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Modelling and qualitative analysis of an illicit drugs model with saturated incidence rate and relapse

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Abstract

Drug abuse is now regarded as a global issue that brings severe consequences on the health, social well-being, and economy. In this study, an illicit drugs model with saturated incidence rate and relapse of individuals who quit using drugs is proposed and analyzed qualitatively. This study aims to determine the behavior of a drug epidemic when the psychological or inhibitory effect and relapse are being considered as well as assist the policymakers in devising effective control measures. The basic reproduction number R_0 is derived and used as a threshold parameter in the global stability analysis. It is found that the drug-free equilibrium is globally asymptotically stable when $R_0 \leqslant 1$. This implies that we can eradicate a drug epidemic when the threshold R_0 is less than or equal to one, irrespective of the initial population size of drug users. On the other hand, the drug persistent equilibrium is globally asymptotically stable when $R_0 > 1$. This indicates that the phenomenon of drug use will remain in a community when the threshold R_0 is greater than one, irrespective of the initial population size of drug users. Next, the sensitivity analysis is performed and the results show that the effective contact rate should be targeted to reduce its value. The numerical simulations are also carried out to illustrate the analytical results and investigate the relationship between the measure of psychological or inhibitory effect and the number of drug users.

Keywords: Illicit drugs model, saturated incidence rate, relapse, global stability analysis, sensitivity analysis.

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1. Introduction

The human usage of drugs or substances has existed since antiquity where they were used for three main purposes, namely religious use, medicinal use, and recreational use [3]. At present, illicit drug use remains a severe global threat that brings numerous health, social, and financial problems. Some examples of common illicit drugs are heroin, methamphetamine, cocaine, cannabis, and opioids. In 2017, it was estimated that 271 million individuals, or 5.5 percent of the world population aged 15-64, had taken drugs in the previous year [24]. Over 2020, there were about 275 million people had used drugs, which had increased by 22 percent compared to 2010 [25]. The number of drug users had been projected to increase by 11 percent worldwide by 2030. Moreover, the number of drug-related deaths was between 0.5 and 1.3 percent of the total number of deaths for individuals aged 15-64 years throughout the world

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in 2012 [6]. Particularly, there were about 211000 drug-related deaths annually which mostly involve juveniles. In 2019, drug use took about half a million lives and the drug users have a higher chance to contract severe illnesses, such as HIV and Hepatitis C [25]. Economically, $471 billion of opioid use disorder and $550 billion of fatal opioid overdose added up to a total of $1021 billion loss in 2017 owing to opioid use [10]. In 2018, the annual drug-related cost in Alaska was estimated at $1055 million, where the productivity loss costs $429 million (40.7%), traffic collision makes up $60 million (5.7%), criminal justice & protective services account for $442 million (41.9%), health care constitutes $106 million (10%), and public assistance & social services are composed of $18 million (1.7%) [13].

There were several previous studies regarding the general drug models. First of all, Kalula and Nyabadza [7] proposed a substance abuse model which incorporates the core and non-core groups. The core group contains the class of permanent quitters while the non-core group is the source of individuals to the core group. A bilinear incidence rate was employed in this model. Besides, Njagarah and Nyabadza [19] studied a drug model with amelioration (heavy drug users can revert to light drug use). In this model, a standard incidence rate was adopted and the permanent quitting of drug users was assumed. Furthermore, Mushayabasa and Tapedzesa [18] studied an illicit drugs model which includes the class of mentally ill individuals. The bilinear incidence rate was employed and the permanent recovery of detected illicit drug users was assumed in this study. Moreover, Ma et al. [11] proposed a drug model which considers the role of media coverage on the transmission of drugs. Thus, the incidence rate of this model contains an exponential decreasing factor that measures the impact of media coverage. In this model, the individuals in treatment will become susceptible again after completing the treatment. Besides general drug models, there were also previous studies regarding heroin models [16, 27], methamphetamine models [17, 20, 21], and synthetic drugs models [12, 22].

Note that the incidence rates of the aforementioned general drug models were assumed to be proportional to the number of drug users. However, it is believed that the incidence rate is a bounded function of the number of drug users. This is owing to the susceptible individuals will change their behavior after noting many drug users experience the adverse effects of drug use. Moreover, a portion of susceptible individuals is believed will not start taking drugs, even though contact with the drug users. In addition, the number of contacts that can result in new drug users is limited when a community is crowded with drug users. Thus, the saturated incidence rate [9, 29] will be adopted in the proposed model. On the other hand, the aforementioned general drug models did not consider the relapse of individuals who quit using drugs. Relapse is a common phenomenon where the relapse rate of individuals with drug use disorders was between 40% and 60% [14]. Hence, the relapse of individuals who quit using drugs will be considered in the proposed model. The main contributions of our study are: 1) the introduction of saturated incidence rate and relapse of individuals who quit using drugs to an illicit drugs model; 2) the qualitative analysis of the newly proposed illicit drugs model; 3) the sensitivity analysis of model parameters on the initial spread of drug use.

