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Conformal quasi-bi-slant Riemannian maps

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Abstract

Conformal maps or horizontally conformal maps are very useful for characterization of harmonic morphisms. Nowadays, many medical problems (directly or indirectly) such as brain imaging (brain surface mapping, [Y. L. Wang, L. M. Lui, X. F. Gu, K. M. Hayashi, T. F. Chan, A. W. Toga, P. M. Thompson, S.-T. Yau, IEEE Transactions on Medical Imaging, **26** (2007), 853–865], [Y. L. Wang, X. F. Gu, K. M. Hayashi, T. F. Chan, P. M. Thompson , S.-T. Yau, Tenth IEEE International Conference on Computer Vision (ICCV'05), **2005** (2005), 1061–1066]) computer graphics ([X. F. Gu, Y. L. Wang, T. F. Chan, P. M. Thompson, S.-T. Yau, IEEE Transactions on Medical Imaging, **23** (2004), 949–958]) etc. can be solved using conformal Riemannian maps. In this paper, as a generalization of conformal Riemannian maps and conformal bi-slant submersions, we introduce conformal quasi-bi-slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds. We study the geometry of leaves of distributions which are involved in the definition of the conformal quasi bi-slant Riemannian maps. We work out conditions for such maps to be integrable, totally geodesic and pluriharmonic. We present two examples for the introduced notion.

Keywords: Almost Hermitian manifolds, Riemannian maps, conformal quasi bi-slant Riemannian maps.

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1. Introduction

In Riemannian geometry, the theory of smooth maps between Riemannian manifolds is a fascinating topic that continually generates new ideas which are very helpful in comparing geometric structures between manifolds. In this point of view, isometric immersions and submersions are basic such maps studied by O'Neill [18] and Gray [10]. In 1992, Fischer introduced the notion of Riemannian maps [8] as a generalization of isometric immersions and Riemannian submersions. More precisely, a smooth map $\pi : (B_1, g_{B_1}) \to (B_2, g_{B_2})$ between Riemannian manifolds such that $0 < \operatorname{rank}\pi < \min\{m, n\}$, where $\dim B_1 = m$ and $\dim B_2 = n$. It satisfies the equation:

$$g_{B_1}(V_1, V_2) = g_{B_2}(\pi_* V_1, \pi_* V_2), \quad \text{for } V_1, V_2 \in \Gamma(\ker \pi_*)^{\perp}.$$

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It follows that isometric immersions and Riemannian submersions are particular cases of Riemannian maps with $\ker \pi_* = \{0\}$ and $(\mathrm{range}\pi_*)^\perp = \{0\}$, respectively. If we denote the kernel space of π_* by $\ker \pi_*$ and the orthogonal complementary space of $\ker \pi_*$ by $(\ker \pi_*)^\perp$ in TB_1, then the TB_1 has the following orthogonal decomposition:

$$TB_1 = \ker \pi_* \oplus (\ker \pi_*)^\perp.$$

Also, if we denote the range of π_* by $\mathrm{range}\pi_*$ and for a point $p \in B_1$ the orthogonal complementary space of $(\mathrm{range}\pi_*)_{\pi(p)}$ by $(\mathrm{range}\pi_*)^\perp_{\pi(p)}$ in $T_{\pi(p)}B_2$, then the tangent space $T_{\pi(p)}B_2$ has the following orthogonal decomposition:

$$T_{\pi(p)}B_2 = (\mathrm{range}\pi_*)_{\pi(p)} \oplus (\mathrm{range}\pi_*)^\perp_{\pi(p)}.$$

A differentiable map $\pi : (B_1, g_{B_1}) \to (B_2, g_{B_2})$ is called a Riemannian map at $p \in B_1$ if the horizontal restriction $\pi^h_{*p} : (\ker \pi_*)^\perp_p \to (\mathrm{range}\pi_*)_{\pi(p)}$ is linear isometry between the inner product space $((\ker \pi_{*p})^\perp, (g_{B_1})_{(p)}|_{(\ker \pi_{*p})^\perp})$ and $(\mathrm{range}\pi_{*\pi_{(p)}}, (g_{B_2})_{(\pi(p))}|_{(\mathrm{range}\pi_{*p})})$ (for details see [5]).

Fischer showed that such maps could be used to solve the generalized eikonal equation, i.e., it satisfies the generalized eikonal equation $\| \pi_* \|^2 = \mathrm{rank}\pi$. Since $\mathrm{rank}\pi$ is an integer valued function and $\| \pi_* \|^2$ is continuous function on the Riemannian manifold so the equality implies that $\mathrm{rank}\pi$ is locally constant and globally constant on connected components. Since energy density $2e(\pi) = \| \pi_* \|^2 = \mathrm{rank}\pi$, i.e., density is quantized to integer if the Riemannian manifold is connected. Thus the eikonal equation is a bridge between geometric optics and physical optics. On the other hand, horizontally conformal maps were defined by Fuglede [9] and Ishihara [14] and these maps are useful for characterization of harmonic morphisms. Horizontally conformal maps (conformal maps) have applications in mathematics as well as in physics. Especially, within the Yang-Mills theory [6], Kaluza-Klein theory [12], supergravity and superstring theories ([7],[13]) redundant robotic chains [4] etc. Thus, the notion of Riemannian maps deserves through study from different perspectives.

Furthermore, Sahin [28] introduced the notion of conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition Theorems. After that, several kinds of conformal Riemannian maps were introduced and studied, some of them are like: conformal Riemannian maps ([28, 31]), conformal anti-invariant Riemannian maps [1], conformal semi-invariant Riemannian maps ([2, 32]), conformal slant Riemannian maps ([3]), etc. Likewise, these maps have been studied widely by many geometers (see also [15, 19–25, 27, 29, 30]) etc.

The present article is organized as follows. Section 2 contains some basic definitions needed throughout this paper. In Section 3, we define conformal quasi bi-slant Riemannian map from almost Hermitian manifolds to Riemannian manifolds and obtain some results on conformal quasi bi-slant Riemannian map from Kähler manifold to Riemannian manifold. In Section 4, some examples for this notion are provided.

2. Preliminaries

An almost Hermitian manifold (N_1, g_1, J) is called a Kähler manifold [36] if

$$(\nabla_{W_1} J) W_2 = 0, \tag{2.1}$$

for $W_1, W_2 \in \Gamma(TN_1)$ with almost complex structure J and almost Hermitian metric g_1 on N_1.

Watson introduced the fundamental tensors of a submersion in [35]. It is known that the fundamental tensor play similar role to that of the second fundamental form of a submersion [16]. O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ [18], for vector fields $V_1, V_2 \in \Gamma(TN_1)$, are defined as

$$\mathcal{A}_{V_1} V_2 = \mathcal{V}\nabla_{\mathcal{H}V_1} \mathcal{H}V_2 + \mathcal{H}\nabla_{\mathcal{H}V_1} \mathcal{V}V_2, \quad \mathcal{T}_{V_1} V_2 = \mathcal{H}\nabla_{\mathcal{V}V_1} \mathcal{V}V_2 + \mathcal{V}\nabla_{\mathcal{V}V_1} \mathcal{H}V_2, \tag{2.2}$$

where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections and ∇ is Levi-Civita connection N_1. On the other hand, from (2.2), we have

$$\nabla_{Y_1} Y_2 = \mathcal{T}_{Y_1} Y_2 + \mathcal{V}\nabla_{Y_1} Y_2, \tag{2.3}$$

$$\nabla_{Y_1} U_1 = \mathcal{H}\nabla_{Y_1} U_1 + \mathcal{T}_{Y_1} U_1, \tag{2.4}$$
$$\nabla_{U_1} Y_1 = \mathcal{A}_{U_1} Y_1 + \mathcal{V}\nabla_{U_1} Y_1, \tag{2.5}$$
$$\nabla_{U_1} U_2 = \mathcal{H}\nabla_{U_1} U_2 + \mathcal{A}_{U_1} U_2, \tag{2.6}$$

for $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$, where $\mathcal{V}\nabla_{Y_1} Y_2 = \widehat{\nabla}_{Y_1} Y_2$. If U_1 is basic, then $\mathcal{A}_{Y_1} U_1 = \mathcal{H}\nabla_{U_1} Y_1$.

It is seen that for $p \in N_1$, $Y_1 \in \mathcal{V}_p$ and $U_1 \in \mathcal{H}_p$ the linear operators $\mathcal{A}_{U_1}$, $\mathcal{T}_{Y_1} : T_p N_1 \to T_p N_1$ are skew-symmetric, that is

$$g_1(\mathcal{A}_{U_1} Z_1, Z_2) = -g_1(Z_1, \mathcal{A}_{U_1} Z_2) \text{ and } g_1(\mathcal{T}_{Y_1} Z_1, Z_2) = -g_1(Z_1, \mathcal{T}_{Y_1} Z_2), \tag{2.7}$$

for each $Z_1, Z_2 \in \Gamma(\, T_p N_1)$.

