J. Math. Computer Sci., 28 (2023), 326–334

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some essential bi-ideals and essential fuzzy bi-ideals in a semigroup

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Abstract

In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

Keywords: Essential bi-ideals, minimal bi-ideals, essential minimal bi-ideals, essential fuzzy minimal bi-ideals.

2020 MSC: 16Y60, 08A72, 03G25, 03E72.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh in 1965 [8]. These concepts were applied in many areas such as medical science, theoretical physics, robotics, computer science, control engineering, information science, measure theory, logic, set theory, topology etc. In 1979, Kuroki [3] defined a fuzzy semigroup and various kinds of fuzzy ideals in semigroups and characterized them.

Essential fuzzy ideals of ring studied by Medhi et al. in 1971 [4]. Later in 2013, Medhi and Saikia [5] stuided concept T-fuzzy essential ideals and proved properties of T-fuzzy essential ideals.

Recently in 2020, Baupradist et al. [1] studied essential ideals and essential fuzzy ideals in semigroups. Together 0-essential ideals and 0-essential fuzzy ideals in semigroups.

In this paper, we give the concepts of essential bi-ideals and essential fuzzy bi-ideals in semigroups. In the main results, we characterized regular, left regular, intra-regular, semisimple semigroups in terms of essential fuzzy ideals and essential fuzzy bi-ideals in semigroups.

2. Preliminaries

In this section, we give some basic definitions and theorems that we need.

A non-empty subset I of a semigroup S is called a subsemigroup of S if $I^2 \subseteq I$. A non-empty subset I of a semigroup S is called a left (right) ideal of S if $SI \subseteq I$ ($IS \subseteq I$). An ideal I of S is a non-empty subset

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doi: 10.22436/jmcs.028.04.02

Received: 2021-10-04 *Revised:* 2022-04-10 *Accepted:* 2022-06-04

which is both a left ideal and a right ideal of S. A subsemigroup I of a semigroup S is called a bi-ideal of S if ISI $\subseteq$ I. It well-know, every ideal of a semigroup S is a bi-ideal of S. For any $a, b \in [0,1]$, we have

$$a \vee b = \max\{a, b\}, \quad \text{and} \quad a \wedge b = \min\{a, b\}.$$

A fuzzy set of a non-empty set T is function from T into unit closed interval $[0,1]$ of real numbers, i.e., $f : T \to [0,1]$.

For any two fuzzy sets of f and g of a non-empty of T, we defined the support of f instead of $\text{supp}(f) = \{u \in T \mid f(u) \neq 0\}$, $f \subseteq g$ if $f(u) \leqslant g(u)$, $(f \vee g)(u) = \max\{f(u), g(u)\} = f(u) \vee g(u)$ and $(f \wedge g)(u) = \min\{f(u), g(u)\} = f(u) \wedge g(u)$ for all $u \in T$.

For two fuzzy sets f and g in a semigroup S, define the product $f \circ g$ as follows : for all $u \in S$,

$$(f \circ g)(u) = \begin{cases} \bigvee\limits_{(y,z) \in F_u} \{\{f(y) \wedge g(z)\} \mid (y,z) \in F_u\}, & \text{if } F_u \neq \emptyset, \\ 0, & \text{if } F_u = \emptyset, \end{cases}$$

where $F_u := \{(y,z) \in S \times S \mid u = yz\}$.

A fuzzy subsemigroup of a semigroup S if $f(uv) \geqslant f(u) \wedge f(v)$ for all $u, v \in S$. A fuzzy left (right) ideal of a semigroup S if $f(uv) \geqslant f(v)$ $(f(uv) \geqslant f(u))$ for all $u, v \in S$. A *fuzzy bi-ideal* of a semigroup S if f is a fuzzy subsemigroup of S and $f(uvw) \geqslant f(u) \wedge f(w)$ for all $u, v, w \in S$. It well-know, every fuzzy ideal of a semigroup S is a fuzzy bi-ideal of S.

The characteristic fuzzy set χ_I of a non-empty set is defined as follows:

$$\chi_I : T \to [0,1], u \mapsto \begin{cases} 1, & \text{if } u \in I, \\ \emptyset, & \text{if } u \notin I. \end{cases}$$

The following of theorems are true.