There are several aims for conducting this study. The first aim is to formulate an illicit drugs model with the saturated incidence rate and relapse of individuals who quit using drugs. Note that most illicit drug users are aged 13 or above [15, 23], and so the formulated model in this paper is only applicable to individuals who are at least 13 years old. Besides, we aim to establish sufficient conditions for the eradication and persistence of the drug epidemic which incorporates the psychological or inhibitory effect and relapse of individuals who quit using drugs. Moreover, we also aim to devise some effective control measures that can control or eradicate the drug epidemic. Lastly, we aim to determine the impact of the measure of psychological or inhibitory effect on the number of drug users.

This paper is organized as follows. In Section 2, an illicit drugs model with saturated incidence rate and relapse of individuals who quit using drugs is formulated. Next, the basic reproduction number will be given in Section 3. In Section 4, the existence of drug persistence equilibrium is shown. Then, the local and global stabilities analysis will be performed in Sections 5 and 6, respectively. In Section 7, the sensitivity analysis of the basic reproduction number is presented. Lastly, the numerical simulations will be provided in Section 8.

2. Mathematical modelling

2.1. Model formulation

A heterogeneous population of size $N(t)$ is divided into the classes of susceptible individuals aged above 13 $S(t)$, drug users $U(t)$, and individuals who quit using drugs $Q(t)$. Hence, the total population size at time t is given by

$$N(t) = S(t) + U(t) + Q(t)$$

For simplifying the notations, the functions of time $S(t)$, $U(t)$, $Q(t)$, and $N(t)$ are denoted as the variables S, U, Q, and N, respectively. In order to have a manageable model, we make the following assumptions:

1. Homogeneous mixing where each individual has the same chance to contact other individuals.
2. The susceptible individuals start taking drugs via effective contact with drug users only.
3. The individuals who quit taking drugs relapse due to their own decisions.

Table 1: Description and dimensions of model parameters.

Parameter	Description	Dimension
Λ	Recruitment rate of susceptible individuals	Individuals $\times$Time^{-1}
μ	Natural death rate	Time^{-1}
c	Average number of contacts per unit time	Time^{-1}
$\hat{\beta}$	Probability that a contact results in a new drug user	Dimensionless
$\beta = c\hat{\beta}$	Effective contact rate	Time^{-1}
k	Measure of psychological or inhibitory effect	Individuals^{-1}
σ	Drug-related death rate	Time^{-1}
α	Recovery rate through abstinence	Time^{-1}
γ	Recovery rate through treatment	Time^{-1}
η	Relapse rate of individuals who quit using drugs	Time^{-1}

Like other models, the proposed model also contains several parameters which are shown in Table 1. With the parameters in Table 1, the movements of individuals from one class to another are illustrated in Figure 1.

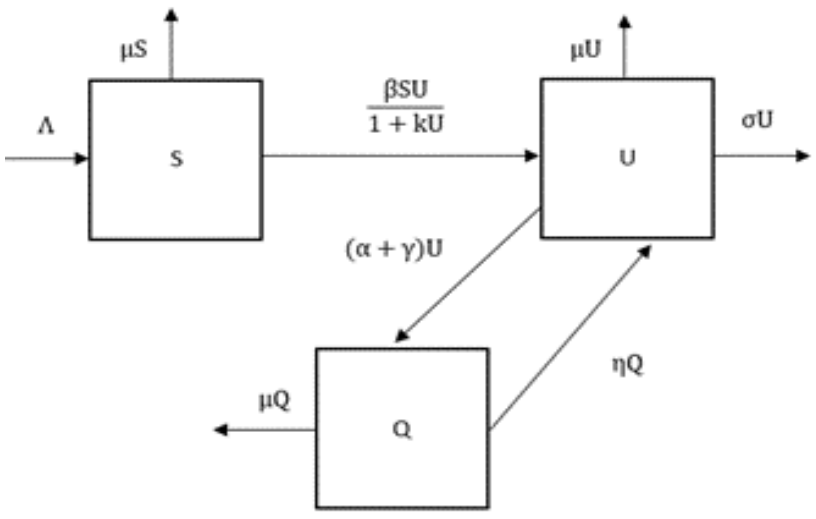

Figure 1: Compartmental diagram of the proposed illicit drugs model.