Let $\pi : (N_1, g_1) \to (N_2, g_2)$ is a smooth map between Riemannian manifolds. Then the differential π_* of π can be observed a section of the bundle $\mathrm{Hom}(TN_1, \pi^{-1}TN_2) \to N_1$, where $\pi^{-1}TN_2$ is the bundle which has fibres $(\pi^{-1}TN_2)_p = T_{\pi(p)}N_2$, has a connection ∇ induced from the Riemannian connection and ∇^{N_1} pullback connection. Then the second fundamental form of π is given by

$$(\nabla \pi_*)(V_1, V_2) = \overset{N_1}{\nabla^{\pi}}_{V_1} \pi_*(V_2) - \pi_*(\mathcal{H}\overset{N_2}{\nabla^{\pi}}_{V_1} V_2), \tag{2.8}$$

for any vector fields $V_1, V_2 \in \Gamma(TN_1)$, where ∇^π is the pullback connection. We recollection that a differentiable map π between two Riemannian manifolds is called totally geodesic if

$$(\nabla \pi_*)(Y_1, Y_2) = 0, \text{ for } Y_1, Y_2 \in \Gamma(TN_1).$$

Definition 2.1. Let (N_1, g_1) and (N_2, g_2) are two Riemannian manifolds with dimensions m and n, respectively. If $\pi : (N_1, g_1) \to (N_2, g_2)$ is a smooth map, then π is a conformal Riemannian map at $p \in N_1$ if $0 < \mathrm{rank}\pi_{*p} < \min\{m, n\}$ and π_{*p} maps $\mathcal{H}_p = (\ker \pi_{*p})^\perp$ conformally onto $\mathrm{range}(\pi_{*p})$, i.e., there exists a number $\lambda^2(p) \neq 0$ such that

$$\lambda^2(p) g_1(V_1, V_2) = g_2(\pi_* V_1, \pi_* V_2),$$

for $V_1, V_2 \in (\ker \pi_{*p})^\perp$. π is called conformal Riemannian map if π is a conformal map at each point $p \in N_1$. A conformal Riemannian map π is proper if $\lambda \neq 1$.

On the other hand, let $\pi : (N_1, g_1) \to (N_2, g_2)$ be a conformal map between Riemannian manifolds. Then, we get

$$(\nabla \pi_*)(V_1, V_2)|_{\mathrm{range}\pi_*} = V_1(\ln \lambda)\pi_*(V_2) + V_2(\ln \lambda)\pi_*(V_1) - g_1(V_1, V_2)\pi_*(\mathrm{grad} \ln \lambda),$$

where $V_1, V_2 \in (\ker \pi_{*p})^\perp$. From equation (2.8), we get

$$\begin{aligned} \overset{N_1}{\nabla^{\pi}}_{V_1} \pi_*(V_2) = {} & \pi_*(\mathcal{H}\overset{N_2}{\nabla^{\pi}}_{V_1} V_2) + V_1(\ln \lambda)\pi_*(V_2) + V_2(\ln \lambda)\pi_*(V_1) \\ & - g_1(V_1, V_2)\pi_*(\mathrm{grad} \ln \lambda) + (\nabla \pi_*)^\perp(V_1, V_2), \end{aligned} \tag{2.9}$$

where $(\nabla \pi_*)^\perp(V_1, V_2)$ is the component of $(\nabla \pi_*)(V_1, V_2)$ on $(\mathrm{range}\pi_*)^\perp$ for $V_1, V_2 \in (\ker \pi_{*p})^\perp$. Thus if we denote the $(\mathrm{range}\pi_*)^\perp$ component of $(\nabla \pi_*)(V_1, V_2)$ by $(\nabla \pi_*)(V_1, V_2)|_{(\mathrm{range}\pi)^\perp}$, we can write $(\nabla \pi_*)(V_1, V_2)$ as

$$(\nabla \pi_*)(V_1, V_2) = (\nabla \pi_*)(V_1, V_2)|_{(\mathrm{range}\pi)} + (\nabla \pi_*)(V_1, V_2)|_{(\mathrm{range}\pi)^\perp},$$

for $V_1, V_2 \in (\ker \pi_{*p})^\perp$.

Definition 2.2. Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) be a Riemannian manifold with dimension m and n, respectively. A map π from an almost Hermitian manifold (N_1, g_1, J) to Riemannian manifold (N_2, g_2) is pluriharmonic map [17] if

$$(\nabla \pi_*)(Z_1, Z_2) + (\nabla \pi_*)(JZ_1, JZ_2) = 0,$$

for $Z_1, Z_2 \in \Gamma(TN_1)$.

3. Conformal quasi bi-slant Riemannian maps

Definition 3.1. Let (N_1, g_1, J) be an almost Hermitian manifold and (N_2, g_2) be a Riemannian manifold. A Riemannian map $\pi : (N_1, g_1, J) \to (N_2, g_2)$ is called a conformal quasi bi-slant Riemannian map if there exist three mutually orthogonal distributions D, D_1, and D_2 such that

(i) $\ker \pi_* = D \oplus D_1 \oplus D_2$;
(ii) $J(D) = D$, i.e., D is invariant;
(iii) $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$;
(iv) for any non-zero vector field $Y_1 \in (D_1)_p$, $p \in N_1$, the angle θ_1 between JY_1 and $(D_1)_p$ is constant and independent of the choice of point p and Y_1 in $(D_1)_p$;
(v) for any non-zero vector field $Y_2 \in (D_2)_q$, $q \in N_1$, the angle θ_2 between JY_2 and $(D_2)_q$ is constant and independent of the choice of point q and Z_2 in $(D_2)_q$.

These angles θ_1 and θ_2 are called slant angles of the Riemannian map.

Let π be conformal quasi bi-slant Riemannian map from an almost Hermitian manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then, we have

$$TN_1 = \ker \pi_* \oplus (\ker \pi_*)^{\perp}.$$

Now, for any vector field $U_1 \in \Gamma(\ker \pi_*)$, we have

$$U_1 = PU_1 + QU_1 + RU_1, \tag{3.1}$$

where P, Q and R are projection morphisms of $\ker \pi_*$ onto D, D_1 and D_2, respectively. For $W_1 \in \Gamma(\ker \pi_*)$, we get

$$JW_1 = \phi W_1 + \omega W_1, \tag{3.2}$$

where $\phi W_1 \in (\Gamma \ker \pi_*)$ and $\omega W_1 \in (\Gamma \ker \pi_*)^{\perp}$. From equations (3.1) and (3.2), we have

$$JU_1 = J(PU_1) + J(QU_1) + J(RU_1) = \phi(PU_1) + \omega(PU_1) + \phi(QU_1) + \omega(QU_1) + \phi(RU_1) + \omega(RU_1).$$

Since $JD = D$, we get $\omega PU_1 = 0$. Therefore, above equation reduces to

$$JU_1 = \phi(PU_1) + \phi QU_1 + \omega QU_1 + \phi RU_1 + \omega RU_1.$$

Now, we have the following decomposition

$$J(\ker \pi_*) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1 \oplus \omega D_2),$$

where $\oplus$ denotes orthogonal direct sum. Further, let $V_1 \in \Gamma(D_1)$ and $V_2 \in \Gamma(D_2)$. Then, we get

$$g_1(V_1, V_2) = 0, \qquad g_1(JV_1, V_2) = g_1(V_1, JV_2) = 0, \qquad g_1(\phi V_1, V_2) = 0, \qquad g_1(V_1, \phi V_2) = 0.$$

If $W_1 \in \Gamma(D)$, $W_2 \in \Gamma(D_1)$ and $W_3 \in \Gamma(D_2)$, then

$$g_1(\phi\, W_1, W_2) = 0, g_1(\phi W_1, W_3) = 0, g_1(\phi W_2, \phi W_3) = 0, g_1(\omega W_2, \omega W_3) = 0.$$

So, we can write $\phi D_1 \cap \phi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}$. Since $\omega D_1 \subseteq (\ker \pi_*)^{\perp}$, $\omega D_2 \subseteq (\ker \pi_*)^{\perp}$, so we can write

$$(\ker \pi_*)^{\perp} = \omega D_1 \oplus \omega D_2 \oplus \mu,$$

where μ is orthogonal complement of $(\omega D_1 \oplus \omega D_2)$ in $(\ker \pi_*)^{\perp}$. Also, for any non-zero vector field $X_1 \in (\ker \pi_*)^{\perp}$, we have

$$JX_1 = BX_1 + CX_1, \tag{3.3}$$

where $BX_1 \in \Gamma(\ker \pi_*)$ and $CX_1 \in \Gamma(\mu)$.

Lemma 3.2. *If π is a conformal quasi bi-slant Riemannian map, then*

$$\phi^2 U_1 + B\omega U_1 = -U_1, \omega\phi U_1 + C\omega U_1 = 0, \quad \omega B U_2 + C^2 U_2 = -U_2, \phi B U_2 + BCU_2 = 0,$$

for $U_1 \in \Gamma(\ker \pi_*)$ *and* $U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.2) and (3.3), we have Lemma 3.4. □

The proof of the following Lemma is exactly the same as that one for quasi bi-slant submersion, see Lemma 3.2 of [26]. So, we omit it.

Lemma 3.3. *If π is a conformal quasi-bi-slant Riemannian map, then*

(i) $\phi^2 Y_1 = -(\cos^2\theta_i) Y_1$;
(ii) $g_1(\phi Y_1, \phi Y_2) = \cos^2\theta_i g_1(Y_1, Y_2)$;
(iii) $g_1(\omega Y_1, \omega Y_2) = \sin^2\theta_i g_1(Y_1, Y_2)$, *for* $Y_1, Y_2 \in \Gamma(D_i)$, *where* $i = 1, 2$.