Theorem 2.1 ([6]). *Let S be a semigroup. Then I is a subsemigroup (left ideal right ideal, bi-ideal) of S if and only if characteristic function χ_I is a fuzzy subsemigroup (left ideal right ideal, bi-ideal) of S.*

Theorem 2.2 ([6]). *Let I and J be subsets of a non-empty set S. Then $\chi_{I \cap J} = \chi_I \wedge \chi_J$ and $\chi_I \circ \chi_J = \chi_{IJ}$.*

Theorem 2.3 ([6]). *Let f be a nonzero fuzzy set of a semigroup S. Then f is a fuzzy subsemigroup (ideal, bi-ideal) of S if and only if* supp(f) *is a subsemigroup (ideal, bi-ideal) of S.*

Next, we will review of essential ideals and fuzzy essential ideals in a semigroup and properties of those.

Definition 2.4. An essential left (right) ideal I of a semigroup S if I is a left (right) ideal of S and $I \cap J \neq \emptyset$ for every left (right) ideal J of S.

Definition 2.5 ([1]). An essential ideal I of a semigroup S if I is an ideal of S and $I \cap J \neq \emptyset$ for every ideal J of S.

Theorem 2.6 ([1]). *Let I be an essential ideal of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential ideal of S.*

Theorem 2.7 ([1]). *Let I and J be essential ideals of a semigroup S. Then $I \cup J$ and $I \cap J$ are essential ideals of S.*

Definition 2.8 ([1]). An essential fuzzy ideal f of a semigroup S if f is a nonzero fuzzy ideal of S and $f \cap g \neq \emptyset$ for every nonzero fuzzy ideal g of S.

Theorem 2.9 ([1]). *Let I be an ideal of a semigroup S. Then I is an essential ideal of S if and only if χ_I is an essential fuzzy ideal of S.*

Theorem 2.10 ([1]). *Let f be a nonzero fuzzy ideal of a semigroup S. Then f is an essential fuzzy ideal of S if and only if* supp(f) *is an essential ideal of S.*

3. Essential subsemigroups and essential fuzzy subsemigroups

In this section, we will study concepts of essential subsemigroups in a semigroup and fuzzy essential subsemigroups in a semigroup and their properties.

Definition 3.1. An essential subsemigroup I of a semigroup S if I is a subsemigroup of S and $I \cap J \neq \emptyset$ for every subsemigroup J of S.

Example 3.2.

(1) Let E be set of all even integers. Then $(E, +)$ and $(\mathbb{N}, +)$ are subsemigroups of $(\mathbb{Z}, +)$. Thus $(E, +) \cap (\mathbb{N}, +) \neq \emptyset$. Hence, $(E, +)$ is an essential subsemigroup of $(\mathbb{Z}, +)$.

(2) Let $A = \{2n \mid n \in \mathbb{Z}\}$ and $B = \{3n \mid n \in \mathbb{Z}\}$. Then $(A, \cdot)$ and $(B, \cdot)$ are subsemigroups of $(\mathbb{Z}, \dot{)}$. Thus $(A, \cdot) \cap (B, \cdot) \neq \emptyset$. Hence $(A, \cdot)$ is an essential subsemigroup.

Theorem 3.3. *Let I be an essential subsemigroup of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential subsemigroup of S.*

Proof. Suppose that I_1 is a subsemigroup of S such that $I_1 \subseteq I$ and let J be any subsemigroup of S. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore I_1 is an essential subsemigroup of S. □

Theorem 3.4. *Let I and J be essential subsemigroups of a semigroup S. Then $I \cup J$ and $I \cap J$ are essential subsemigroups of S.*

Proof. Since $I \subseteq I \cup J$ and I is an essential subsemigroup, we have $I \cup J$ is an essential subsemigroup of S, by Theorem 3.3.

Since I and J are essential subsemigroups of S we have I and J are subsemigroups of S. Thus $I \cap J$ is a subsemigroups of S.

Let K be a subsemigroup of S. Then $I \cap K \neq \emptyset$. Thus there exists $u, v \in I \cap K$. Let $u, v \in J$. Then $uv \in (I \cap J) \cap K$. Thus $(I \cap J) \cap K \neq \emptyset$. Hence $I \cap J$ is an essential subsemigroup of S. □

Definition 3.5. An essential fuzzy subsemigroup f of a semigroup S if f is a nonzero fuzzy subsemigroup of S and $f \cap g \neq \emptyset$ for every nonzero fuzzy subsemigroup g of S.