According to the assumptions, Table 1 and Figure 1, we have the following set of governing equations which is a nonlinear system of ordinary differential equations:

$$\frac{dS}{dt} = \Lambda - \frac{\beta SU}{1+kU} - \mu S, \tag{2.1}$$

$$\frac{dU}{dt} = \frac{\beta SU}{1+kU} - A_1 U + \eta Q, \tag{2.2}$$

$$\frac{dQ}{dt} = A_2 U - A_3 Q, \tag{2.3}$$

where $A_1 = \alpha+\gamma+\sigma+\mu$, $A_2 = \alpha+\gamma$, and $A_3 = \eta+\mu$. Given $\beta = 0.34$ and $k = 0.4$ [5], the function $g(U) = \beta U/(1+kU)$ is plotted in Figure 2. From Figure 2, we can observe that the curve of function $g(U) = \beta U/(1+kU)$ increases and approaches the saturated level $\beta/k = 0.85$, when the number of drug users U gets larger. Note that the qualitatively similar curves would also be obtained for different values of β and k. Thus, function $g(U) = \beta U/(1+kU)$ is said to be bounded and monotonically increasing. Accordingly, function $Sg(U) = \beta SU/(1+kU)$ is called the saturated incidence rate. It is worth noting that βU represents the force of spread of drug use which is similar to the force of infection in epidemiology. On the other hand, $1/(1+kU)$ measures the psychological or inhibitory effect from the behavioral change of susceptible individuals or from the crowding effect of drug users.

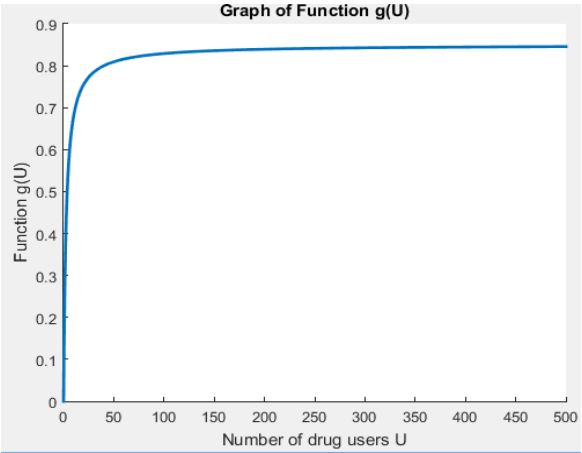

Figure 2: Graph of function $g(U) = \frac{\beta U}{1+kU}$.

2.2. Basic properties

It is crucial to note that all variables S, U, and Q of model (2.1)-(2.3) must be positive and bounded since they represent various subpopulations.

2.2.1. Positivity of solutions

Lemma 2.1. *If* $S(0) \geqslant 0$, $U(0) \geqslant 0$, *and* $Q(0) \geqslant 0$, *then variables* S, U, *and* Q *of model* (2.1)-(2.3) *are positive for* $t > 0$.

Proof. We first prove the positivity of variable S. From equation (2.1), we have

$$\frac{dS}{dt} + \left(\mu + \frac{\beta U}{1+kU}\right) S > 0. \tag{2.4}$$

Multiplying inequality (2.4) with integrating factor $\exp(\mu t + \int_0^t \frac{\beta U(s)}{1+kU(s)} ds)$, we have

$$\frac{d}{dt}\left[S \exp\left(\mu t + \int_0^t \frac{\beta U(s)}{1+kU(s)} ds\right)\right] > 0. \tag{2.5}$$

Changing the independent variable of inequality (2.5) from t to s and integrating it from $s = 0$ to $s = t$ gives

$$\int_0^t \frac{d}{ds}\left[S(s)\exp\left(\mu s + \int_0^s \frac{\beta U}{1+kU}dt\right)\right]ds > 0.$$

Thus, we obtain

$$S > S(0)\exp\left[-\left(\mu t + \int_0^t \frac{\beta U(s)}{1+kU(s)}ds\right)\right] \geqslant 0.$$

Similarly, the positivity of variables U and Q can be proved in the same way. So, variables S, U, and Q of model (2.1)-(2.3) are positive for $t > 0$, given any non-negative initial conditions: $S(0) \geqslant 0$, $U(0) \geqslant 0$, and $Q(0) \geqslant 0$. □

2.2.2. Positively invariant region

Lemma 2.2. *The region*

$$\omega = \left\{(S, U, Q) \in \mathbb{R}^3_+ |\ 0 \leqslant N \leqslant \frac{\Lambda}{\mu}\right\}$$

with non-negative initial conditions $S(0) \geqslant 0$, $U(0) \geqslant 0$, *and* $Q(0) \geqslant 0$, *is positively invariant and attracting with respect to model* (2.1)-(2.3) *for* $t > 0$.