Lemma 3.4. *If π is a conformal quasi-bi-slant Riemannian map, then*

$$\mathcal{V}\nabla_{Y_1}\phi Y_2 + \mathcal{T}_{Y_1}\omega Y_2 = \phi\mathcal{V}\nabla_{Y_1} Y_2 + B\mathcal{T}_{Y_1} Y_2, \tag{3.4}$$
$$\mathcal{T}_{Y_1}\phi Y_2 + \mathcal{H}\nabla_{Y_1}\omega Y_2 = \omega\mathcal{V}\nabla_{Y_1} Y_2 + C\mathcal{T}_{Y_1} Y_2, \tag{3.5}$$
$$\mathcal{V}\nabla_{U_1} BU_2 + \mathcal{A}_{U_1} CU_2 = \phi\mathcal{A}_{U_1} U_2 + B\mathcal{H}\nabla_{U_1} U_2, \tag{3.6}$$
$$\mathcal{A}_{U_1} BU_2 + \mathcal{H}\nabla_{U_1} CU_2 = \omega\mathcal{A}_{U_1} U_2 + C\mathcal{H}\nabla_{U_1} U_2, \tag{3.7}$$
$$\mathcal{V}\nabla_{Y_1} BU_1 + \mathcal{T}_{Y_1} CU_1 = \phi\mathcal{T}_{Y_1} U_1 + B\mathcal{H}\nabla_{Y_1} U_1, \tag{3.8}$$
$$\mathcal{T}_{Y_1} BU_1 + \mathcal{H}\nabla_{Y_1} CU_1 = \omega\mathcal{T}_{Y_1} U_1 + C\mathcal{H}\nabla_{Y_1} U_1, \tag{3.9}$$
$$\mathcal{V}\nabla_{U_1}\phi Y_1 + \mathcal{A}_{U_1}\omega Y_1 = B\mathcal{A}_{U_1} Y_1 + \phi\mathcal{V}\nabla_{U_1} Y_1, \tag{3.10}$$
$$\mathcal{A}_{U_1}\phi Y_1 + \mathcal{H}\nabla_{U_1}\omega Y_1 = C\mathcal{A}_{U_1} Y_1 + \omega\mathcal{V}\nabla_{U_1} Y_1, \tag{3.11}$$

for any $Y_1, Y_2 \in \Gamma(\ker \pi_*)$ *and* $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (2.1), (2.3), (2.4), (2.7), (2.8), (3.2), and (3.3), we get equations (3.4)-(3.11). □

Now, we define

$$\begin{aligned} (\nabla_{V_1}\phi)V_2 &= \mathcal{V}\nabla_{V_1}\phi V_2 - \phi\mathcal{V}\nabla_{V_1} V_2, & (\nabla_{V_1}\omega)V_2 &= \mathcal{H}\nabla_{V_1}\omega V_2 - \omega\mathcal{V}\nabla_{V_1} V_2, \\ (\nabla_{U_1} C)U_2 &= \mathcal{H}\nabla_{U_1} CU_2 - C\mathcal{H}\nabla_{U_1} U_2, & (\nabla_{U_1} B)U_2 &= \mathcal{V}\nabla_{U_1} BU_2 - B\mathcal{H}\nabla_{U_1} U_2, \end{aligned} \tag{3.12}$$

for any $V_1, V_2 \in \Gamma(\ker \pi_*)$ and $U_1, U_2 \in \Gamma(\ker \pi_*)^\perp$.

Lemma 3.5. *If π is a conformal quasi-bi-slant Riemannian map, then*

$$\begin{aligned} (\nabla_{W_1}\phi)W_2 &= B\mathcal{T}_{W_1} W_2 - \mathcal{T}_{W_1}\omega W_2, & (\nabla_{W_1}\omega)W_2 &= C\mathcal{T}_{W_1} W_2 - \mathcal{T}_{W_1}\phi W_2, \\ (\nabla_{Z_1} C)Z_2 &= \omega\mathcal{A}_{Z_1} Z_2 - \mathcal{A}_{Z_1} BZ_2, & (\nabla_{Z_1} B)Z_2 &= \phi\mathcal{A}_{Z_1} Z_2 - \mathcal{A}_{Z_1} CZ_2, \end{aligned}$$

for any vectors $W_1, W_2 \in \Gamma(\ker \pi_*)$ *and* $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$.

Proof. Using equations (3.4), (3.5), (3.6), (3.7), and (3.12), we get all equations of Lemma 3.5. □

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on N_1, respectively, then

$$B\mathcal{T}_{Y_1} Y_2 = \mathcal{T}_{Y_1}\omega Y_2, C\mathcal{T}_{Y_1} Y_2 = \mathcal{T}_{Y_1}\phi Y_2,$$

for any $Y_1, Y_2 \in \Gamma(TN_1)$.

Theorem 3.6. D *is integrable if and only if*

$$\mathcal{V}\nabla_{V_1}JV_2 - \mathcal{V}\nabla_{V_2}JV_1 \in \Gamma(D), \phi(\mathcal{T}_{V_1 - V_2}\omega Z_1) \in \Gamma(D_1 \oplus D_2),$$

for $V_1, V_2 \in \Gamma(D)$ *and* $Z_1 \in \Gamma(D_1 \oplus D_2)$.

Proof. Using equations (2.1), (2.3), (2.4), and (3.2), we have

$$\begin{aligned}
g_1([V_1, V_2], Z_1) &= g_1(\nabla_{V_1}JV_2, JZ_1) - g_1(\nabla_{V_2}JV_1, JZ_1)\\
&= g_1(\nabla_{V_1}JV_2, \phi Z_1) - g_1(\nabla_{V_2}JV_1, \phi Z_1) + g_1(\nabla_{V_1}JV_2, \omega Z_1) - g_1(\nabla_{V_2}JV_1, \omega Z_1)\\
&= g_1(\mathcal{V}\nabla_{V_1}JV_2 - \mathcal{V}\nabla_{V_2}JV_1, \phi Z_1) - g_1(JV_2, \nabla_{V_1}\omega Z_1) + g_1(JV_1, \nabla_{V_2}\omega Z_1)\\
&= g_1(\mathcal{V}\nabla_{V_1}JV_2 - \mathcal{V}\nabla_{V_2}JV_1, \phi Z_1) + g_1(V_2, \phi\mathcal{T}_{V_1}\omega Z_1) - g_1(V_1, \phi\mathcal{T}_{V_2}\omega Z_1),
\end{aligned}$$

for $V_1, V_2 \in \Gamma(D)$ and $Z_1 \in \Gamma(D_1 \oplus D_2)$, which completes the proof. □

Theorem 3.7. D_1 *is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\Big\{g_2((\nabla\pi_*)(U_1, PW_1), \pi_*(\omega\phi U_2)) - g_2((\nabla\pi_*)(U_2, PW_1), \pi_*(\omega\phi U_1))\\
&\quad + g_2((\nabla\pi_*)(U_1, \omega U_2) - (\nabla\pi_*)(U_2, \omega U_1), \pi_*(\omega RW_1))\\
&\quad - g_2(\pi_*(\omega U_2), \pi_*(U_1, JPW_1)) + g_2(\pi_*(\omega U_1), \pi_*(U_2, JPW_1))\Big\}\\
&= g_1(\mathcal{H}\nabla_{U_1}\omega U_2 - \mathcal{H}\nabla_{U_2}\omega U_1, \omega RW_1) + g_1(\mathcal{V}\nabla_{U_1}\phi U_2 - \mathcal{V}\nabla_{U_2}\phi U_1, \phi RW_1)\\
&\quad + g_1(\mathcal{T}_{U_1}\omega U_2 - \mathcal{T}_{U_2}\omega U_1, \phi RW_1),
\end{aligned}$$

for $U_1, U_2 \in \Gamma(D_1)$ *and* $W_1 \in \Gamma(D \oplus D_2)$.