Theorem 3.6. *Let I be a subsemigroup of a semigroup S. Then I is an essential subsemigroup of S if and only if χ_I is an essential fuzzy subsemigroup of S.*

Proof. Suppose that I is an essential subsemigroup of S and let g be a nonzero fuzzy subsemigroup of S. Then $\text{supp}(g)$ is subsemigroup of S. By assumption we have I is a subsemgroup of S. Thus $I \cap \text{supp}(g) \neq 0$. So there exists $u \in I \cap \text{supp}(g)$. It implies that $(\chi_I \cap g)(u) \neq 0$. Hence, $\chi_I \cap g \neq 0$. Therefore, χ_I is an essential fuzzy subsemigroup of S.

Conversely, assume that χ_I is an essential fuzzy subsemigroup of S and let J be a subsemigroup of S. Then χ_J is a nonzero fuzzy subsemigroup of S. Sicne χ_I is an essential fuzzy subsemigroup of S we have χ_I is a fuzzy subsemigorup of S. Thus, $\chi_I \cap \chi_J \neq 0$. So by Theorem 2.2, $\chi_{I \cap J} \neq \emptyset$. Hence, $I \cap J \neq \emptyset$. Therefore I is an essential subsemigroup of S. □

Theorem 3.7. *Let f be a nonzero fuzzy subsemigroup of a semigroup S. Then f is an essential fuzzy subsemigroup of S if and only if $\text{supp}(f)$ is an essential subsemigroup of S.*

Proof. Assume that f is an essential fuzzy subsemigroup of S. Then $\text{supp}(f)$ is a subsemigroup of S. Let I be a subsemigroup of S. Then by Theorem 2.1, χ_I is a subsemigroup of S. Since f is an essential fuzzy subsemigroup of S we have f is a fuzzy subsemigroup of S. Thus $f \wedge \chi_I \neq 0$. So there exists $u \in S$ such that $(f \wedge \chi_I)(u) \neq 0$. It implies that $f(u) \neq 0$ and $\chi_I \neq 0$. Hence, $u \in \text{supp}(f) \cap I$ so $\text{supp}(f) \cap I \neq \emptyset$ it implies that $\text{supp}(f)$ is an essential subsemigroup of S.

Conversely, assume that $\text{supp}(f)$ is an essential ideal of S and let g be a nonzero fuzzy subsemigroup of S. Then $\text{supp}(g)$ is a subsemigroup of S. Thus $\text{supp}(f) \cap \text{supp}(g) \neq \emptyset$. So there exists

$$u \in \text{supp}(f) \cap \text{supp}(g).$$

This implies that $f(u) \neq 0$ and $g(u) \neq 0$ for all $u \in S$. Hence, $(f \wedge g)(u) \neq 0$ for all $u \in S$. Therefore, $f \wedge g \neq 0$. We conclude that f is an essential fuzzy subsemigroup of S. □

Theorem 3.8. *Let f be an essential fuzzy subsemigroup of a semigroup S. If f_1 is a fuzzy subsemigroup of S such that $f \subseteq f_1$, then f_1 is also an essential fuzzy subsemigroup of S.*

Proof. Let f_1 be a fuzzy subsemigroup of S such that $f \subseteq f_1$ and let g be any fuzzy subsemigroup of S. Thus, $f \wedge g \neq 0$. So $f_1 \wedge g \neq 0$. Hence f_1 is an essential fuzzy subsemigroup of S. □

Theorem 3.9. *Let f_1 and f_2 be essential fuzzy subsemigroups of a semigroup S. Then $f_1 \vee f_2$ and $f_1 \wedge f_2$ are essential fuzzy subsemigroups of S.*

Proof. Let f_1 and f_2 be essential fuzzy subsemigroups of S. Then by Theorem 3.8, $f_1 \vee f_2$ is an essential fuzzy subsemigroup of S. Since f_1 and f_2 are essential fuzzy subsemigroups of S we have $f_1 \cap f_2$ is a fuzzy subsemigroup of S. Let g be a nonzero fuzzy subsemigroup of S. Then $f_1 \wedge g \neq 0$. Thus there exists $u \in S$ such that $f_1(u) \neq 0$ and $(g)(u) \neq 0$. Since $f_2 \neq 0$ and let $v \in S$ such that $f_2(v) \neq 0$. Since f_1 and f_2 are fuzzy subsemigroups of S we have $f_1(uv) \geqslant f_1(u) \wedge f_1(v) > 0$ and $f_2(uv) \geqslant f_2(u) \wedge f_2(v) > 0$. Thus $(f_1 \wedge f_2)(uv) = f_1(uv) \wedge f_2(uv) \neq 0$. Since g is a fuzzy subsemigroup of S and $g(u) \neq 0$ we have $g(uv) \neq 0$ for all $u, v \in S$. Thus $[(f_1 \wedge f_2) \wedge g](uv) \neq 0$. Hence $[(f_1 \wedge f_2) \wedge g] \neq 0$. Therefore $f_1 \wedge f_2$ is an essential fuzzy subsemigroup of S. □