Proof. Adding equations (2.1)-(2.3) gives

$$\frac{dN}{dt} = \Lambda - \mu N - \sigma U.$$

Since $\sigma U \geqslant 0$, we have

$$\frac{dN}{dt} + \mu N \leqslant \Lambda. \tag{2.6}$$

Multiplying inequality (2.6) with integrating factor $\exp(\mu t)$ gives

$$\frac{d}{dt}(N\exp(\mu t)) \leqslant \Lambda\exp(\mu t). \tag{2.7}$$

Changing the independent variable of inequality (2.7) from t to s and integrating it from $s = 0$ to $s = t$, we have

$$\int_0^t \frac{d}{ds}[N(s)\exp(\mu s)]ds \leqslant \int_0^t \Lambda\exp(\mu s)ds.$$

After some algebraic manipulations and given Lemma 2.1, we obtain

$$0 \leqslant N \leqslant \frac{\Lambda}{\mu} + \left[N(0) - \frac{\Lambda}{\mu}\right]\exp(-\mu t). \tag{2.8}$$

As $t \to \infty$, inequality (2.8) becomes

$$0 \leqslant N \leqslant \frac{\Lambda}{\mu}.$$

If $N(0) \leqslant \Lambda/\mu$, then we have $\lim_{t\to\infty} N = \Lambda/\mu$. This indicates the upper bound of total population size N is Λ/μ. Whereas, if $N(0) > \Lambda/\mu$, then N will decrease to Λ/μ as $t \to \infty$. This indicates the solution (S, U, Q) of model (2.1)-(2.3) will enter or approach region $\omega = \left\{(S, U, Q) \in \mathbb{R}^3_+ |\ 0 \leqslant N \leqslant \Lambda/\mu\right\}$ over time. As a result, region ω is positively invariant and attracting with respect to model (2.1)-(2.3) for $t > 0$. Therefore, model (2.1)-(2.3) is said to be mathematically and epidemiologically well-posed in region ω. Accordingly, it is sufficient to study the dynamics of solutions of model (2.1)-(2.3) that start in region ω only. □

3. Drug-free equilibrium and basic reproduction number

We first point out that model (2.1)-(2.3) always has the drug-free equilibrium for any parameter values:

$$\mathbf{E_0} = \left(\frac{\Lambda}{\mu}, 0, 0\right). \tag{3.1}$$

Then, the next-generation matrix method [26] is employed to derive a crucial threshold, which is called the basic reproduction number R_0. With the derivation of R_0 in the **Appendix 1** and some algebraic manipulations, we obtain

$$R_0 = \frac{\beta\Lambda(\eta+\mu)}{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]} = \frac{\beta\Lambda A_3}{\mu[\mu A_2 + A_3(\sigma+\mu)]}. \tag{3.2}$$

The basic reproduction number R_0 can be defined as the expected number or the average number of secondary cases of drug use caused by a drug user during his/ her period of drug use, when he/she was introduced into a fully susceptible population. Accordingly, the initial spread of drug use is directly related to R_0. From equation (3.2), it is worth noting that R_0 is independent of the measure of psychological or inhibitory effect k whilst R_0 is dependent on the relapse rate of individuals who quit using drugs η. Thus, there is no relationship between parameter k and the initial spread of drug use. In contrast, parameter η affects the initial spread of drug use.

4. Existence of drug persistent equilibrium

In this section, the drug persistent equilibrium is found by letting the right-hand side of model (2.1)-(2.3) equals zero. With some algebraic manipulations, we have

$$S^* = \frac{\Lambda(1+kU^*)}{\beta U^* + \mu(1+kU^*)}, \tag{4.1}$$

$$Q^* = \frac{A_2}{A_3}U^*. \tag{4.2}$$

Substituting equations (4.1) and (4.2) into equation (2.2) gives

$$U^* = \frac{\beta\Lambda A_3 - \mu[\mu A_2 + A_3(\sigma+\mu)]}{(\beta+\mu k)[\mu A_2 + A_3(\sigma+\mu)]} = \frac{\mu}{\beta+\mu k}(R_0 - 1). \tag{4.3}$$

It is crucial to note that $U^* > 0$ if and only if $R_0 > 1$. Therefore, if $R_0 > 1$, then the drug persistent equilibrium is

$$\mathbf{E}^* = (S^*, U^*, Q^*),$$

where S^*, U^*, and Q^* are given in equations (4.1), (4.3), and (4.2), respectively. The existence of equilibria of model (2.1)-(2.3) can be summarised as follows:

1. If $R_0 \leqslant 1$, then only the drug-free equilibrium $\mathbf{E_0} = (S_0, U_0, Q_0)$ exists.
2. If $R_0 > 1$, then the drug-free equilibrium $\mathbf{E_0} = (S_0, U_0, Q_0)$ and a unique drug persistent equilibrium $\mathbf{E}^* = (S^*, U^*, Q^*)$ exist.

Based on the above results, we can conclude that model (2.1)-(2.3) does not undergo backward bifurcation as the unique drug persistent equilibrium exists only when $R_0 > 1$.