Proof. For $U_1, U_2 \in \Gamma(D_1)$ and $W_1 \in \Gamma(D \oplus D_2)$, we have

$$g_1([U_1, U_2], W_1) = g_1(\nabla_{U_1}U_2, W_1) - g_1(\nabla_{U_2}U_1, W_1).$$

Using equations (2.1), (2.3), (2.4), (3.1), (3.2), and Lemma 3.3, we have

$$\begin{aligned}
g_1([U_1, U_2], W_1) &= g_1(\nabla_{U_1}JU_2, JW_1) - g_1(\nabla_{U_2}JU_1, JW_1)\\
&= \cos^2\theta_1 g_1(\nabla_{U_1}U_2, PW_1) - \cos^2\theta_1 g_1(\nabla_{U_2}U_1, PW_1) + g_1(\omega\phi U_2, \mathcal{T}_{U_1}PW_1)\\
&\quad - g_1(\omega\phi U_1, \mathcal{T}_{U_2}PW_1) + g_1(\mathcal{V}\nabla_{U_1}\phi U_2 - \mathcal{V}\nabla_{U_2}\phi U_1, \phi RW_1) + g_1(\mathcal{T}_{U_1}\omega U_2\\
&\quad - \mathcal{T}_{U_2}\omega U_1, \omega RW_1) - g_1(\omega U_2, \nabla_{U_1}JPW_1) + g_1(\omega U_1, \nabla_{U_2}JPW_1)\\
&\quad + g_1(\mathcal{T}_{U_1}\omega U_2 - \mathcal{T}_{U_2}\omega U_1, \phi RW_1) + g_1(\mathcal{H}\nabla_{U_1}\omega U_2 - \mathcal{H}\nabla_{U_2}\omega U_1, \omega RW_1).
\end{aligned}$$

Since π is conformal Riemannian map, using equations (2.8) and (2.9), we have

$$\begin{aligned}
&g_1([U_1, U_2], W_1) - \cos^2\theta_1 g_1([U_1, U_2], W_1)\\
&= g_1(\mathcal{H}\nabla_{U_1}\omega U_2 - \mathcal{H}\nabla_{U_2}\omega U_1, \omega RW_1) + g_1(\mathcal{V}\nabla_{U_1}\phi U_2 - \mathcal{V}\nabla_{U_2}\phi U_1, \phi RW_1)\\
&\quad - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(U_1, PW_1), \pi_*(\omega\phi U_2)) + \frac{1}{\lambda^2}g_2((\nabla\pi_*)(U_2, PW_1), \pi_*(\omega\phi U_1))\\
&\quad - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(U_1, \omega U_2) - (\nabla\pi_*)(U_2, \omega U_1), \pi_*(\omega RW_1)) + \frac{1}{\lambda^2}g_2(\pi_*(\omega U_2), \pi_*(U_1, JPW_1))\\
&\quad - \frac{1}{\lambda^2}g_2(\pi_*(\omega U_1), \pi_*(U_2, JPW_1)) + g_1(\mathcal{T}_{U_1}\omega U_2 - \mathcal{T}_{U_2}\omega U_1, \phi RW_1),
\end{aligned}$$

which completes the proof. □

The proof of the following theorem is similar as the Theorem 3.7.

Theorem 3.8. D_2 *is integrable if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\{g_2((\nabla\pi_*)(Y_1,PX_1),\pi_*(\omega\phi Y_2)) - g_2((\nabla\pi_*)(Y_2,PX_1),\pi_*(\omega\phi Y_1))\\
&\quad + g_2((\nabla\pi_*)(Y_1,\omega Y_2) - (\nabla\pi_*)(Y_2,\omega Y_1),\pi_*(\omega RX_1))\\
&\quad - g_2(\pi_*(\omega Y_2),\pi_*(Y_1,JPX_1)) + g_2(\pi_*(\omega Y_1),\pi_*(Y_2,JPX_1))\}\\
&= g_1(\mathcal{H}\nabla_{Y_1}\omega Y_2 - \mathcal{H}\nabla_{Y_2}\omega Y_1,\omega RX_1) + g_1(\mathcal{V}\nabla_{Y_1}\phi Y_2 - \mathcal{V}\nabla_{Y_2}\phi Y_1,\phi RX_1)\\
&\quad + g_1(\mathcal{T}_{Y_1}\omega Y_2 - \mathcal{T}_{Y_2}\omega Y_1,\phi RX_1),
\end{aligned}$$

for $Y_1, Y_2 \in \Gamma(D_2)$ *and* $X_1 \in \Gamma(D \oplus D_1)$.

Theorem 3.9. $(\ker\pi_*)^\perp$ *is integrable if and only if*

$$\begin{aligned}
&g_1(CY_2,\operatorname{grad}\ln\lambda)g_1(Y_1,\omega\eta) - g_1(Y_1,CY_2)g_1(\operatorname{grad}\ln\lambda,\omega\eta)\\
&\quad - g_1(CY_1,\operatorname{grad}\ln\lambda)g_1(Y_2,\omega\eta) + g_1(Y_2,CY_1)g_1(\operatorname{grad}\ln\lambda,\omega\eta) - g_1(\mathcal{V}\nabla_{Y_1}BY_2 - \mathcal{V}\nabla_{Y_2}BY_1,\phi\eta)\\
&= \frac{1}{\lambda^2}\{g_2((\nabla\pi_*)(Y_2,BY_1) - (\nabla\pi_*)(Y_1,BY_2),\pi_*(\omega\eta))\\
&\quad - g_2((\nabla\pi_*)(Y_2,\phi\eta),\pi_*(CY_1)) + g_2((\nabla\pi_*)(Y_1,\phi\eta),\pi_*(CY_2))\\
&\quad + g_2(\nabla_{Y_1}\pi_*(CY_2) - \nabla_{Y_2}\pi_*(CY_1),\pi_*(\omega\eta))\},
\end{aligned}$$

for $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$ *and* $\eta \in \Gamma(\ker\pi_*)$.

Proof. We note that $\Gamma(\ker\pi_*)^\perp$ is integrable if and only if $g_1([Y_1,Y_2],\eta) = 0$, for all $Y_1, Y_2 \in \Gamma(\ker\pi_*)^\perp$ and $\eta \in \Gamma(\ker\pi_*)$. Now, using equations (2.1), (2.5), (2.6), (3.2), and (3.3), we have

$$\begin{aligned}
g_1([Y_1,Y_2],\eta) &= g_1(\nabla_{Y_1}Y_2,\eta) - g_1(\nabla_{Y_2}Y_1,\eta),\\
&= g_1(\mathcal{V}\nabla_{Y_1}BY_2,\phi\eta) - g_1(\mathcal{A}_{Y_1}\phi\eta,CY_2) + g_1(\mathcal{A}_{Y_1}BY_2,\omega\eta) + g_1(\mathcal{H}\nabla_{Y_1}BY_2,\omega\eta)\\
&\quad - g_1(\mathcal{V}\nabla_{Y_2}BY_1,\phi\eta) + g_1(\mathcal{A}_{Y_2}\phi\eta,CY_1) - g_1(\mathcal{A}_{Y_2}BY_1,\omega\eta) - g_1(\mathcal{H}\nabla_{Y_2}BY_1,\omega\eta).
\end{aligned}$$

Since π is conformal Riemannian map, using equations (2.8) and (2.9), we have

$$\begin{aligned}
g_1([Y_1,Y_2],\eta) &= g_1(\mathcal{V}\nabla_{Y_1}BY_2 - \mathcal{V}\nabla_{Y_2}BY_1,\phi\eta) - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Y_1,BY_2) - (\nabla\pi_*)(Y_2,BY_1),\pi_*(\omega\eta))\\
&\quad - \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Y_2,\phi\eta),\pi_*(CY_1)) + \frac{1}{\lambda^2}g_2((\nabla\pi_*)(Y_1,\phi\eta),\pi_*(CY_2))\\
&\quad + \frac{1}{\lambda^2}g_2(\nabla_{Y_1}\pi_*(CY_2) - \nabla_{Y_2}\pi_*(CY_1),\pi_*(\omega\eta)) - g_1(CY_2,\operatorname{grad}\ln\lambda)g_1(Y_1,\omega\eta)\\
&\quad + g_1(Y_1,CY_2)g_1(\operatorname{grad}\ln\lambda,\omega\eta) + g_1(CY_1,\operatorname{grad}\ln\lambda)g_1(Y_2,\omega\eta)\\
&\quad - g_1(Y_2,CY_1)g_1(\operatorname{grad}\ln\lambda,\omega\eta),
\end{aligned}$$

which completes the proof. □

Theorem 3.10. $(\ker\pi_*)^\perp$ *defines a totally geodesic foliation on* N_1 *if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\{g_2(\nabla_{Z_1}\pi_*Z_2,\pi_*(\omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta)) - \frac{1}{\lambda^2}g_2(\nabla_{Z_1}\pi_*(CZ_2),\pi_*(\omega\eta))\right\}\\
&= g_1(\mathcal{A}_{Z_1}Z_2, P\eta + \cos^2\theta_1 Q\eta + \cos^2\theta_2 R\eta) + g_1(\mathcal{A}_{Z_1}BZ_2,\omega\eta)\\
&\quad + g_1(Z_1,\operatorname{grad}\ln\lambda)g_1(Z_2,\omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta)\\
&\quad + g_1(Z_1,\operatorname{grad}\ln\lambda)g_1(Z_2,\omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta)\\
&\quad - g_1(Z_1,Z_2)g_1(\operatorname{grad}\ln\lambda,\omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta)\\
&\quad - g_1(Z_1,\omega\eta)g_1(CZ_2,\operatorname{grad}\ln\lambda) + g_1(Z_1,CZ_2)g_1(\omega\eta,\operatorname{grad}\ln\lambda),
\end{aligned}$$

for $Z_1, Z_2 \in \Gamma(\ker\pi_*)^\perp$ *and* $\eta \in \Gamma(\ker\pi_*)$.