4. Essential bi-ideals and essential fuzzy bi-ideals

In this section, we defined essential bi-ideals and essential fuzzy bi-ideal in semigroup and its integrated properties .

Definition 4.1. An essential bi-ideal I of a semigroup S if I is a bi-ideal of S and $I \cap J \neq \emptyset$ for every bi-ideal J of S.

Example 4.2. Let $S = \{\Psi, \Omega, \Upsilon, \Pi\}$ be semigroup with the following Cayley table.

$\cdot$	Ψ	Ω	Υ	Π
Ψ	Ψ	Ψ	Ψ	Ψ
Ω	Ψ	Ψ	Ψ	Ψ
Υ	Ψ	Ψ	Ω	Ψ
Π	Ψ	Ψ	Ω	Ω

Then $\{\Psi\}$, $\{\Psi, \Omega\}$, $\{\Psi, \Omega, \Upsilon\}$, $\{\Psi, \Omega, \Pi\}$, and $\{\Psi, \Omega, \Upsilon, \Pi\}$ are bi-ideal of S. Thus $\{\Psi\} \cap \{\Psi, \Omega\} \neq \emptyset$ and

$$\{\Psi, \Omega, \Pi\} \cap \{\Psi, \Omega, \Upsilon, \Pi\} \neq \emptyset.$$

Hence $\{\Psi\}$ and $\{\Psi, \Omega, \Pi\}$ are essential bi-ideals of S.

Theorem 4.3. *Let I be an essential bi-ideal of a semigroup S. If I_1 is an ideal of S containing I, then I_1 is also an essential bi-ideal of S.*

Proof. Suppose that I_1 is a bi-ideal of S such that $I_1 \subseteq I$ and let J be any bi-ideal of S. Thus, $I \cap J \neq \emptyset$. Hence, $I_1 \cap J \neq \emptyset$. Therefore I_1 is an essential bi-ideal of S. □

Theorem 4.4. *Let I and J be essential bi-ideals of a semigroup S. Then $I \cup J$ and $I \cap J$ are essential bi-ideals of S.*

Proof. Since I and J are essential bi-ideals of a semigroup S we have I and J are essential subsemigroups of a semigroup S. Thus by Theorem 3.4, $I \cup J$ and $I \cap J$ are essential subsemigroups of S. Since $I \subseteq I \cup J$ and I is an essential bi-ideal we have $I \cup J$ is an essential bi-ideal of S.

Let K be a bi-ideal of S. Then $I \cap K \neq \emptyset$. Thus there exists u, v and $w \in I \cap K$. Let u, v and $w \in J$. Then $uvw \in (I \cap J) \cap K$. Thus $(I \cap J) \cap K \neq \emptyset$. Hence $I \cap J$ is an essential bi-ideal of S. □

Definition 4.5. An essential fuzzy bi-ideal f of a semigroup S if f is a nonzero fuzzy bi-ideal of S and $f \wedge g \neq 0$ for every nonzero fuzzy bi-ideal g of S.

Theorem 4.6. *Let* I *be a bi-ideal of a semigroup* S. *Then* I *is an essential bi-ideal of* S *if and only if* χ_I *is an essential fuzzy bi-ideal of* S.

Proof. Suppose that I is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then by Theorem 3.6, supp(g) is subsemigroup of S and χ_I is an essential fuzzy subsemigroup of S. Thus there exists $u, v, w \in I \cap \text{supp}(g)$ such that $(f \wedge \chi_I)(uvw) \neq 0$. It implies that $\chi_I \wedge g \neq 0$. Therefore, χ_I is an essential fuzzy bi-ideal of S.

