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Positive periodic solution of a discrete commensal symbiosis model with Beddington-DeAngelis functional response

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Abstract

A non-autonomous discrete commensal symbiosis model with Beddington-DeAngelis functional response is proposed and studied in this paper. Sufficient conditions are obtained for the existence of positive periodic solution of the system.

Keywords: Commensal symbiosis model, positive periodic solution, Beddington-DeAngelis functional response.

2020 MSC: 34C25, 92D25, 34D20, 34D40.

1. Introduction

In the past decade, numerous works ([6, 13, 14, 31, 32, 37–39]) on the mutualism model has been published and many excellent works concerned with the persistence, existence of positive periodic solution, and stability of the system were obtained. However, only recently did scholars paid attention to the commensal symbiosis model, an interaction in which one population obtains benefits while the other population is neither benefited nor harmed ([1, 2, 4, 5, 7, 8, 11, 12, 15–30, 33–36, 41, 42]).

Though there are numerous works on commensalism model, most of them were focus on the continuous case, and to this day, there are only few works on discrete commensalism model (see[5, 20, 25, 33–35]). It is well known that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations.

In 2015, Xue et al. [34] first time proposed a discrete commensalism model

$$x_1(k+1) = x_1(k)\exp\left\{r_1\left(1-\frac{x(k)}{K_1}+\frac{\alpha y(k)}{K_1}\right)\right\}, \quad x_2(k+1) = x_2(k)\exp\left\{r_2\left(1-\frac{y(k)}{K_2}\right)\right\}. \tag{1.1}$$

They investigated the local and global stability property of the system. Xie et al. [33] proposed the following discrete commensal symbiosis model

$$x_1(k+1) = x_1(k)\exp\left\{a_1(k)-b_1(k)x_1(k)+c_1(k)x_2(k)\right\},\ x_2(k+1) = x_2(k)\exp\left\{a_2(k)-b_2(k)x_2(k)\right\}.$$

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Under the assumption that $\{b_i(k)\}, i = 1,2, \{c_1(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$ are ω-periodic sequences, which satisfies $\overline{a}_i = \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} a_i(k) > 0, i = 1,2$. They investigated the positive periodic solution of the system. Li et al. [20] argued that a suitable relationship between two species should be a nonlinear one. They adopt the Holling II functional response to system (1.1), and proposed the following two species discrete commensal symbiosis model

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left\{a_1(k) - b_1(k)x_1(k) + \frac{c_1(k)x_2(k)}{e_1(k)+f_1(k)x_2(k)}\right\},\\ x_2(k+1) &= x_2(k)\exp\left\{a_2(k) - b_2(k)x_2(k)\right\}. \end{aligned} \tag{1.2}$$

The authors of [20] also obtained a set of sufficient conditions which ensure the existence of the positive periodic solution of the system.

It brings to our attention that in system (1.2), the functional response takes the form

$$\frac{c_1(k)x_2(k)}{e_1(k)+f_1(k)x_2(k)},$$

which is independent of the first species. Such an assumption seems curiously since generally speaking, in the process of obtain the benefit from the second species, mutual interference between the first species will be happen and this reduce the overall benefit from the second species.

Now, if we introduce the Beddington-DeAngelis functional response [3, 11, 21, 40] to two species commensalism model, we can propose the following model

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\left\{a_1(k) - b_1(k)x_1(k) + \frac{c_1(k)x_2(k)}{e_1(k)+f_1(k)x_2(k)+f_2(k)x_1(k)}\right\},\\ x_2(k+1) &= x_2(k)\exp\left\{a_2(k) - b_2(k)x_2(k)\right\}, \end{aligned} \tag{1.3}$$

where $\{b_i(k)\}, \{f_i(k)\}, i = 1,2, \{c_1(k)\}\{e_1(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$ are ω-periodic sequences, which satisfies $\overline{a}_i = \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} a_i(k) > 0, i = 1,2$. $x(k)$ and $y(k)$ represent the densities of the first and second species of k-generation, respectively. We assume that the coefficients of the system (1.3) are all periodic sequences with a common integer period. Such an assumption seems reasonable in view of seasonal factors, e.g., mating habits, availability of food, weather conditions, harvesting, and hunting, etc. Here, in model (1.3), the second species has positive effect on the first species, which obeys the Beddington-DeAngelis type functional response, i.e.,

$$\frac{c_1(k)x_2(k)}{e_1(k)+f_1(k)x_2(k)+f_2(k)x_1(k)}.$$

The Beddington-DeAngelis functional response is similar to the Holling type II functional response in system (1.2) but has an extra term $f_2(k)x_1(k)$ in the denominator which models mutual interference between the first species. We argued that in general, the intra-competition among the first species may reduce the influence of the second species, so the functional response in commensalism models must be both species involved, which means the Beddington-DeAngelis type functional response is more reasonable.