Proof. For $Z_1, Z_2 \in \Gamma(\ker \pi_*)^\perp$ and $\eta \in \Gamma(\ker \pi_*)$, using equations (2.1), (3.2), and Lemma 3.3, we have

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, \eta) &= g_1(\nabla_{Z_1} JZ_2, JP\eta) + g_1(\nabla_{Z_1} JZ_2, JQ\eta) + g_1(\nabla_{Z_1} JRZ_2, JR\eta), \\ &= g_1(\nabla_{Z_1} Z_2, P\eta + \cos^2\theta_1 Q\eta + \cos^2\theta_2 R\eta) \\ &\quad - g_1(\nabla_{Z_1} Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) + g_1(\nabla_{Z_1} JZ_2, \omega Q\eta + \omega R\eta). \end{aligned}$$

Since $\omega P\eta + \omega Q\eta + \omega R\eta = \omega\eta, \omega P\eta = 0$ and using equations (2.5) and (2.6), we get

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, \eta) &= g_1(\mathcal{A}_{Z_1} Z_2, P\eta + \cos^2\theta_1 Q\eta + \cos^2\theta_2 R\eta) + g_1(\mathcal{A}_{Z_1} BZ_2, \omega\eta) \\ &\quad - g_1(\mathcal{H}\nabla_{Z_1} Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) + g_1(\mathcal{H}\nabla_{Z_1} CZ_2, \omega\eta). \end{aligned}$$

Since π is conformal Riemannian map, using equations (2.8) and (2.9), we have

$$\begin{aligned} g_1(\nabla_{Z_1} Z_2, \eta) = {} & g_1(\mathcal{A}_{Z_1} Z_2, P\eta + \cos^2\theta_1 Q\eta + \cos^2\theta_2 R\eta) + g_1(\mathcal{A}_{Z_1} BZ_2, \omega\eta) \\ & - \frac{1}{\lambda^2} g_2(\nabla_{Z_1} \pi_* Z_2, \pi_*(\omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta)) + \frac{1}{\lambda^2} g_2(\nabla_{Z_1} \pi_*(CZ_2), \pi_*(\omega\eta)) \\ & + g_1(Z_1, \operatorname{grad}\ln\lambda) g_1(Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) \\ & + g_1(Z_1, \operatorname{grad}\ln\lambda) g_1(Z_2, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) \\ & - g_1(Z_1, Z_2) g_1(\operatorname{grad}\ln\lambda, \omega\phi P\eta + \omega\phi Q\eta + \omega\phi R\eta) \\ & - g_1(Z_1, \omega\eta) g_1(CZ_2, \operatorname{grad}\ln\lambda) + g_1(Z_1, CZ_2) g_1(\omega\eta, \operatorname{grad}\ln\lambda), \end{aligned}$$

which completes the proof. □

Theorem 3.11. $(\ker \pi_*)$ *defines a totally geodesic foliation on* N_1 *if and only if*

$$\mathcal{T}_{W_1} PW_2 + \cos^2\theta_1 \mathcal{T}_{W_1} QW_2 + \cos^2\theta_2 \mathcal{T}_{W_1} RW_2 - \mathcal{H}\nabla_{W_1} \omega\phi W_2 - \omega\mathcal{T}_{W_1} \omega W_2 - C\mathcal{H}\nabla_{W_1} \omega W_2 = 0,$$

for $W_1, W_2 \in \Gamma(\ker \pi_*)$.

Proof. For $W_1, W_2 \in \Gamma(\ker \pi_*)$, using equations (2.1), (2.3), (2.4), (2.8), (3.2), (3.3), and Lemma 3.3, we have

$$\begin{aligned} (\nabla\pi_*)(W_1, W_2) &= \pi_*(J\nabla_{W_1} JW_2), \\ &= \pi_*(J\nabla_{W_1} \phi PW_2 + J\nabla_{W_1} \omega PW_2 + J\nabla_{W_1} \phi QW_2 + J\nabla_{W_1} \omega QW_2 \\ &\quad + J\nabla_{W_1} \phi RW_2 + J\nabla_{W_1} \omega RW_2), \\ &= \pi_*(-\mathcal{T}_{W_1} PW_2 - \mathcal{V}\nabla_{W_1} PW_2 - \cos^2\theta_1 \mathcal{T}_{W_1} QW_2 - \cos^2\theta_1 \mathcal{V}\nabla_{W_1} QW_2 - \cos^2\theta_2 \mathcal{T}_{W_1} RW_2 \\ &\quad - \cos^2\theta_1 \mathcal{V}\nabla_{W_1} RW_2 + \mathcal{T}_{W_1} \omega\phi PW_2 + \mathcal{H}\nabla_{W_1} \omega\phi PW_2 + \mathcal{T}_{W_1} \omega\phi QW_2 + \mathcal{T}_{W_1} \omega\phi RW_2 \\ &\quad + \mathcal{H}\nabla_{W_1} \omega\phi QW_2 + \mathcal{H}\nabla_{W_1} \omega\phi RW_2 + \phi\mathcal{T}_{W_1} \omega QW_2 + \omega\mathcal{T}_{W_1} \omega QW_2 + B\mathcal{H}\nabla_{W_1} \omega QW_2 \\ &\quad + C\mathcal{H}\nabla_{W_1} \omega QW_2 + \phi\mathcal{T}_{W_1} \omega RW_2 + \omega\mathcal{T}_{W_1} \omega RW_2 + B\mathcal{H}\nabla_{W_1} \omega RW_2 + C\mathcal{H}\nabla_{W_1} \omega RW_2 \\ &\quad + \phi\mathcal{T}_{W_1} \omega PW_2 + \omega\mathcal{T}_{W_1} \omega PW_2 + B\mathcal{H}\nabla_{W_1} \omega PW_2 + C\mathcal{H}\nabla_{W_1} \omega PW_2). \end{aligned}$$

Since $PW_2 + QW_2 + RW_2 = W_2$, $\omega PW_2 + \omega QW_2 + \omega RW_2 = \omega W_2$ and $\omega PW_2 = 0$, we get

$$\begin{aligned} (\nabla\pi_*)(W_1, W_2) &= \pi_*(-\mathcal{T}_{W_1} PW_2 - \mathcal{V}\nabla_{W_1} PW_2 - \cos^2\theta_1 \mathcal{T}_{W_1} QW_2 - \cos^2\theta_1 \mathcal{V}\nabla_{W_1} QW_2 \\ &\quad - \cos^2\theta_2 \mathcal{T}_{W_1} RW_2 - \cos^2\theta_2 \mathcal{V}\nabla_{W_1} RW_2 + \mathcal{T}_{W_1} \omega\phi W_2 + \mathcal{H}\nabla_{W_1} \omega\phi W_2 \\ &\quad + \phi\mathcal{T}_{W_1} \omega W_2 + \omega\mathcal{T}_{W_1} \omega W_2 + B\mathcal{H}\nabla_{W_1} \omega W_2 + C\mathcal{H}\nabla_{W_1} \omega W_2), \end{aligned}$$

the proof follows from the above equation. □

Theorem 3.12. D *defines a totally geodesic foliation on* N_1 *if and only if*

$$\omega\mathcal{V}\nabla_{Y_1} JY_2 + C\mathcal{T}_{Y_1} JY_2 = 0,$$

for $Y_1, Y_2 \in \Gamma(D)$.

Proof. Since D is invariant distribution, we get $JY_1 = \phi Y_1$, i.e., $\omega Y_1 = 0$. Using equations (2.1), (2.3), (2.8), (3.2), and (3.3), we have

$$(\nabla\pi_*)(Y_1, Y_2) = \pi_*(J\nabla_{Y_1}JY_2) = \pi_*(\phi\mathcal{V}\nabla_{Y_1}JY_2 + \omega\mathcal{V}\nabla_{Y_1}JY_2 + B\mathcal{T}_{Y_1}JY_2 + C\mathcal{T}_{Y_1}JY_2),$$

for all $Y_1, Y_2 \in \Gamma(D)$, which completes the proof. □

Theorem 3.13. D_1 *defines a totally geodesic foliation on* N_1 *if and only if*

$$\mathcal{H}\nabla_{Y_1}\omega\phi Y_2 + C\mathcal{H}\nabla_{Y_1}\omega Y_2 + \omega\mathcal{T}_{Y_1}\omega Y_2 = 0,$$

for $Y_1, Y_2 \in \Gamma(D_1)$.

Proof. For all $Y_1, Y_2 \in \Gamma(D_1)$, using equations (2.1), (2.3), (2.4), (2.8), (3.2), and Lemma 3.3, we have

$$(\nabla\pi_*)(Y_1, Y_2) = \pi_*(J\nabla_{Y_1}\phi Y_2 + J\nabla_{Y_1}\omega Y_2) = \pi_*(-\cos^2\theta_1\nabla_{Y_1}Y_2 + \nabla_{Y_1}\omega\phi Y_2 + J\mathcal{H}\nabla_{Y_1}\omega Y_2 + J\mathcal{T}_{Y_1}\omega Y_2).$$

Now, using equations (2.4) and (3.3), we have

$$\sin^2\theta_1(\nabla\pi_*)(Y_1, Y_2) = \pi_*(\mathcal{H}\nabla_{Y_1}\omega\phi Y_2 + \mathcal{T}_{Y_1}\omega\phi Y_2 + B\mathcal{H}\nabla_{Y_1}\omega Y_2 + C\mathcal{H}\nabla_{Y_1}\omega Y_2 + \phi\mathcal{T}_{Y_1}\omega Y_2 + \omega\mathcal{T}_{Y_1}\omega Y_2),$$

which completes the proof. □

Theorem 3.14. D_2 *defines a totally geodesic foliation on* N_1 *if and only if*

$$\mathcal{H}\nabla_{U_1}\omega\phi U_2 + C\mathcal{H}\nabla_{U_1}\omega U_2 + \omega\mathcal{T}_{U_1}\omega U_2 = 0,$$

for all $U_1, U_2 \in \Gamma(D_2)$.