Conversely, assume that χ_I is an essential fuzzy bi-ideal of S and let J be a bi-ideal of S. Then χ_I is an essential fuzzy subsemigroup of S and J is a subsemigroup of S. Thus by Theorem 3.6, I is an essential subsemigroup of S. Since J be a bi-ideal of S we have χ_J is a nonzero fuzzy bi-ideal of S. Then, $\chi_I \wedge \chi_J \neq 0$. Thus, $\chi_{I \cap J} \neq \emptyset$. Hence, $I \cap J \neq \emptyset$. Therefore I is an essential bi-ideal of S. □

Theorem 4.7. *Let* f *be a nonzero fuzzy bi-ideal of a semigroup* S. *Then* f *is an essential fuzzy bi-ideal of* S *if and only if* supp(f) *is an essential bi-ideal of* S.

Proof. Assume that f is an essential fuzzy bi-ideal of S. Then f is an essential fuzzy subsemigroup of S. Thus by Theorem 3.7, supp(f) is an essential subsemigorup of S. Let I be a bi-ideal of S. Then by Thoerem 2.1, χ_I is a bi-ideal of S. Thus $f \wedge \chi_I \neq \emptyset$. Thus there exists $u \in S$ such that $(f \wedge \chi_I)(u) \neq \emptyset$. It implies that $f(u) \neq 0$ and $\chi_I \neq 0$. Hence, $u \in \text{supp}(f) \cap I$ so $\text{supp}(f) \cap I \neq \emptyset$ it implies that supp(f) is an essential bi-ideal of S.

Conversely, assume that supp(f) is an essential bi-ideal of S and let g be a nonzero fuzzy bi-ideal of S. Then supp(f) is an essential bi-ideal of S. Since g be a nonzero fuzzy bi-ideal of S we have f is an essential fuzzy subsemigroup of S and supp(g) is a subsemigroup of S, by Theorem 3.7. This implies that $\text{supp}(f) \cap \text{supp}(g) \neq \emptyset$. So there exists $u \in \text{supp}(f) \cap \text{supp}(g)$, this implies that $f(u) \neq 0$ and $g(u) \neq 0$. Hence, $(f \wedge g)(u) \neq 0$. Therefore, $f \wedge g \neq 0$. We conclude that f is an essential fuzzy bi-ideal of S. □

Theorem 4.8. *Let* f *be an essential fuzzy bi-ideal of a semigroup* S. *If* f_1 *is a fuzzy bi-ideal of S such that* $f \subseteq f_1$, *then* f_1 *is also an essential fuzzy bi-ideal of* S.

Proof. Let f_1 be a fuzzy bi-ideal of S such that $f \subseteq f_1$ and let g be any fuzzy bi-ideal of S. Thus $f \wedge g \neq 0$. So $f_1 \wedge g \neq 0$. Hence, f_1 is an essential fuzzy bi-ideal of S. □

Theorem 4.9. *Let* f_1 *and* f_2 *be essential fuzzy bi-ideals of a semigroup* S. *Then* $f_1 \vee f_2$ *and* $f_1 \wedge f_2$ *are essential fuzzy bi-ideals of* S.

Proof. Let f_1 and f_2 be essential fuzzy bi-ideal of S. Then by Theorem 4.8, $f_1 \vee f_2$ is an essential fuzzy bi-ideal of S. Since f_1 and f_2 are essential fuzzy bi-ideals of S we have f_1 and f_2 is an essential fuzzy subsemigroup of S. Thus $f_1 \wedge f_2$ is an essential fuzzy subsemigroup of S. Let g be a nonzero fuzzy bi-ideal of S. Then $f_1 \wedge g \neq 0$. Thus there exists $u, w \in S$ such that $f_1(uw) \neq 0$ and $(g)(uw) \neq 0$. Since $f_2 \neq 0$ and let $v \in S$ such that $f_2(v) \neq 0$. Since f_1 and f_2 are fuzzy subsemigroups of S we have

$$f_1(uvw) \geqslant f_1(u) \wedge f_1(w) > 0,$$

and

$$f_2(uvw) \geqslant f_2(u) \wedge f_2(w) > 0.$$

Thus $(f_1 \wedge f_2)(uvw) = f_1(uvw) \wedge f_2(uvw) \neq 0$. Since g is a fuzzy subsemigroup of S and $g(v) \neq 0$ we have $g(uvw) \neq 0$ for all $u, v \in S$. Thus $[(f_1 \wedge f_2) \wedge g](uvw) \neq 0$. Hence $[(f_1 \wedge f_2) \wedge g] \neq 0$. Therefore $f_1 \wedge f_2$ is an essential fuzzy bi-ideal of S. □

The following theorem we will use the basic knowledge of ideal and bi-ideal in semigroups to prove essential bi-ideal in semigroup.