5. Local stability analysis

The local stability analysis is performed in the first place instead of directly performing the global stability analysis. This is because the global properties of models with complicated dynamics are laborious to be determined and the conditions of local stability and global stability are not necessarily the same. Accordingly, we first perform the linear stability analysis [28] to have a look at the behavior of the solutions near to the equilibria of model (2.1)-(2.3).

5.1. Drug-free equilibrium

Theorem 5.1. *If $R_0 < 1$, then the drug-free equilibrium $\mathbf{E_0} = (\Lambda/\mu, 0, 0)$ is locally asymptotically stable.*

Proof. The Jacobian evaluated at the drug-free equilibrium $\mathbf{E_0} = (\Lambda/\mu, 0, 0)$ is

$$\mathbf{J}(\mathbf{E_0}) = \begin{pmatrix} -\mu & -\beta\frac{\Lambda}{\mu} & 0 \\ 0 & \beta\frac{\Lambda}{\mu} - A_1 & \eta \\ 0 & A_2 & -A_3 \end{pmatrix}.$$

So, the characteristic equation for $\mathbf{J}(\mathbf{E_0})$ is

$$(\mu + \lambda)(\lambda^2 + a_1\lambda + a_2) = 0,$$

where $a_1 = A_1 + A_3 - \beta\Lambda/\mu$ and $a_2 = A_1A_3 - \eta A_2 - A_3\beta\Lambda/\mu$. It is crucial to note that a zero eigenvalue would be obtained when $R_0 = 1$, since $a_2 = 0$. Thus, if $R_0 = 1$, then the drug-free equilibrium $\mathbf{E}_0$ is said to be nonhyperbolic, since matrix $\mathbf{J}(\mathbf{E_0})$ has a zero eigenvalue [1]. As a result, when $R_0 = 1$, the linear stability analysis fails to help us to draw any conclusion on the local stability of $\mathbf{E}_0$, based on the Hartman-Grobman theorem [8]. In the following, we proceed to find the signs of the real parts of eigenvalues of matrix $\mathbf{J}(\mathbf{E_0})$ for $R_0 \neq 1$. The sign of the first eigenvalue can be found easily which is

$$\lambda_1 = -\mu < 0.$$

For the remaining eigenvalues, the Routh-Hurwitz criteria are employed to determine the signs of their real parts. The computations of determinants of all Hurwitz matrices are shown as follows.

$$\begin{aligned} \det(\mathbf{H_1}) = a_1 &= \frac{\mu A_2 + A_3(\sigma+\mu)}{A_3}\left[\frac{A_3(A_1+A_3)}{\mu A_2 + A_3(\sigma+\mu)} - \frac{\beta\Lambda A_3}{\mu[\mu A_2 + A_3(\sigma+\mu)]}\right] \\ &= \frac{\mu A_2 + A_3(\sigma+\mu)}{A_3}[\frac{A_3(A_1+A_3)}{\mu A_2 + A_3(\sigma+\mu)} - R_0]. \end{aligned}$$

If $R_0 < A_3(A_1+A_3)/[\mu A_2 + A_3(\sigma+\mu)]$, then we have

$$\det(\mathbf{H_1}) > 0.$$

Next, we find

$$\det(\mathbf{H_2}) = \begin{vmatrix} a_1 & 0 \\ 1 & a_2 \end{vmatrix} = \det(\mathbf{H_1})(A_1A_3 - \eta A_2 - A_3\beta\frac{\Lambda}{\mu}).$$

It is worth noting that

$$A_1A_3 - \eta A_2 - A_3\beta\frac{\Lambda}{\mu} = [\mu A_2 + A_3(\sigma+\mu)](1 - R_0).$$

Thus, we have

$$\det(\mathbf{H_2}) = \det(\mathbf{H_1})[\mu A_2 + A_3(\sigma+\mu)](1 - R_0).$$

If $R_0 < 1$ and $R_0 < A_3(A_1+A_3)/[\mu A_2 + A_3(\sigma+\mu)]$ (so that $\det(\mathbf{H_1}) > 0$), then we obtain

$$\det(\mathbf{H_2}) > 0.$$

Note that

$$\frac{A_3(A_1+A_3)}{\mu A_2 + A_3(\sigma+\mu)} = \frac{\mu(\alpha+\gamma) + (\eta+\mu)(\sigma+2\mu) + \eta(\alpha+\gamma+\eta) + \mu\eta}{\mu(\alpha+\gamma) + (\eta+\mu)(\sigma+\mu)} > 1.$$