Proof. The proof of the above theorem follows the similar approach as the proof of Theorem 3.14. □

Theorem 3.15. *Let* π *be a conformal quasi-bi-slant Riemannian map from a Kähler manifold* (N_1, g_1, J) *to a Riemannian manifold* (N_2, g_2). *Then, any two of following assertions imply the third one:*

(i) *the horizontal distribution* $(\ker\pi_*)^\perp$ *defines totally geodesic foliation on* N_1*;*
(ii) *the map* π *is a horizontally homothetic map;*
(iii) $\nabla^\pi_{JX_1}\pi_*(CX_2) = \pi_*(J[JX_1, X_2]) + (\nabla\pi_*)(CX_1, CX_2)^\perp + \pi_*(\mathcal{A}_{CX_2}BX_1 + \mathcal{A}_{CX_1}BX_2 + \mathcal{T}_{BX_1}BX_2)$, *for* $X_1, X_2 \in (\ker\pi_*)^\perp$.

Proof. For all $X_1, X_2 \in (\ker\pi_*)^\perp$, using equations (2.1), (2.3), (2.5), (2.8), (2.9), and (3.3), we get

$$\begin{aligned}\pi_*(\nabla_{JX_1}JX_2) = {} & \nabla^\pi_{JX_1}\pi_*(CX_2) - (\nabla\pi_*)(CX_1, BX_2) - (\nabla\pi_*)(BX_1, BX_2)\\ & - (\nabla\pi_*)(BX_1, CX_2) - (\nabla\pi_*)(CX_1, CX_2),\\ = {} & \nabla^\pi_{JX_1}\pi_*(CX_2) - (\nabla\pi_*)(CX_1, CX_2)^\perp - \pi_*(\mathcal{A}_{CX_2}BX_1 + \mathcal{A}_{CX_1}BX_2 + \mathcal{T}_{BX_1}BX_2)\\ & - CX_1(\ln\lambda)\pi_*(CX_2) - CX_2(\ln\lambda)\pi_*(CX_1) + g_1(CX_1, CX_2)\pi_*(\mathrm{grad}\ln\lambda).\end{aligned} \tag{3.13}$$

On the other hand, we get

$$\nabla_{JX_1}JX_2 = J[JX_1, X_2] + J\nabla_{X_2}JX_1, \quad \nabla_{X_2}X_1 = J[JX_1, X_2] - \nabla_{JX_1}JX_2. \tag{3.14}$$

From equations (3.13) and (3.14), we have

$$\begin{aligned}\pi_*(\nabla_{X_2}X_1) = {} & \pi_*(J[JX_1, X_2]) - \nabla^\pi_{JX_1}\pi_*(CX_2) + (\nabla\pi_*)(CX_1, CX_2)^\perp\\ & + \pi_*(\mathcal{A}_{CX_2}BX_1 + \mathcal{A}_{CX_1}BX_2 + \mathcal{T}_{BX_1}BX_2) + CX_1(\ln\lambda)\pi_*(CX_2)\\ & + CX_2(\ln\lambda)\pi_*(CX_1) - g_1(CX_1, CX_2)\pi_*(\mathrm{grad}\ln\lambda).\end{aligned} \tag{3.15}$$

Now, taking assertions (i) and (ii) in equation (3.15), we get the (iii). Taking assertions (ii) and (iii) and equation (3.15), we get $\pi_*(\nabla_{X_2}X_1) = 0$. Hence, the horizontal distribution $(\ker\pi_*)^\perp$ defines totally geodesic foliation on N_1. Further, using assertions (i) and (iii) and equation (3.15), we have

$$CX_1(\ln\lambda)\pi_*(CX_2) + CX_2(\ln\lambda)\pi_*(CX_1) - g_1(CX_1, CX_2)\pi_*(\text{grad}\ln\lambda) = 0,$$

for $CX_1 \in \Gamma(\mu)$. Taking $X_2 = X_1$ in the above equation, we obtain

$$CX_1(\ln\lambda)\pi_*(CX_1) + CX_1(\ln\lambda)\pi_*(CX_1) - g_1(CX_1, CX_1)\pi_*(\text{grad}\ln\lambda) = 0. \tag{3.16}$$

Taking inner product in equation (3.16) with $\pi_*(CX_1)$, we get

$$\lambda^2 CX_1(\ln\lambda)g_1(CX_1, CX_1) = 0. \tag{3.17}$$

It gives λ is a constant on μ, for all $Y_1 \in \Gamma(\ker\pi_*)$ and $\omega Y_1 \in \Gamma(\omega D_1 \oplus \omega D_2)$. Similarly, taking inner product in equation (3.16) with $\pi_*(\omega Y_1)$, we have

$$\lambda^2 \omega Y_1(\ln\lambda)g_1(CX_1, CX_1) = 0. \tag{3.18}$$

It means λ is a constant on $\Gamma(\omega D_1 \oplus \omega D_2)$. Therefore, λ is constant on horizontal distribution. Thus, from equations (3.17) and (3.18), we obtain (iii) one. □

Theorem 3.16. *Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). Then the map π, defines totally geodesic foliations on N_1 if and only if*

(i) *the map π is a horizontally homothetic map; and*

(ii) $\overset{N_2}{\nabla}{}^{\pi}_{X}\pi_*(X_2) + (\nabla\pi_*)^\perp(X_1, X_2) = \pi_*(\mathcal{A}_{X_1}Y_2 + \mathcal{T}_{Y_1}Y_2 + \mathcal{H}\overset{N_1}{\nabla}{}^{\pi}_{Y_1}X_2) + \overset{N_2}{\nabla}{}^{\pi}_{X_1}\pi_*(X_2)$, *provided for $X, Y \in \Gamma(TN_1)$, where X_1, X_2 and Y_1, Y_2 are horizontal and vertical parts of X and Y, respectively.*

Proof. From equations (2.4), (2.5), (2.6), and (2.8), we get

$$\begin{aligned}(\nabla\pi_*)(X, Y) &= \overset{N_2}{\nabla}{}^{\pi}_{X}\pi_*(X_2) - \pi_*(\overset{N_1}{\nabla}{}^{\pi}_{X_1}Y_2 + \overset{N_1}{\nabla}{}^{\pi}_{Y_1}Y_2 + \overset{N_1}{\nabla}{}^{\pi}_{Y_1}X_2 + \overset{N_1}{\nabla}{}^{\pi}_{X_1}X_2)\\ &= \overset{N_2}{\nabla}{}^{\pi}_{X}\pi_*(X_2) - \pi_*(\mathcal{A}_{X_1}Y_2 + \mathcal{T}_{Y_1}Y_2 + \mathcal{H}\overset{N_1}{\nabla}{}^{\pi}_{Y_1}X_2) - \pi_*(\overset{N_1}{\nabla}{}^{\pi}_{X_1}X_2),\end{aligned}$$

for $X, Y \in \Gamma(TN_1)$, X_1, X_2 and Y_1, Y_2 are horizontal and vertical parts of X and Y, respectively. Using equation (2.9), we get

$$\begin{aligned}(\nabla\pi_*)(X, Y) = {}&\overset{N_2}{\nabla}{}^{\pi}_{X}\pi_*(X_2) - \pi_*(\mathcal{A}_{X_1}Y_2 + \mathcal{T}_{Y_1}Y_2 + \mathcal{H}\overset{N_1}{\nabla}{}^{\pi}_{Y_1}X_2) - \overset{N_2}{\nabla}{}^{\pi}_{X_1}\pi_*(X_2)\\ &+ X_1(\ln\lambda)\pi_*(X_2) + X_2(\ln\lambda)\pi_*(X_1) - g_1(X_1, X_2)\pi_*(\text{grad}\ln\lambda) + (\nabla\pi_*)^\perp(X_1, X_2).\end{aligned} \tag{3.19}$$

Since π defines totally geodesic foliation on N_1, we have (3.19). When we take π is a horizontally homothetic map then from equation (3.19), we get

$$X_1(\ln\lambda)\pi_*(X_2) + X_2(\ln\lambda)\pi_*(X_1) - g_1(X_1, X_2)\pi_*(\text{grad}\ln\lambda) = 0. \tag{3.20}$$

From equation (3.20), we have

$$\lambda^2 X_2(\ln\lambda)g_1(X_1, X_1) = 0, \tag{3.21}$$

for $X_1 \in \Gamma(\ker\pi_*)^\perp$. From equation (3.21) , λ is a constant on horizontal distribution. Since π is a horizontally homothetic map, we get assertion (i). Further, from equation (3.19), we have

$$\overset{N_2}{\nabla}{}^{\pi}_{X}\pi_*(X_2) = \pi_*(\mathcal{A}_{X_1}Y_2 + \mathcal{T}_{Y_1}Y_2 + \mathcal{H}\overset{N_1}{\nabla}{}^{\pi}_{Y_1}X_2) + \overset{N_2}{\nabla}{}^{\pi}_{X_1}\pi_*(X_2) - (\nabla\pi_*)^\perp(X_1, X_2). \tag{3.22}$$

From equation (3.22), we get (ii). □

Theorem 3.17. *Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If π is a D-pluriharmonic map then one of the below assertions imply the second one:*

(i) D *defines totally geodesic foliation on* N_1*;*

(ii) $C\mathcal{T}_{JX_1}X_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{JX_1}X_2 = 0$, *for* $X_1, X_2 \in \Gamma(D)$.