The aim of this paper is to obtain a set of sufficient conditions which ensure the existence of positive periodic solution of system (1.2). To the best of our knowledge, this is the first time that such kind of commensal symbiosis model is proposed and studied.

2. Main result

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin ([10]).

Lemma 2.1 (Continuation theorem)**.** *Let* L *be a Fredholm mapping of index zero and let* N *be* L*-compact on* $\bar{\Omega}$*. Suppose*

(a) *for each* $\lambda \in (0,1)$*, every solution* x *of* $Lx = \lambda Nx$ *is such that* $x \notin \partial\Omega$*;*
(b) $QNx \neq 0$ *for each* $x \in \partial\Omega \cap \mathrm{Ker}(L)$ *and*

$$\deg\{JQN, \Omega \cap \mathrm{Ker}(L), 0\} \neq 0.$$

Then the equation $Lx = Nx$ *has at least one solution lying in* $\mathrm{Dom}(L) \cap \bar{\Omega}$*.*

Let Z, Z^+, R, and R^+ denote the sets of all integers, nonnegative integers, real unumbers, and nonnegative real numbers, respectively. For convenience, in the following discussion, we will use the notation below throughout this paper:

$$I_\omega = \{0, 1, \dots, \omega - 1\}, \ \bar{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \ \ g^u = \max_{k \in I_\omega} g(k), \ \ g^l = \min_{k \in I_\omega} g(k),$$

where $\{g(k)\}$ is an ω-periodic sequence of real numbers defined for $k \in Z$.

Lemma 2.2 ([9])**.** *Let* $g : Z \to R$ *be* ω*-periodic, i.e.,* $g(k+\omega) = g(k)$*. Then for any fixed* $k_1, k_2 \in I_\omega$*, and any* $k \in Z$*, one has*

$$g(k) \leqslant g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|, \quad g(k) \geqslant g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Lemma 2.3. *The system of algebraic equations*

$$\bar{a}_1 - \bar{b}_1 x_1 + \frac{\bar{c}_1 x_2}{\bar{e}_1 + \bar{f}_1 x_2 + \bar{f}_2 x_1} = 0, \quad \bar{a}_2 - \bar{b}_2 x_2 = 0 \tag{2.1}$$

has a unique positive solution $(x_1^*, x_2^*) \in R_2^+$*, where*

$$\begin{aligned} x_1^* &= \frac{-A_2 + \sqrt{A_2^2 - 4A_1A_3}}{2A_1}, \ x_2^* = \frac{\bar{a}_2}{\bar{b}_2}, \\ A_1 &= b_1 b_2 f_2 > 0, \\ A_2 &= -a_1 b_2 f_2 + a_2 b_1 f_1 + b_1 b_2 e_1, \\ A_3 &= -a_1 f_1 a_2 - a_1 e_1 b_2 - c_1 a_2 < 0. \end{aligned} \tag{2.2}$$

Proof. From the second equation of (2.1), it immediately follows that

$$x_2 = \frac{\bar{a}_2}{\bar{b}_2}. \tag{2.3}$$

Substituting (2.3) into the first equation of (2.1) and simplify, we finally obtain

$$A_1 x_1^2 + A_2 x_1 + A_3 = 0, \tag{2.4}$$

where A_1, A_2, and A_3 are defined as in (2.2), since $A_1 > 0$, $A_3 < 0$, it immediately follows that the equation (2.4) has a unique positive solution x_1^*. Consequently, system (2.1) has a unique positive solution $(x_1^*, x_2^*) \in R_2^+$. This ends the proof of Lemma 2.3. □

We now reach the position to establish our main result.