As a result, we have

$$R_0 < 1 < \frac{A_3(A_1+A_3)}{\mu A_2 + A_3(\sigma+\mu)}.$$

Therefore, the determinants of the Hurwitz matrices $\mathbf{H_1}$ and $\mathbf{H_2}$ are greater than zero when $R_0 < 1$. This implies the real parts of all eigenvalues of the Jacobian evaluated at $\mathbf{E_0} = (\Lambda/\mu, 0, 0)$ are negative. As a result, the drug-free equilibrium $\mathbf{E_0}$ is locally asymptotically stable when $R_0 < 1$. □

5.2. Drug persistent equilibrium

Theorem 5.2. *If $R_0 > 1$, then the drug persistent equilibrium $\mathbf{E}^* = (S^*, U^*, Q^*)$ is locally asymptotically stable.*

Proof. The Jacobian evaluated at drug persistent equilibrium $\mathbf{E}^* = (S^*, U^*, Q^*)$ is

$$\mathbf{J}(\mathbf{E}^*) = \begin{pmatrix} -(\frac{\beta U^*}{1+kU^*} + \mu) & -\frac{\beta S^*}{1+kU^*} + \frac{k\beta S^* U^*}{(1+kU^*)^2} & 0 \\ \frac{\beta U^*}{1+kU^*} & \frac{\beta S^*}{1+kU^*} - \frac{k\beta S^* U^*}{(1+kU^*)^2} - A_1 & \eta \\ 0 & A_2 & -A_3 \end{pmatrix}.$$

The characteristic equation for $\mathbf{J}(\mathbf{E}^*)$ is found to be

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= \frac{\beta U^*}{1+kU^*} + \mu - \frac{\beta S^*}{1+kU^*} + \frac{k\beta S^* U^*}{(1+kU^*)^2} + A_1 + A_3, \\ b_2 &= \left(\frac{\beta U^*}{1+kU^*} + \mu\right)\left[-\frac{\beta S^*}{1+kU^*} + \frac{k\beta S^* U^*}{(1+kU^*)^2} + A_1 + A_3\right] \\ &\quad + A_3\left[-\frac{\beta S^*}{1+kU^*} + \frac{k\beta S^* U^*}{(1+kU^*)^2} + A_1\right] - \eta A_2 + \frac{\beta^2 S^* U^*}{(1+kU^*)^3}, \\ b_3 &= \left(\frac{\beta U^*}{1+kU^*} + \mu\right)\left\{A_3[-\frac{\beta S^*}{1+kU^*} + \frac{k\beta S^* U^*}{(1+kU^*)^2} + A_1] - \eta A_2\right\} + \frac{\beta^2 S^* U^* A_3}{(1+kU^*)^3}. \end{aligned}$$

The Routh-Hurwitz criteria are again adopted to determine the signs of real parts of all eigenvalues of the Jacobian evaluated at $\mathbf{E}^*$. The computations of the determinants of Hurwitz matrices are shown as follows.

$$\det(\mathbf{H_1}) = b_1 = \frac{\beta U^*}{1+kU^*} + \mu + A_3 + \frac{k\beta S^* U^*}{(1+kU^*)^2} + (A_1 - \frac{\beta S^*}{1+kU^*}).$$

Note that

$$\frac{\beta S^*}{1+kU^*} = \frac{\mu(\alpha+\gamma) + (\eta+\mu)(\sigma+\mu)}{\eta+\mu} \tag{5.1}$$

and

$$A_1 = \frac{\mu(\alpha+\gamma) + (\eta+\mu)(\sigma+\mu) + \eta(\alpha+\gamma)}{\eta+\mu}. \tag{5.2}$$

By comparing equations (5.1) and (5.2), we can conclude that

$$A_1 > \frac{\beta S^*}{1+kU^*}. \tag{5.3}$$

Accordingly, if $R_0 > 1 (U^* > 0)$, then we have

$$\det(\mathbf{H_1}) > 0.$$

Next, we find

$$\begin{aligned} \det(\mathbf{H_2}) = \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} &= \left[\frac{k\beta S^* U^*}{(1+kU^*)^2} + (A_1 - \frac{\beta S^*}{1+kU^*})\right] b_2 + (\frac{\beta U^*}{1+kU^*} + \mu)^2 [(A_1 - \frac{\beta S^*}{1+kU^*}) \\ &\quad + \frac{k\beta S^* U^*}{(1+kU^*)^2} + A_3] + (\frac{\beta U^*}{1+kU^*} + \mu)\frac{\beta^2 S^* U^*}{(1+kU^*)^3} \end{aligned}$$

$$+A_3\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\left(A_1-\frac{\beta S^*}{1+kU^*}\right)+\frac{k\beta S^*U^*}{(1+kU^*)^2}+A_3\right]$$
$$+\frac{k\beta S^*U^*(A_3)^2}{(1+kU^*)^2}+A_3\left(A_1A_3-\eta A_2-\frac{\beta S^*A_3}{1+kU^*}\right).$$