Proof. By definition of pluriharmonic map, we have

$$0 = (\nabla\pi_*)(X_1, X_2) + (\nabla\pi_*)(JX_1, JX_2),$$

for $X_1, X_2 \in \Gamma(D)$. From equations (2.1), (2.4), (2.5), (2.8), (3.2), and (3.3), we get

$$\begin{aligned}
\pi_*(\overset{N_1}{\nabla}_{X_1}X_2) &= -\pi_*(J(\overset{N_1}{\nabla}_{JX_1}X_2)),\\
\pi_*(\overset{N_1}{\nabla}_{X_1}X_2) &= -\pi_*(J(\mathcal{T}_{JX_1}X_2 + \mathcal{V}\overset{N_1}{\nabla}_{JX_1}X_2)),\\
\pi_*(\overset{N_1}{\nabla}_{X_1}X_2) &= -\pi_*(B\mathcal{T}_{JX_1}X_2 + C\mathcal{T}_{JX_1}X_2 + \phi\mathcal{V}\overset{N_1}{\nabla}_{JX_1}X_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{JX_1}X_2).
\end{aligned} \tag{3.23}$$

Taking assertion (i) in equation (3.23), we obtain (ii) as, $C\mathcal{T}_{JX_1}X_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{JX_1}X_2 = 0$. Similarly, taking assertion (ii) in equation (3.23), we get (i) one. □

Theorem 3.18. *Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If π is a D_1-pluriharmonic map then any two of the following assertions imply the third one:*

(i) D_1 *defines totally geodesic foliation on* N_1*;*

(ii) λ *is constant on* ωD_1 *and* $(\nabla\pi_*)^\perp(\omega Y_1, \omega Y_2) = 0$*;*

(iii) $\cos^2\theta_1(C\mathcal{T}_{\phi Y_1}Y_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{\phi Y_1}Y_2) = (C\mathcal{H}\overset{N_1}{\nabla}_{\phi Y_1}\omega\phi Y_2 + \omega\mathcal{T}_{\phi Y_1}\omega\phi Y_2) - (\mathcal{A}_{\omega Y_2}\omega Y_1 + \mathcal{A}_{\omega Y_1}\phi Y_2)$, *for* $Y_1, Y_2 \in \Gamma(D_1)$.

Proof. By definition of pluriharmonic map, we get

$$0 = (\nabla\pi_*)(Y_1, Y_2) + (\nabla\pi_*)(JY_1, JY_2),$$

for $Y_1, Y_2 \in \Gamma(D_1)$. Using equations (2.1), (2.5), (2.9), and Lemma 3.3, we get

$$\begin{aligned}
\pi_*(\overset{N_1}{\nabla}_{Y_1}Y_2) &= -\pi_*(\overset{N_1}{\nabla}_{\phi Y_1}\phi Y_2) - \pi_*(\overset{N_1}{\nabla}_{\omega Y_2}\phi Y_1) - \pi_*(\overset{N_1}{\nabla}_{\omega Y_1}\phi Y_2) - \pi_*(\overset{N_1}{\nabla}_{\omega Y_1}\omega Y_2),\\
&= \pi_*(J\overset{N_1}{\nabla}_{\phi Y_1}J\phi Y_2) - \pi_*(\mathcal{V}\overset{N_1}{\nabla}_{\omega Y_2}\phi Y_1 + \mathcal{A}_{\omega Y_2}\phi Y_1 + \mathcal{V}\overset{N_1}{\nabla}_{\omega Y_1}\phi Y_2 + \mathcal{A}_{\omega Y_1}\phi Y_2)\\
&\quad + (\nabla\pi_*)^\perp(\omega Y_1, \omega Y_2) + \omega Y_1(\ln\lambda)\pi_*(\omega Y_2) + \omega Y_2(\ln\lambda)\pi_*(\omega Y_1)\\
&\quad - g_1(\omega Y_1, \omega Y_2)\pi_*(\operatorname{grad}\ln\lambda),\\
&= -\cos^2\theta_1\pi_*(B\mathcal{T}_{\phi Y_1}Y_2 + C\mathcal{T}_{\phi Y_1}Y_2 + \phi\mathcal{V}\overset{N_1}{\nabla}_{\phi Y_1}Y_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{\phi Y_1}Y_2)\\
&\quad + \pi_*(B\mathcal{H}\overset{N_1}{\nabla}_{\phi Y_1}\omega\phi Y_2 + C\mathcal{H}\overset{N_1}{\nabla}_{\phi Y_1}\omega\phi Y_2 + \phi\mathcal{T}_{\phi Y_1}\omega\phi Y_2 + \omega\mathcal{T}_{\phi Y_1}\omega\phi Y_2)\\
&\quad - \pi_*(\mathcal{V}\overset{N_1}{\nabla}_{\omega Y_2}\phi Y_1 + \mathcal{A}_{\omega Y_2}\phi Y_1 + \mathcal{V}\overset{N_1}{\nabla}_{\omega Y_1}\phi Y_2 + \mathcal{A}_{\omega Y_1}\phi Y_2)\\
&\quad + (\nabla\pi_*)^\perp(\omega Y_1, \omega Y_2) + \omega Y_1(\ln\lambda)\pi_*(\omega Y_2) + \omega Y_2(\ln\lambda)\pi_*(\omega Y_1)\\
&\quad - g_1(\omega Y_1, \omega Y_2)\pi_*(\operatorname{grad}\ln\lambda).
\end{aligned} \tag{3.24}$$

Now, taking assertions (i) and (ii) in equation (3.24), we get

$$\begin{aligned}\pi_*(\overset{N_1}{\nabla}_{Y_1}Y_2) &= 0,\\ \omega Y_1(\ln\lambda)\pi_*(\omega Y_2) + \omega Y_2(\ln\lambda)\pi_*(\omega Y_1) - g_1(\omega Y_1, \omega Y_2)\pi_*(\operatorname{grad}\ln\lambda) &= 0,\\ (\nabla\pi_*)^\perp(\omega Y_1, \omega Y_2) &= 0,\end{aligned}$$

respectively. We obtain (iii) as

$$\cos^2\theta_1(C\mathcal{T}_{\phi Y_1}Y_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{\phi Y_1}Y_2) = (C\mathcal{H}\overset{N_1}{\nabla}_{\phi Y_1}\omega\phi Y_2 + \omega\mathcal{T}_{\phi Y_1}\omega\phi Y_2) - (\mathcal{A}_{\omega Y_2}\omega Y_1 + \mathcal{A}_{\omega Y_1}\phi Y_2).$$

Taking assertions (ii) and (iii) in equation (3.24), we get (i) . Lastly, suppose that (i) and (iii) are satisfied in equation (3.24). Then, we get

$$\omega Y_1(\ln\lambda)\pi_*(\omega Y_2) + \omega Y_2(\ln\lambda)\pi_*(\omega Y_1) - g_1(\omega Y_1, \omega Y_2)\pi_*(\operatorname{grad}\ln\lambda) = 0. \tag{3.25}$$

Taking inner product in equation (3.25) with $\pi_*(\omega Y_1)$, we get

$$\lambda^2\omega Y_2(\ln\lambda)g_1(\omega Y_1, \omega Y_1) = 0, \tag{3.26}$$

for all $\omega Y_1 \in \Gamma(D_1)$. We have $\omega Y_2(\ln\lambda) = 0$, from equation (3.26), i.e., $\omega D_1(\ln\lambda) = 0$. Hence, we obtain assertion (ii). Thus, we complete the proof. □

In a similar way as above theorem, we obtain the following theorem.

