tag{3.1}$$

is oscillatory, then (1.1) has property B.

Proof. Let $\{x(\ell)\}$ be a positive solution of (1.1) for all $\ell \geqslant \ell_0$. Then from Lemma 2.3, there exists an integer $\ell_1 \geqslant \ell_0$ such that either $x(\ell) \in W_1$ or $x(\ell) \in W_3$ for all $\ell \geqslant \ell_1$. If $x(\ell) \in W_1$, then by virtue of (1.1) and (2.4), we have

$$D_3x(\ell) \geqslant \left(\frac{a(\ell)}{d(\ell+1)}B(\ell+1) + b(\ell)\right)g(x(\sigma(\ell+1))) = Q(\ell)g(x(\sigma(\ell+1))).$$

Summing up the last inequality from ℓ to ∞, we find

$$-D_2x(\ell) \geqslant \sum_{s=\ell}^{\infty} Q(s)g(x(\sigma(s+1))) \geqslant \left(\sum_{s=\ell}^{\infty} Q(s)\right)g(x(\sigma(\ell+1))).$$

Again summing the above inequality form ℓ to ∞ yields

$$D_1x(\ell) \geqslant \sum_{s=\ell}^{\infty} \frac{g(x(\sigma(s+1)))}{c(s)} \sum_{t=s}^{\infty} Q(t) \geqslant \left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)g(x(\sigma(\ell+1))).$$

Therefore $\{x(\ell)\}$ is a positive solution of the advanced difference inequality

$$\Delta\, x(\ell) - c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(x(\sigma(\ell+1)) \geqslant 0.$$

By Lemma 2.6, we see that (3.1) also has positive solution which is a contradiction. Hence $x(\ell) \in W_3$, which means that (1.1) has property B. Hence proved. □

Corollary 3.2. *Assume that* $g(u) = u^{\alpha}$ *and let* (2.3) *hold. If*

$$\liminf_{\ell\to\infty} \sum_{s=\ell+1}^{\sigma(\ell+1)-1} c^{\frac{-1}{\alpha}}(s)\left(\sum_{t=s}^{\infty} \bar{Q}(t)\right)^{\frac{1}{\alpha}} > \frac{1}{e}, \tag{3.2}$$

then (1.1) *has property B.*

Proof. Using [13, Theorem 2.1] and (3.2), we see that equation (3.1) is oscillatory. Hence by Theorem 3.1 the conclusion follows. □

Corollary 3.3. *Assume that* $g(u) = u^{\beta}$ *where* β *is a ratio of odd positive integers with* $\beta > \alpha$ *and let* (2.3) *hold. If*

$$\sum_{\ell=\ell_1}^{\infty} c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{t=\ell}^{\infty} \bar{Q}(t)\right)^{\frac{1}{\alpha}} = \infty, \tag{3.3}$$

then (1.1) *has property B.*

Proof. Proceeding as in the proof of Theorem 3.1, we obtain

$$\Delta x(\ell) - c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}} x^{\frac{\beta}{\alpha}}(\sigma(\ell+1)) \geqslant 0,$$

or

$$\Delta x(\ell) \geqslant c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}} x^{\frac{\beta}{\alpha}}(\ell+1).$$

Hence

$$\int_{x(\ell)}^{x(\ell+1)} \frac{ds}{s^{\frac{\beta}{\alpha}}} \geqslant \frac{\Delta x(\ell)}{x^{\beta}(\ell+1)} \geqslant c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from n_1 to ∞, we obtain

$$\frac{1}{(\frac{\beta}{\alpha}-1)x^{\frac{\beta}{\alpha}-1}(\ell_1)} \geqslant \sum_{\ell=\ell_1}^{\infty} c^{\frac{-1}{\alpha}}(\ell)\left(\sum_{s=\ell}^{\infty} \bar{Q}(s)\right)^{\frac{1}{\alpha}},$$

which contradicts (3.3). Hence $x(\ell) \in W_3$, which means that (1.1) has property B. Hence proved.

□

In the following theorems we obtain criteria for the oscillation of all solutions of (1.1) by eliminating the remaining class of nonoscillatory solution of (1.1).