Theorem 4.10. *Every essential ideal of semigroup S is an essential bi-ideal of S.*

Proof. The proof is obvious. □

Theorem 4.11. *Every essential fuzzy ideal of semigroup S is an essential fuzzy bi-ideal of S.*

Proof. The proof is obvious. □

5. Characterizing some semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals

In this section, we will characterize regular, left regular, intra-regular, semsimiple semigroups by using essential fuzzy ideals and essential fuzzy bi-ideals in semigroups. The following lemmas will be used to prove Theorem 5.3.

Lemma 5.1. *Let S be a semigroup. If f is an essential fuzzy right ideal and g is an essential fuzzy left ideal of S then $f \circ g \subseteq f \wedge g$.*

Proof. Assume that f and g is an essential fuzzy right ideal and an essential fuzzy left ideal of S respectively. Then f and g is a fuzzy right ideal and a fuzzy left ideal of S respectively. Let $u \in S$. If $F_u = \emptyset$, then $(f \circ g)(u) = 0 \leqslant ((f(u) \wedge g(u)) = (f \wedge g)(u)$. If $F_u \neq \emptyset$, then

$$\begin{aligned}(f \circ g)(u) &= (\bigvee_{(i,j) \in F_u} \{f(i) \wedge g(j)\} \leqslant \bigvee_{(i,j) \in F_u} \{(f(ij) \wedge g(ij))\} \\ &= (f(u) \wedge g(u)) = (f \wedge g)(u).\end{aligned}$$

Hence, $(f \circ g)(u) \leqslant (f \wedge g)(u)$. Therefore, $f \circ g \subseteq f \wedge g$. □

Lemma 5.2 ([6]). *A semigroup S is regular if and only if $RL = R \cap L$, for every right ideal R and left ideal L of S.*

The following theorem show an equivalent conditional statement for a regular semigroup.

Theorem 5.3. *A semigroup S is regular if and only if $f \circ g = f \wedge g$ for every essential fuzzy right ideal f and essential fuzzy left ideal g of S.*

Proof. ($\Rightarrow$): Let f and g be an essential fuzzy right ideal and an essential fuzzy left ideal of S respectively. Then f and g is a fuzzy right ideal and a fuzzy left ideal of S respectively. Then by Lemma 5.1, $f \circ g \subseteq f \wedge g$. Let $u \in S$. Then there exists $x \in S$ such that $u = uxu$. Thus

$$\begin{aligned}(f \circ g)(u) &= (\bigvee_{(y,z) \in F_u} \{f(y) \wedge g(z)\}) = (\bigvee_{(y,z) \in F_{uxu}} \{f(y) \wedge g(z)\} \\ &\leqslant f(ux) \wedge g(u) \leqslant f(u) \wedge g(u) = (f \wedge g)(u).\end{aligned}$$

Hence, $(f \wedge g)(u) \geqslant (f \circ g)(u)$, and so $(f \wedge g)(u) \subseteq (f \circ g)(u)$. Therefore, $f \circ g = f \wedge g$.

($\Leftarrow$): Let R and L be a right ideal and a left ideal of S respectively. Then by Theorem 2.1, χ_R and χ_L is an essential fuzzy right ideal and an essential fuzzy left ideal of S respectively. By supposition and Theorem 2.2, we have

$$\begin{aligned}\chi_{RL}(u) &= (\chi_R \circ \chi_L)(u) = (\chi_R \wedge \chi_L)(u) \\ &= \chi_{R \wedge L}(u) = 1.\end{aligned}$$

Thus $u \in RL$, and so $RL = R \cap L$. It follows that by Lemma 5.2, S is regular. □

Lemma 5.4 ([6]). *A semigroup* S *is regular if and only if* $R_1 \cap R_2 \cap B \subseteq R_1R_2B$*, for every right ideals* R_1, R_2 *and every bi-ideal* B *of* S.

Theorem 5.5. *A semigroup* S *is regular if and only if* $f \wedge g \wedge h \subseteq f \circ g \circ h$*, for every essential fuzzy right ideals* f, g *and every essential fuzzy bi-ideal* h *of* S.