Theorem 3.19. *Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If π is a D_2-pluriharmonic map then any two of the following assertions imply the third one:*

(i) *D_2 defines totally geodesic foliation on N_1;*
(ii) *λ is constant on ωD_2 and $(\nabla\pi_*)(\omega Z_1, \omega Z_2) = 0$;*
(iii) $\cos^2\theta_2(C\mathcal{T}_{\phi Z_1}Z_2 + \omega\mathcal{V}\overset{N_1}{\nabla}_{\phi Z_1}Z_2) = (C\mathcal{H}\overset{N_1}{\nabla}_{\phi Z_1}\omega\phi Z_2 + \omega\mathcal{T}_{\phi Z_1}\omega\phi Z_2) - (\mathcal{A}_{\omega Z_2}\omega Z_1 + \mathcal{A}_{\omega Z_1}\phi Z_2)$, *for $Z_1, Z_2 \in \Gamma(D_2)$.*

Theorem 3.20. *Let π be a conformal quasi-bi-slant Riemannian map from a Kähler manifold (N_1, g_1, J) to a Riemannian manifold (N_2, g_2). If π is a $(\ker\pi_*)^\perp$-pluri-harmonic map then any two of the following assertions imply the third one:*

(i) *$(\ker\pi_*)^\perp$ defines totally geodesic foliation on N_1;*
(ii) *λ is a constant on μ;*
(iii) $\overset{N_2}{\nabla}{}^{\pi}_{X_1}\pi_*(X_2) = \pi_*(\mathcal{T}_{BX_1}BX_2 + \mathcal{A}_{CX_1}BX_2 + \mathcal{A}_{CX_2}BX_1) + (\nabla\pi_*)^\perp(CX_1, CX_2)$, *for $X_1, X_2 \in \Gamma(\ker\pi_*)^\perp$.*

Proof. By definition of pluri-harmonic map, we have

$$(\nabla\pi_*)(X_1, X_2) + (\nabla\pi_*)(JX_1, JX_2) = 0,$$

for $X_1, X_2 \in \Gamma(\ker\pi_*)^\perp$. Now, using equations (2.1), (2.3) ,(2.5), and (2.9), we get

$$\begin{aligned}0 = {}&\overset{N_2}{\nabla}{}^{\pi}_{X_1}\pi_*(X_2) - \pi_*(\overset{N_1}{\nabla}_{X_1}X_2) - \pi_*(\mathcal{T}_{BX_1}BX_2 + \mathcal{V}\overset{N_1}{\nabla}_{BX_1}BX_2\\ &+ \mathcal{A}_{CX_1}BX_2 + \mathcal{A}_{CX_2}BX_1 + \mathcal{V}\overset{N_1}{\nabla}_{CX_1}BX_2 + \mathcal{V}\overset{N_1}{\nabla}_{CX_2}BX_1)\\ &+ CX_1(\ln\lambda)\pi_*(CX_2) + CX_2(\ln\lambda)\pi_*(CX_1) - g_1(CX_1, CX_2)\pi_*(\operatorname{grad}\ln\lambda) + (\nabla\pi_*)^\perp(CX_1, CX_2).\end{aligned} \tag{3.27}$$

When (i) and (ii) are satisfied, from equation (3.27), we get

$$\begin{aligned} \pi_*(\overset{N_1}{\nabla}_{X_1}X_2) &= 0,\\ CX_1(\ln\lambda)\pi_*(CX_2) + CX_2(\ln\lambda)\pi_*(CX_1) - g_1(CX_1, CX_2)\pi_*(\operatorname{grad}\ln\lambda) &= 0. \end{aligned} \tag{3.28}$$

So, we get assertion (iii).

When assertions (ii) and (iii) are satisfied in equation (3.27), we obtain $\pi_*(\overset{N_1}{\nabla}_{X_1}X_2) = 0$, which means that $(\ker\pi_*)^\perp$ defines totally geodesic foliation on N_1, for all $X_1, X_2 \in \Gamma(\ker\pi_*)^\perp$. Hence, assertions (ii) and (iii) imply assertion (i). Further, when assertions (i) and (iii) are satisfied in equation (3.27), we obtain equation (3.28). From equation (3.27), we get

$$\begin{aligned} \lambda^2 CX_1(\ln\lambda)g_1(CX_2, CX_2) + \lambda^2 CX_2(\ln\lambda)g_1(CX_2, CX_1) - \lambda^2 g_1(CX_1, CX_2)CX_2(\ln\lambda) &= 0,\\ \lambda^2 CX_1(\ln\lambda)g_1(CX_2, CX_2) &= 0, \end{aligned}$$

for all $CX_1, CX_2 \in \Gamma(\mu)$. Here, we have $CX_1(\ln\lambda) = 0$ which implies that λ is a constant on μ. So we have assertion (ii), which completes the proof. □

4. Example

Note that given an Euclidean space R^{2k} with coordinates $(x_1, x_2, \dots, x_{2k-1}, x_{2k})$, we can canonically choose an almost complex structure J on R^{2k} as follows:

$$J(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + \cdots + a_{2k-1}\frac{\partial}{\partial x_{2k-1}} + a_{2k}\frac{\partial}{\partial x_{2k}}) = -a_2\frac{\partial}{\partial x_1} + a_1\frac{\partial}{\partial x_2} + \cdots - a_{2k}\frac{\partial}{\partial x_{2k-1}} + a_{2k-1}\frac{\partial}{\partial x_{2k}},$$

where $a_1, a_2, \dots, a_{2k}$ are C^∞ functions defined on R^{2k}. Throughout this section, we will use this notation.

Example 4.1. Define a map $\pi : R^{10} \to R^6$ by

$$\pi(x_1, x_2, \dots, x_{10}) = e^5(x_1 - x_3, x_4, 2021, x_6 + x_8, x_7, 2022),$$

which is a conformal quasi-bi-slant Riemannian map, such that

$$X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3},\ X_2 = \frac{\partial}{\partial x_2},\ X_3 = \frac{\partial}{\partial x_5},\ X_4 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8},\ X_5 = \frac{\partial}{\partial x_9},\ X_6 = \frac{\partial}{\partial x_{10}},\ \ker\pi_* = D \oplus D_1 \oplus D_2,$$

where

$$\begin{aligned} D &=< X_5 = \frac{\partial}{\partial x_9}, X_6 = \frac{\partial}{\partial x_{10}} >,\\ D_1 &=< X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2} >,\\ D_2 &=< X_3 = \frac{\partial}{\partial x_5}, X_4 = \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8} >,\\ (\ker\pi_*)^\perp &=< H_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}, H_2 = \frac{\partial}{\partial x_4}, H_3 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8}, H_4 = \frac{\partial}{\partial x_7} >,\\ \pi_*H_1 = e^5\frac{\partial}{\partial v_1},&\ \pi_*H_2 = e^5\frac{\partial}{\partial v_2},\ \pi_*H_3 = e^5\frac{\partial}{\partial v_3},\ \pi_*H_4 = e^5\frac{\partial}{\partial v_4}, \end{aligned}$$

with quasi-bi-slant angle $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{4}$. Hence, we have

$$\begin{aligned} g_2(\pi_*H_1, \pi_*H_1) = (e^5)^2 g_1(H_1, H_1),\ g_2(\pi_*H_2, \pi_*H_2) = (e^5)^2 g_1(H_2, H_2),\\ g_2(\pi_*H_3, \pi_*H_3) = (e^5)^2 g_1(H_3, H_3),\ g_2(\pi_*H_4, \pi_*H_4) = (e^5)^2 g_1(H_4, H_4). \end{aligned}$$

Thus, π is a conformal quasi-bi-slant Riemannian map with $\lambda = e^5$.

Example 4.2. Let $\pi : (\mathbb{R}^{10}, g_{10} = e^{x_6}(dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2 + dx_9^2 + dx_{10}^2)) \to (\mathbb{R}^4, g_4 = (dv_1^2 + dv_2^2 + dv_3^2 + dv_4^2))$ defined by

$$\pi(x_1, \dots, x_{10}) = (2020, \frac{x_3 - \sqrt{3}x_5}{2}, x_6, \frac{x_7 + x_9}{\sqrt{2}}),$$

which is a conformal quasi-bi-slant Riemannian map such that

$$X_1 = \frac{\partial}{\partial x_1},\ X_2 = \frac{\partial}{\partial x_2},\ X_3 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}),\ X_4 = \frac{\partial}{\partial x_4},\ X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}),\ X_6 = \frac{\partial}{\partial x_8},\ X_7 = \frac{\partial}{\partial x_{10}},$$

$\ker \pi_* = D \oplus D_1 \oplus D_2,$

where

$$\begin{aligned}
D &=< X_1 = \frac{\partial}{\partial x_1}, X_2 = \frac{\partial}{\partial x_2} >,\\
D_1 &=< X_3 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_5}), X_4 = \frac{\partial}{\partial x_4} >,\\
D_2 &=< X_5 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_9}), X_6 = \frac{\partial}{\partial x_8}, X_7 = \frac{\partial}{\partial x_{10}} >,\\
(\ker \pi_*)^\perp &=< H_1 = \frac{1}{2}(\frac{\partial}{\partial x_3} - \sqrt{3}\frac{\partial}{\partial x_5}), H_2 = \frac{\partial}{\partial x_6}, H_3 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}) >,\\
\pi_* H_1 &= \frac{\partial}{\partial v_2},\ \pi_* H_2 = \frac{\partial}{\partial v_3},\ \pi_* H_3 = \frac{\partial}{\partial v_4},
\end{aligned}$$

with conformal quasi-bi-slant angle $\theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{\pi}{4}$. Hence, we have

$$g_2(\pi_* H_1, \pi_* H_1) = e^{-2x_6} g_1(H_1, H_1),\quad g_2(\pi_* H_2, \pi_* H_2) = e^{-2x_6} g_1(H_2, H_2),\quad g_2(\pi_* H_3, \pi_* H_3) = e^{-2x_6} g_1(H_3, H_3).$$

Thus, π is conformal quasi-bi-slant Riemannian map with $\lambda = e^{-x_6}$.

The defined map π in above examples satisfies Lemma 3.3 for any $Y_1, Y_2 \in \Gamma(D)$. One can easily observe that $(\nabla \pi_*)(Y_1, Y_2) = 0$, i.e., D defines totally geodesic foliation on $\mathbb{R}^{10}$. So, the map π satisfies Theorems 3.13 and 3.18.

Similarly the distribution D_1 satisfies Theorems 3.14 and 3.19 and the distribution D_2 satisfies Theorems 3.15 and 3.20.

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