Theorem 3.4. *Assume that all conditions of Corollary* 3.2 *are satisfied and* (2.5) *hold. If*

$$\limsup_{\ell\to\infty} \frac{1}{E_{12}(\sigma(\ell),\ell_1)} \sum_{s=\ell_1}^{\sigma(\ell)-1} \left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(t)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}} > 1, \tag{3.4}$$

then (1.1) *is oscillatory.*

Proof. Let $\{x(\ell)\}$ be a positive solution of (1.1) for $\ell \geqslant \ell_0$. If follows from Lemma 2.3 that there exists an integer $\ell_1 \geqslant \ell_0$ such that $x(\ell) \in W_1$ or $x(\ell) \in W_3$ for all $\ell \geqslant \ell_1$. From Corollary 3.2, we see that (1.1) has property B, that is, $x(\ell) \in W_3$. Summing up (1.1) from ℓ to $j-1$, we have

$$D_2x(j) \geqslant D_2x(j) - D_2x(\ell) \geqslant \sum_{s=\ell}^{j-1} \frac{a(s)}{d(s+1)} D_1x(s+1) + \sum_{s=\ell}^{j-1} b(s)g(x(\sigma(s))). \tag{3.5}$$

From Lemma 2.3, there exists an integer $\ell_2 \geqslant \ell_1$ such that $\frac{x(\ell)}{E_{12}(\ell,\ell_1)}$ is nondecreasing for all $\ell \geqslant \ell_2$, and hence

$$D_2x(j) \geqslant \sum_{s=\ell}^{j-1} b(s)g(x(\sigma(s))) \geqslant g\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1} b(s)g(E_{12}(\sigma(s),\ell_1)), \quad \ell \geqslant \ell_2. \tag{3.6}$$

Summing (3.6) in j, one obtains

$$D_1x(j) \geqslant g\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1}\frac{1}{c(s)}\sum_{t=\ell}^{s-1} b(t)g(E_{12}(\sigma(t),\ell_1)). \tag{3.7}$$

Using (3.7) in (3.5), we have

$$\begin{aligned} D_2x(j) &\geqslant g\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1}\frac{a(s)}{d(s+1)}\bar{b}(s,\ell) + \sum_{s=\ell}^{j-1} b(s)g(x(\sigma(s))) \\ &\geqslant g\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1} E(s,\ell). \end{aligned}$$

Summing in j once more, we get

$$D_1x(j) \geqslant g\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1}\frac{1}{c(s)}\sum_{t=\ell}^{s-1} E(t,\ell).$$

Summing the last inequality finally in j we find

$$x(j) \geqslant g^{\frac{1}{\alpha}}\left(\frac{x(\sigma(\ell))}{E_{12}(\sigma(\ell),\ell_1)}\right)\sum_{s=\ell}^{j-1}\left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(s)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}}. \tag{3.8}$$

Setting $j = \sigma(\ell)$ and $g(u) = u^{\alpha}$, we obtain

$$1 \geqslant \frac{1}{E_{12}(\sigma(\ell),\ell_1)}\sum_{s=\ell}^{\sigma(\ell)-1}\left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(s)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}}.$$

Taking the lim sup on both sides of the resulting inequality we are led to the contradiction with (3.4). Thus $W_3 = \phi$ and so (1.1) is oscillatory. Hence proved. □

Theorem 3.5. *Assume that all conditions of Corollary 3.3 are satisfied and (2.5) holds. If*

$$\limsup_{\ell\to\infty}\left(\frac{1}{E_{12}(\sigma(\ell),\ell_1)}\right)^{\frac{\beta}{\alpha}}\sum_{s=\ell}^{\sigma(\ell)-1}\left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(t)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}} > 0, \tag{3.9}$$

then (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.4, we arrive at (3.8). Now setting $j = \sigma(\ell)$ and $g(u) = u^{\beta}$ with $\beta > \alpha$ in (3.8), we obtain

$$\frac{x(\sigma(\ell))}{x^{\frac{\beta}{\alpha}}(\sigma(\ell))} \geqslant \left(\frac{1}{E_{12}(\sigma(\ell),\ell_1)}\right)^{\frac{\beta}{\alpha}}\sum_{s=\ell}^{\sigma(\ell)-1}\left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(t)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}}.$$

Taking lim sup on both sides of the inequality we see that

$$0 \geqslant \limsup_{\ell\to\infty}\left(\frac{1}{E_{12}(\sigma(\ell),\ell_1)}\right)^{\frac{\beta}{\alpha}}\sum_{s=\ell}^{\sigma(\ell)-1}\left(\frac{1}{d(s)}\sum_{t=\ell}^{s-1}\frac{1}{c(t)}\sum_{i=\ell}^{t-1} E(i,\ell)\right)^{\frac{1}{\alpha}},$$

which contradicts (3.9). Thus $W_3 = \phi$ and so (1.1) is oscillatory. Hence proved. □