Proof. ($\Rightarrow$): Let f, g be two essential fuzzy right ideals, h be an essential fuzzy bi-ideal of S. Then f, g be two fuzzy right ideals, h is a fuzzy bi-ideal of S. Let $u \in S$ Since S is regular, there exists $x \in S$ such that $u = uxu$. Thus

$$\begin{aligned}(f \circ g \circ h)(u) &= (\bigvee_{(i,j)\in F_u} \{f(i) \wedge (g \circ h)(j)\} = \bigvee_{(i,j)\in F_{uxu}} \{f(i) \wedge (g \circ h)(j)\} \\ &\geqslant (f(ux) \wedge (g \circ h)(u) = f(ux) \wedge (\bigvee_{(p,q)\in F_u} \{g(p) \wedge h(q)\}) \\ &= f(ux) \wedge (\bigvee_{(p,q)\in F_{uxu}} \{g(p) \wedge h(q)\}) \geqslant f(ux) \wedge (g(ux) \wedge h(u) \\ &= f(u) \wedge (g(u) \wedge h(u)) = (f \wedge g \wedge h)(u).\end{aligned}$$

Hence, $(f \wedge g \wedge h)(u) \leqslant (f \circ g \circ h)(u)$. Therefore, $f \wedge g \wedge h \subseteq f \circ g \circ h$.

($\Leftarrow$): Let R_1, R_2 be two right ideals and let B be a bi-ideal of S. Then by Theorem 2.1, χ_{R_1} and χ_{R_2} are essential fuzzy right ideals and χ_B is an essential fuzzy bi-ideal of S. Thus χ_{R_1} and χ_{R_2} are fuzzy right ideals and χ_B is a fuzzy bi-ideal of S. By supposition and Lemma 2.2, we have

$$\begin{aligned}1 &= (\chi_{R_1\cap R_2\cap B})(u) = (\chi_{R_1}) \wedge (\chi_{R_2}) \wedge (\chi_B)(u) \\ &\subseteq (\chi_{R_1}) \circ (\chi_{R_2}) \circ (\chi_B)(u) = \chi_{R_1R_2B}(u).\end{aligned}$$

Thus, $u \in R_1R_2B$ and so, $R_1 \cap R_2 \cap B \subseteq R_1R_2B$. It follows that by Lemma 5.4, S is regular. □

Definition 5.6 ([6]). A semigroup S called left regular if for each element $u \in S$, there exists an element $x \in S$ such that $u = xu^2$.

Lemma 5.7 ([6]). *A semigroup* S *is left regular if and only if* $I \cap B \subseteq IB$*, for every ideal* I *of* S *and every bi-ideal* B *of* S.

Theorem 5.8. *A semigroup* S *is left regular if and only if* $f \wedge g \subseteq f \circ g$*, for every essential fuzzy ideal* f *and every essential fuzzy bi-ideal* g *of* S.

Proof. ($\Rightarrow$): Assume that f and g is an essential fuzzy ideals and an essential fuzzy bi-ideal of S respectively. Then f and g is a fuzzy ideals and a fuzzy bi-ideal of S respectively. Let $u \in S$. Since S is left rgular, there exist $x \in S$ such that $u = xu^2$. Thus

$$\begin{aligned}(f \circ g)(u) &= (\bigvee_{(i,j)\in F_u} \{f(i) \wedge g(j)\} = (\bigvee_{(i,j)\in F_{(u=xuu)}} \{f(i) \wedge g(j)\} \\ &\geqslant f(xu) \wedge g(u) \geqslant f(u) \wedge g(u) = (f \wedge g)(u).\end{aligned}$$

Hence, $(f \wedge g)(u) \leqslant (f \circ g)(u)$. Therefore, $f \wedge g \subseteq f \circ g$.

($\Leftarrow$): Let I and B be an ideal and a bi-ideal of S respectively. Then by Theorem 2.1, $\succ_I$ and $\succ_J$ is an essential fuzzy ideal and an essential fuzzy bi-ideal of S respectively. Thus $\succ_I$ and $\succ_J$ is a fuzzy ideal and a fuzzy bi-ideal of S respectively. By supposition and Lemma 2.2, we have

$$\chi_{I\cap B}(u) = (\chi_I \wedge \chi_B)(u) \subseteq (\chi_I \circ \chi_B)(u) = \chi_{IB}(u) = 1.$$

Thus, $u \in IB$ and so, $IB \subseteq I \cap B$. It follows that by Lemma 5.8, S is left regualr. □

The following definition and lemma will be used to prove in Theorem 5.11.