It is crucial to note that

$$A_1A_3-\eta A_2-\frac{\beta S^*A_3}{1+kU^*}=0. \tag{5.4}$$

Thus, we have

$$\det(\mathbf{H_2})=\left[\frac{k\beta S^*U^*}{(1+kU^*)^2}+\left(A_1-\frac{\beta S^*}{1+kU^*}\right)\right](b_2)+\left(\frac{\beta U^*}{1+kU^*}+\mu\right)$$
$$\times\left\{\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\left(A_1-\frac{\beta S^*}{1+kU^*}\right)+\frac{k\beta S^*U^*}{(1+kU^*)^2}+A_3\right]+\frac{\beta^2S^*U^*}{(1+kU^*)^3}\right\}$$
$$+A_3\left\{\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\left(A_1-\frac{\beta S^*}{1+kU^*}\right)+\frac{k\beta S^*U^*}{(1+kU^*)^2}+A_3\right]+\frac{k\beta S^*U^*A_3}{(1+kU^*)^2}\right\}.$$

Based on inequality (5.3), we can conclude that $\det(\mathbf{H_2})>0$ if and only if b_2 has a positive sign and $R_0>1(U^*>0)$. According to inequality (5.3) and equation (5.4), we obtain

$$b_2=\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\left(A_1-\frac{\beta S^*}{1+kU^*}\right)+\frac{k\beta S^*U^*}{(1+kU^*)^2}+A_3\right]+\frac{k\beta S^*U^*A_3}{(1+kU^*)^2}+\frac{\beta^2S^*U^*}{(1+kU^*)^3}>0. \tag{5.5}$$

Therefore, based on inequalities (5.3) and (5.5), if $R_0>1(U^*>0)$, then we have

$$\det(\mathbf{H_2})>0.$$

Finally, the determinant of the last Hurwitz matrix is

$$\det(\mathbf{H_3})=\begin{vmatrix} b_1 & b_3 & 0\\ 1 & b_2 & 0\\ 0 & b_1 & b_3\end{vmatrix}=\det(\mathbf{H_2})\left\{\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\left(A_1A_3-\eta A_2-\frac{\beta S^*A_3}{1+kU^*}\right)+\frac{k\beta S^*U^*A_3}{(1+kU^*)^2}\right]+\frac{\beta^2S^*U^*A_3}{(1+kU^*)^3}\right\}.$$

Based on equation (5.4), we obtain

$$\det(\mathbf{H_3})=\det(\mathbf{H_2})\left\{\left(\frac{\beta U^*}{1+kU^*}+\mu\right)\left[\frac{k\beta S^*U^*A_3}{(1+kU^*)^2}\right]+\frac{\beta^2S^*U^*A_3}{(1+kU^*)^3}\right\}.$$

If $R_0>1$ ($U^*>0$ and $\det(\mathbf{H_2})>0$), then we have

$$\det(\mathbf{H_3})>0.$$

Therefore, the determinants of the Hurwitz matrices $\mathbf{H_1}$, $\mathbf{H_2}$, and $\mathbf{H_3}$ are greater than zero when $R_0>1$ ($U^*>0$). This implies all the eigenvalues of the Jacobian evaluated at $\mathbf{E}^*=(S^*,U^*,Q^*)$ have negative real parts. As a result, the drug persistent equilibrium $\mathbf{E}^*=(S^*,U^*,Q^*)$ is locally asymptotically stable when $R_0>1$. □

6. Global stability analysis

Based on the results of local stability analysis, it seems that the periodic orbit does not exist for model (2.1)-(2.3). Thus, the non-existence of periodic orbit will be proved by performing the global stability analysis. Particularly, Liapunov's direct method [8] is employed to establish the conditions of global asymptotic stability of drug-free and drug persistent equilibria of model (2.1)-(2.3).

6.1. Drug-free equilibrium

Theorem 6.1. *If* $R_0 \leqslant 1$, *then the drug-free equilibrium* $\mathbf{E_0} = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ *of model* (2.1)-(2.3) *is globally asymptotically stable.*