Theorem 2.4. *System* (1.3) *admits at least one positive* ω*-periodic solution.*

Proof. Let

$$x_i(k) = \exp\{u_i(k)\}, \quad i = 1, 2,$$

so that system (1.3) becomes

$$\begin{aligned}
u_1(k+1) - u_1(k) &= a_1(k) - b_1(k)\exp\{u_1(k)\} + \frac{c_1(k)\exp\{u_2(k)\}}{e_1(k) + f_1(k)\exp\{u_2(k)\} + f_2(k)\exp\{u_1(k)\}}, \\
u_2(k+1) - u_2(k) &= a_2(k) - b_2(k)\exp\{u_2(k)\}.
\end{aligned}$$

Define

$$l_2 = \Big\{ y = \big\{y(k)\big\}, y(k) = (y_1(k), y_2(k))^T \in R^2 \Big\}.$$

For $a = (a_1, a_2)^T \in R^2$, define $|a| = \max\{|a_1|, |a_2|\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω sequences equipped with the usual normal form $\|y\| = \max\limits_{k \in I_\omega} |y(k)|$. It is not difficult to show that l^ω is a finite-dimensional Banach space. Let

$$l_0^\omega = \{y = \{y(k)\} \in l^\omega : \sum_{k=0}^{\omega-1} y(k) = 0\}, \ l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) = h \in R^2, k \in Z\},$$

then l_0^ω and l_c^ω are both closed linear subspace of l^ω, and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \quad \dim(l_c^\omega) = 2.$$

Now let us define $X = Y = l^\omega$, $(Ly)(k) = y(k+1) - y(k)$. It is trivial to see that L is a bounded linear operator and

$$\ker(L) = l_c^\omega, \quad \mathrm{Im}(L) = l_0^\omega, \quad \dim \mathrm{Ker}(L) = 2 = \mathrm{Codim}\mathrm{Im}(L).$$

Then it follows that L is a Fredholm mapping of index zero. Let

$$N(u_1, u_2)^T = (N_1, N_2)^T := N(u, k),$$

where

$$\begin{aligned}
N_1 &= a_1(k) - b_1(k)\exp\{u_1(k)\} + \frac{c_1(k)\exp\{u_2(k)\}}{e_1(k) + f_1(k)\exp\{u_2(k)\} + f_2(k)\exp\{u_1(k)\}}, \\
N_2 &= a_2(k) - b_2(k)\exp\{u_2(k)\}, \\
Px &= \frac{1}{\omega}\sum_{s=0}^{\omega-1} x(s), x \in X, \quad Qy = \frac{1}{\omega}\sum_{s=0}^{\omega-1} y(s), y \in Y.
\end{aligned}$$

It is not difficult to show that P and Q are two continuous projectors such that

$$\mathrm{Im}(P) = \mathrm{Ker}(L) \quad \text{and} \quad \mathrm{Im}(L) = \mathrm{Ker}(Q) = \mathrm{Im}(I - Q).$$

Furthermore, the generalized inverse (to L) K_p: ImL $\rightarrow$KerP$\cap$DomL exists and is given by

$$K_p(z) = \sum_{s=0}^{k-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1}(\omega - s)z(s).$$

Thus

$$QNx = \frac{1}{\omega}\sum_{k=0}^{\omega-1} N(x, k), \quad Kp(I - Q)Nx = \sum_{s=0}^{k-1} N(x, s) + \frac{1}{\omega}\sum_{s=0}^{\omega-1} sN(x, s) - \Big(\frac{k}{\omega} + \frac{\omega - 1}{2\omega}\Big)\sum_{s=0}^{\omega-1} N(x, s).$$

Obviously, QN and $K_p(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, it is not difficult to show that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on any open bounded set $\Omega \subset X$. The isomorphism J of $\mathrm{Im}(Q)$ onto $\mathrm{Ker}(L)$ can be the identity mapping, since $\mathrm{Im}(Q)$=$\mathrm{Ker}(L)$.

Now we are at the point to search for an appropriate open, bounded subset Ω in X for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{aligned} u_1(k+1)-u_1(k) &= \lambda\Big[a_1(k)-b_1(k)\exp\{u_1(k)\}+\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\Big],\\ u_2(k+1)-u_2(k) &= \lambda[a_2(k)-b_2(k)\exp\{u_2(k)\}]. \end{aligned} \tag{2.5}$$

Suppose that $y = (y_1(k), y_2(k))^T \in X$ is an arbitrary solution of system (2.5) for a certain $\lambda \in (0,1)$. Summing on both sides of (2.5) from 0 to $\omega-1$ with respect to k, we reach

$$\begin{aligned} &\sum_{k=0}^{\omega-1}\Big[a_1(k)-b_1(k)\exp\{u_1(k)\}+\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\Big]=0,\\ &\sum_{k=0}^{\omega-1}[a_2(k)-b_2(k)\exp\{u_2(k)\}]=0. \end{aligned}$$