4. Examples

Example 4.1. Let us take the 3rd-order linear advanced difference equation

$$\Delta^3 x(\ell) - \frac{1}{(\ell+1)(\ell+2)}\Delta x(\ell+1) - \frac{b}{\ell(\ell+1)(\ell+2)}x(2\ell) = 0, \quad \ell \geqslant 1, \tag{4.1}$$

where $b > 0$. Here $c(\ell) = d(\ell) = 1$, $a(\ell) = \dfrac{1}{(\ell+1)(\ell+2)}$, $b(\ell) = \dfrac{b}{\ell(\ell+1)(\ell+2)}$, $\sigma(\ell) = \lambda\ell$, $g(m) = m$ and $\alpha = 1$. A simple calculation shows that conditions (H_1)–(H_5) hold and $A(\ell) = \dfrac{1}{\ell}$ and

$$\bar{Q}(\ell) \geqslant \frac{3b}{4}\frac{1}{(\ell+1)(\ell+2)}.$$

Thus Condition (2.3) holds, and Condition (3.2) becomes

$$\begin{aligned}
\liminf_{\ell\to\infty} \sum_{s=\ell+1}^{2\ell+1}\left(\sum_{t=s}^{\infty}\bar{Q}(t)\right) &\geqslant \liminf_{\ell\to\infty} \sum_{s=\ell+1}^{2\ell+1}\frac{3b}{4}\left(\frac{1}{s+1}\right) \\
&\geqslant \liminf_{\ell\to\infty} \frac{3b}{4}\left(\frac{1}{2\ell+2}\right)(\ell+1) \\
&= \frac{3b}{8} > \frac{1}{e}.
\end{aligned}$$

Hence by Corollary 3.2, Equation (4.1) has property B if $\dfrac{3b}{8} > \dfrac{1}{e}$. Moreover, the condition (2.5) holds. The condition (3.4) clearly holds for $b > 0$. Hence by Theorem 3.4, Equation (4.1) is oscillatory if $b > \dfrac{8}{3e}$.

Example 4.2. Let us take the 3^rd-order advanced difference equation

$$\Delta^2(\ell(\Delta x(\ell))^3) - \frac{1}{\ell+2}(\Delta x(\ell+1))^3 - \frac{1}{\ell(\ell+1)(\ell+2)}x^3(2\ell) = 0, \quad \ell \geqslant 1. \tag{4.2}$$

Here $c(\ell) = 1, d(\ell) = 1, a(\ell) = \dfrac{1}{\ell+2}, b(\ell) = \dfrac{1}{\ell(\ell+1)(\ell+2)}, \sigma(\ell) = 2\ell$ and $f(m) = m^3$. A simple calculation shows that assumptions (H_1)–(H_5) are satisfied, and

$$A(\ell) = \frac{1}{\ell}, \quad \text{and} \quad \bar{Q}(\ell) \geqslant \frac{3}{4(\ell+1)(\ell+2)}.$$

Thus Condition (2.3), which takes the form

$$\sum_{\ell-1}^{\infty}\left(\frac{A(\ell)}{d(\ell)}\right) = \sum_{\ell=1}^{\infty}\frac{1}{n^{\frac{2}{3}}} = \infty,$$

is satisfied. Now Condition (3.2) becomes

$$\begin{aligned}
\liminf_{\ell\to\infty} \sum_{s=\ell+1}^{2\ell+1}\left(\sum_{t=s}^{\infty}\bar{Q}(t)\right)^{\frac{1}{3}} &\geqslant \liminf_{\ell\to\infty}\left(\frac{3}{4}\right)^{\frac{1}{3}} \sum_{s=\ell+1}^{2\ell+1}\frac{1}{(s+1)^{\frac{1}{3}}} \\
&\geqslant \lim_{\ell\to\infty}\left(\frac{3}{8}\right)^{\frac{1}{3}}(\ell+1)^{\frac{2}{3}} = \infty > \frac{1}{e}.
\end{aligned}$$

Therefore by Corollary 3.2, Equation (4.2) has Property B. Moreover $E_{12}(n,1) \approx \ell, \bar{b}(s,\ell) \approx 8(s^2+s-\ell s)$, and $E(s,\ell) \approx 16 + \dfrac{8}{s} - \dfrac{8\ell}{s}$. Now, Condition (3.4) holds. Hence by Theorem 3.4, Equation (4.2) is oscillatory.

5. Conclusion

In [5, 7, 8, 24–26], the authors investigated the oscillatory and asymptotic properties of (1.1) and the present work can be considered as a complement of the earlier results. In contrary to [5, 7, 8, 24, 26], these criteria do not depend on the solutions of auxiliary equation (2.1). The problem is open for researchers to get the solution structure and corresponding oscillatory criteria for the equation

$$\Delta(c(\ell)\Delta(d(\ell)(\Delta x(\ell))^{\alpha})) + a(\ell)(\Delta x(\ell+1))^{\alpha} - b(\ell)g(x(\sigma(\ell)) = 0,$$

where $\sigma(\ell)$ is such that either $\sigma(\ell) \leqslant \ell$ or $\sigma(\ell) \geqslant \ell + 1$.

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