Definition 5.9 ([6]). A semigroup S is called *intra-regular* if for each $u \in S$, there exist $a, b \in S$ such that $u = au^2b$.

Lemma 5.10 ([6]). *A semigroup S is intra-regular if and only if $L \cap R \subseteq LR$, for every left ideal L and every right ideal R of S.*

Theorem 5.11. *A semigroup S is intra-regular if and only if $f \wedge g \subseteq f \circ g$, for every essential left ideal f and essential right ideal g of S.*

Proof. ($\Rightarrow$): Assume that f and g is an essential fuzzy left ideal and an essential right ideal of S respectively. Then f and g is a left ideal and a right ideal of S respectively. Let $u \in S$. Since S is intra-regular, there exist $a, b \in S$ such that $u = au^2b$. Thus

$$\begin{aligned}(f \circ g)(u) = (\bigvee_{(i,j)\in F_u} \{f(i) \wedge g(j)\} = (\bigvee_{(i,j)\in F_{auub}} \{f(i) \wedge g(j)\} \\ \geqslant f(au) \wedge g(ub) \geqslant f(u) \wedge g(u) = (f \wedge g)(u).\end{aligned}$$

It implies that, $(f \wedge g)(u) \leqslant (f \circ g)(u)$. Hence, $f \wedge g \subseteq f \circ g$.

($\Leftarrow$): Let R and L be a right ideal and a left ideal of S respectively. Then by Theorem 2.1, χ_R and χ_L is an essential fuzzy right ideal and an essential fuzzy left ideal of S respectively. Thus χ_R and χ_L is a fuzzy right ideal and a fuzzy left ideal of S. By supposition and Lemma 2.2, we have

$$\chi_{R\cap L}(u) = (\chi_R \wedge \chi_L)(u) \supseteq \chi_{RL}(u) = (\chi_R \circ \chi_L)(u) = 1.$$

Thus $u \in LR$, and so $L \cap R \subseteq LR$. It follows that by Lemma 5.10, S is intra-regular. □

The following definition and lemma will be used to prove in Theorem 5.15.

Definition 5.12 ([6]). A semigroup S is called *semisimple* if every ideal of S is idempotent.

Remark 5.13. A semigroup S is semisimple if and only if $u \in (SuS)(SuS)$ for every $u \in S$, that is there exist $w, y, z \in S$ such that $u = wuyuz$.

Lemma 5.14 ([6]). *A semigroup S is semisimple if and only if $I \cap J = IJ$, for every ideals I and J of S.*

Theorem 5.15. *A semigroup S is semisimple if and only if $f \wedge g = f \circ g$, for every essential fuzzy ideals f and g of S.*

Proof. ($\Rightarrow$) Assume that f and g are essential fuzzy ideals of S. Then f and g are fuzzy ideals of S. Then by Theorem 5.1, $f \circ g \subseteq f \cap g$. Let $u \in S$. Since S is semisimple, there exist $w, x, y, z \in S$ such that $u = (xuy)(wuz)$. Thus

$$\begin{aligned}(f \circ g)(u) = \bigvee_{(i,j)\in F_u} \{f(i) \wedge g(j)\} = \bigvee_{(i,j)\in F_{(xuy)(wuz)}} \{f(i) \wedge g(j)\} \\ \geqslant f(xuy) \wedge g(wuz) \geqslant f(xu) \wedge g(uz) \\ \geqslant f(u) \wedge g(u) = (f \wedge g)(u).\end{aligned}$$

Hence, $(f \cap g)(u) \leqslant (f \circ g)(u)$, and so $f \wedge g \subseteq f \circ g$. Therefore, $f \wedge g = f \circ g$.

($\Leftarrow$): Let I and J be ideals of S. Then by Theorem 2.1, χ_I and χ_J are essential fuzzy ideals of S. Thus χ_I and χ_J are fuzzy ideals of S. By supposition and Lemma 2.2, we have

$$\chi_{IJ}(u) = (\chi_I \circ \chi_J)(u) = (\chi_I \wedge \chi_J)(u) = \chi_{I\cap J}(u) = 1.$$

Thus $u \in IJ$, and so $IJ = I \cap J$. It follows that by Lemma 5.14, S is semisimple. □

Acknowledgment

This work is partially supported by School of Science, University of Phayao The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

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