That is,

$$\sum_{k=0}^{\omega-1} b_1(k)\exp\{u_1(k)\} = \bar{a}_1\omega + \sum_{k=0}^{\omega-1}\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}, \tag{2.6}$$

$$\sum_{k=0}^{\omega-1} b_2(k)\exp\{u_2(k)\} = \bar{a}_2\omega. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\begin{aligned} \sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)| &= \lambda\sum_{k=0}^{\omega-1}\Big|a_1(k)-b_1(k)\exp\{u_1(k)\}+\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\Big|\\ &\leqslant \sum_{k=0}^{\omega-1}|a_1(k)|+\sum_{k=0}^{\omega-1}\Big(b_1(k)\exp\{u_1(k)\}+\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\Big)\\ &= \sum_{k=0}^{\omega-1}|a_1(k)|+\bar{a}_1\omega+2\sum_{k=0}^{\omega-1}\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\\ &= (\bar{A}_1+\bar{a}_1)\omega+2\sum_{k=0}^{\omega-1}\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}}\\ &\leqslant (\bar{A}_1+\bar{a}_1)\omega+2\sum_{k=0}^{\omega-1}\frac{c_1(k)}{f_1(k)} = (\bar{A}_1+\bar{a}_1)\omega+2\overline{\Big(\frac{c_1}{f_1}\Big)}\omega,\\ \sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)| &= \lambda\sum_{k=0}^{\omega-1}|a_2(k)-b_2(k)\exp\{u_2(k)\}|\\ &\leqslant \sum_{k=0}^{\omega-1}|a_2(k)|+\sum_{k=0}^{\omega-1}b_2(k)\exp\{u_2(k)\} \leqslant \sum_{k=0}^{\omega-1}|a_2(k)|+\bar{a}_2\omega \leqslant (\bar{A}_2+\bar{a}_2)\omega. \end{aligned} \tag{2.8}$$

where $\bar{A}_1 = \frac{1}{\omega}\sum_{k=0}^{\omega-1}|a_1(k)|$, $\bar{A}_2 = \frac{1}{\omega}\sum_{k=0}^{\omega-1}|a_2(k)|$, $\overline{(\frac{c_1}{f_1})} = \frac{1}{\omega}\sum_{k=0}^{\omega-1}\frac{c_1(k)}{f_1(k)}$.

Since $\{u(k)\} = \{(u_1(k), u_2(k))^T\} \in X$, there exist $\eta_i, \delta_i, i = 1, 2$ such that

$$u_i(\eta_i) = \min_{k\in I_\omega} u_i(k), \ \ u_i(\delta_i) = \max_{k\in I_\omega} u_i(k).$$

By (2.7), we have

$$\exp\{u_2(\eta_2)\} \sum_{k=0}^{\omega-1} b_2(k) \leqslant \bar{a}_2\omega.$$

So

$$u_2(\eta_2) \leqslant \ln \frac{\bar{a}_2}{\bar{b}_2}. \tag{2.9}$$

It follows from Lemma 2.2, (2.8), and (2.9) that

$$u_2(k) \leqslant u_2(\eta_2) + \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \leqslant \ln \frac{\bar{a}_2}{\bar{b}_2} + (\bar{A}_2 + \bar{a}_2)\omega, \tag{2.10}$$

from (2.7) we also have

$$\exp\{u_2(\delta_2)\} \sum_{k=0}^{\omega-1} b_2(k) \geqslant \bar{a}_2\omega,$$

and so

$$u_2(\delta_2) \geqslant \ln \frac{\bar{a}_2}{\bar{b}_2}. \tag{2.11}$$

It follows from Lemma 2.2, (2.8), and (2.11) that

$$u_2(k) \geqslant u_2(\delta_2) - \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| \geqslant \ln \frac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2 + \bar{a}_2)\omega,$$

which together with (2.10) leads to

$$|u_2(k)| \leqslant \max\left\{|\ln \frac{\bar{a}_2}{\bar{b}_2} + (\bar{A}_2 + \bar{a}_2)\omega|, |\ln \frac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2 + \bar{a}_2)\omega|\right\} \overset{\text{def}}{=} H_2.$$

It follows from (2.6) that

$$\begin{aligned}\sum_{k=0}^{\omega-1} b_1(k) \exp\{u_1(\eta_1)\} &= \bar{a}_1\omega + \sum_{k=0}^{\omega-1} \frac{c_1(k) \exp\{u_2(k)\}}{e_1(k) + f_1(k) \exp\{u_2(k)\} + f_2(k) \exp\{u_1(k)\}} \\ &\leqslant \bar{a}_1\omega + \sum_{k=0}^{\omega-1} \frac{c_1(k)}{f_1(k)} = \bar{a}_1\omega + \overline{(\frac{c_1}{f_1})}\omega,\end{aligned}$$

and so,

$$u_1(\eta_1) \leqslant \ln \frac{\Delta_1}{\bar{b}_1}, \tag{2.12}$$

where

$$\Delta_1 = \bar{a}_1 + \overline{(\frac{c_1}{f_1})}.$$

It follows from Lemma 2.2, (2.8), and (2.12) that

$$u_1(k) \leqslant u_1(\eta_1) + \sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| \leqslant \ln \frac{\Delta_1}{\bar{b}_1} + (\bar{A}_1 + \bar{a}_1)\omega + 2\overline{(\frac{c_1}{f_1})}\omega \overset{\text{def}}{=} M_1. \tag{2.13}$$

It follows from (2.6) that

$$\sum_{k=0}^{\omega-1} b_1(k)\exp\{u_1(\delta_1)\} = \bar{a}_1\omega + \sum_{k=0}^{\omega-1}\frac{c_1(k)\exp\{u_2(k)\}}{e_1(k)+f_1(k)\exp\{u_2(k)\}+f_2(k)\exp\{u_1(k)\}} \geqslant \bar{a}_1\omega,$$

and so,

$$u_1(\delta_1) \geqslant \ln\frac{\bar{a}_1}{\bar{b}_1}. \tag{2.14}$$

It follows from Lemma 2.2, (2.8), and (2.14) that

$$u_1(k) \geqslant u_1(\delta_1) - \sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)| \geqslant \ln\frac{\bar{a}_1}{\bar{b}_1} - (\bar{A}_1+\bar{a}_1)\omega - 2\overline{(\frac{c_1}{f_1})}\omega \overset{\text{def}}{=} M_2. \tag{2.15}$$

It follows from (2.13) and (2.15) that

$$|u_1(k)| \leqslant \max\Big\{|M_1|,|M_2|\Big\} \overset{\text{def}}{=} H_1.$$

Clearly, H_1 and H_2 are independent on the choice of λ. Already, in Lemma 2.3, we had showed that the system of algebraic equations

$$\bar{a}_1 - \bar{b}_1x_1 + \frac{\bar{c}_1x_2}{\bar{e}_1+\bar{f}_1x_2+\bar{f}_2x_1} = 0, \quad \bar{a}_2 - \bar{b}_2x_2 = 0$$

has a unique positive solution $(x_1^*, x_2^*) \in R_2^+$.

Let $H = H_1 + H_2 + H_3$, where $H_3 > 0$ is taken sufficiently enough large such that $\|(\ln\{x_1^*\},\ \ln\{x_2^*\})^T\| = |\ln\{x_1^*\}| + |\ln\{x_2^*\}| < H_3$. Let $H = H_1 + H_2 + H_3$, and define

$$\Omega = \Big\{u(t) = (u_1(k), u_2(k))^T \in X : \|u\| < H\Big\}.$$

It is clear that Ω verifies requirement (a) in Lemma 2.1. When $u \in \partial\Omega \cap \mathrm{Ker}(L) = \partial\Omega \cap R^2$, u is constant vector in R^2 with $\|u\| = B$. Then

$$QNu = \begin{pmatrix} \bar{a}_1 - \bar{b}_1\exp\{u_1\} + \dfrac{\bar{c}_1\exp\{u_2\}}{\bar{e}_1+\bar{f}_1\exp\{u_2\}+\bar{f}_2\exp\{u_1\}} \\ \bar{a}_2 - \bar{b}_2\exp\{u_2\} \end{pmatrix} \neq 0.$$

Moreover, direct calculation shows that

$$\deg\{JQN, \Omega\cap\mathrm{Ker}(L), 0\} = \mathrm{sgn}\Big(\Gamma\Big) = 1 \neq 0,$$

where

$$\Gamma = \Big(\bar{b}_1 + \frac{\bar{c}_1\bar{f}_2\exp\{u_2^*\}}{(\bar{f}_1\exp\{u_2^*\}+\bar{f}_2\exp\{u_1^*\}+\bar{e}_1)^2}\Big)\bar{b}_2\exp\{u_1^*\}\exp\{u_2^*\},$$

where deg(.) is the Brouwer degree and the J is the identity mapping since $\mathrm{Im}(Q) = \mathrm{Ker}(L)$.

By now we have proved that Ω verifies all the requirements in Lemma 2.1. Hence (2.1) has at least one solution $(u_1^*(k), u_2^*(k))^T$ in $\mathrm{Dom}(L)\cap\bar{\Omega}$. And so, system (1.3) admits a positive periodic solution $(x_1^*(k), x_2^*(k))^T$, where $x_i^*(k) = \exp\{u_i^*(k)\}, i = 1, 2$. This completes the proof of Theorem 2.4. □

3. Numeric simulations

Now let us consider the following example.

Example 3.1.

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big\{1 - 2x_1(k) - \frac{(1+0.5\sin(\pi k))x_2(k)}{1+x_2(k)+f_2(k)x_1(k)}\Big\}, \\ x_2(k+1) &= x_2(k)\exp\Big\{1 - \big(1+0.5\cos(\pi k+\frac{\pi}{3})\big)x_2(k)\Big\}. \end{aligned} \tag{3.1}$$

Corresponding to system (1.3), here we choose $a_1(k) = 1, b_1(k) = 2, c(k) = 1+0.5\sin(\pi k), e_1(k) =$

$1, f_1(k) = 1$, $a_2(k) = 1, b_2(k) = 1 + 0.5\cos(\pi k + \frac{\pi}{3})$. One could easily check that the condition of Theorem 2.4 holds, and consequently, system (3.1) admits at least one positive 2-period solution. Now let us choose $f_2(k) = 0.1, 10$, and 100, respectively shown in Figures 1-3, numeric simulations also support this assertion.

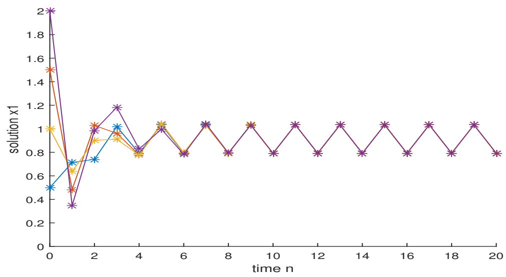

Figure 1: Dynamic behaviors of the first component x_1 in system (3.1) with the initial condition $(x(0), y(0)) = (0.5, 0.5)$, $(1, 1)$, $(1.5, 1.5)$, and $(2, 2)$, $f_2(k) = 0.1$, respectively.

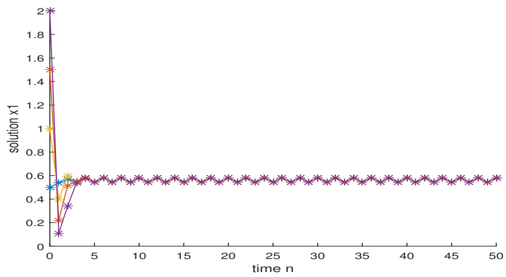

Figure 2: Dynamic behaviors of the first component x_1 in system (3.1) with the initial condition $(x(0), y(0)) = (0.5, 0.5)$, $(1, 1)$, $(1.5, 1.5)$, and $(2, 2)$, $f_2(k) = 10$, respectively.

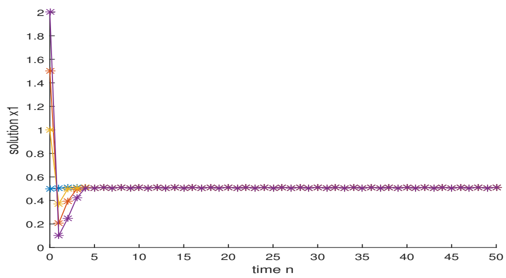

Figure 3: Dynamic behaviors of the first component x_1 in system (3.1) with the initial condition $(x(0), y(0)) = (0.5, 0.5)$, $(1, 1)$, $(1.5, 1.5)$, and $(2, 2)$, $f_2(k) = 100$, respectively.

4. Discussion

In this paper, we propose a commensalism model with Beddington-DeAngelis functional response, we argued that in the process of obtaining the benefit from the second species, mutual interference among the first species will happen and this reduces the overall benefit from the second species. Beddington-DeAngelis functional response is a suitable way to describe this factor.

Theorem 2.4 shows that Beddington-DeAngelis functional response has no influence on the existence of the positive periodic solution. However, numeric simulations (Figures 1-3) show that with the increasing of mutual interference, the fluctuation of the first species will reduce. This could make the system more observable and the amount of the species could be easy to control in appropriate quantities.

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