Proof. Given a combination of Volterra-type and linear Liapunov function as

$$V = A_3\left(S - S_0 - S_0 \ln \frac{S}{S_0}\right) + A_3 U + \eta Q, \tag{6.1}$$

where $S_0 = \Lambda/\mu$, the time derivative of equation (6.1) is

$$\begin{aligned}
V' &= A_3\left(1 - \frac{S_0}{S}\right)\left(\Lambda - \frac{\beta S U}{1+kU} - \mu S\right) + A_3\left(\frac{\beta S U}{1+kU} - A_1 U + \eta Q\right) + \eta(A_2 U - A_3 Q) \\
&= A_3\left(1 - \frac{S_0}{S}\right)\left(\mu S_0 - \frac{\beta S U}{1+kU} - \mu S\right) + \frac{A_3 \beta S U}{1+kU} - A_1 A_3 U + \eta A_2 U \\
&= A_3\left(1 - \frac{S_0}{S}\right)\left[-\frac{\beta S U}{1+kU} - \mu(S - S_0)\right] + \frac{A_3 \beta S U}{1+kU} - U(A_1 A_3 - \eta A_2) \\
&= -\frac{A_3 \mu S(S - S_0)}{S} + \frac{A_3 \mu S_0 (S - S_0)}{S} + \frac{\beta S_0 A_3 U}{1+kU} - \frac{A_3 \beta S U}{1+kU} + \frac{A_3 \beta S U}{1+kU} - U(A_1 A_3 - \eta A_2) \\
&= -A_3 \mu \frac{(S - S_0)^2}{S} + \frac{\beta S_0 A_3 U}{1+kU} - U(A_1 A_3 - \eta A_2) \\
&= -A_3 \mu \frac{(S - S_0)^2}{S} + U\left[\frac{\beta \Lambda A_3}{\mu(1+kU)} - (A_1 A_3 - \eta A_2)\right] \\
&\leqslant -A_3 \mu \frac{(S - S_0)^2}{S} + U\left[\frac{\beta \Lambda A_3}{\mu} - (A_1 A_3 - \eta A_2)\right] \\
&= -A_3 \mu \frac{(S - S_0)^2}{S} - U(A_1 A_3 - \eta A_2)\left[1 - \frac{\beta \Lambda A_3}{\mu(A_1 A_3 - \eta A_2)}\right] \\
&= -A_3 \mu \frac{(S - S_0)^2}{S} - U[\mu A_2 + A_3(\sigma + \mu)]\left\{1 - \frac{\beta \Lambda A_3}{\mu[\mu A_2 + A_3(\sigma + \mu)]}\right\} \\
&= -A_3 \mu \frac{(S - S_0)^2}{S} - U[\mu A_2 + A_3(\sigma + \mu)](1 - R_0).
\end{aligned}$$

Thus, we obtain $V' \leqslant 0$ when $R_0 \leqslant 1$. Note that $V' = 0$ if and only if $S = S_0 = \Lambda/\mu$ and $U = U_0 = 0$. Accordingly, if $R_0 \leqslant 1$, then $V' < 0$ for $S \neq S_0 = \Lambda/\mu$, $U \neq U_0 = 0$, and $Q \neq Q_0 = 0$. Therefore, the drug-free equilibrium $\mathbf{E_0} = (\Lambda/\mu, 0, 0)$ is said to be globally asymptotically stable when $R_0 \leqslant 1$. As a result, if $R_0 \leqslant 1$, then each solution $\mathbf{x} = (S, U, Q)$ of model (2.1)-(2.3) which starts in region ω will approach $\mathbf{E_0} = (\Lambda/\mu, 0, 0)$ as $t \to \infty$. □

6.2. Drug persistent equilibrium

Theorem 6.2. *If* $R_0 > 1$, *then the drug persistent equilibrium* $\mathbf{E}^* = (S^*, U^*, Q^*)$ *of model* (2.1)-(2.3) *is globally asymptotically stable.*

Proof. A Volterra-type of Liapunov function is proposed:

$$\hat{V} = (1 + kU^*)\left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \left(U - U^* - U^* \ln \frac{U}{U^*}\right) + \frac{\eta Q^*}{A_2 U^*}\left(Q - Q^* - Q^* \ln \frac{Q}{Q^*}\right). \tag{6.2}$$

The time derivative of equation (6.2) is

$$\begin{aligned}\hat{V}' = (1+kU^*)\left(\frac{S-S^*}{S}\right)\left(\Lambda - \frac{\beta SU}{1+kU} - \mu S\right) + \left(\frac{U-U^*}{U}\right)\left(\frac{\beta SU}{1+kU} - A_1 U + \eta Q\right) \\ + \frac{\eta Q^*}{A_2 U^*}\left(\frac{Q-Q^*}{Q}\right)(A_2 U - A_3 Q).\end{aligned} \tag{6.3}$$

From model (2.1)-(2.3), we have

$$\Lambda = \frac{\beta S^* U^*}{1+kU^*} + \mu S^*, \quad A_1 = \frac{\beta S^*}{1+kU^*} + \eta\frac{Q^*}{U^*}, \quad A_3 = A_2\frac{U^*}{Q^*}. \tag{6.4}$$

Substituting (6.4) into equation (6.3), we obtain

$$\begin{aligned}\hat{V}' &= (1+kU^*)\left(\frac{S-S^*}{S}\right)\left[\left(\frac{\beta S^* U^*}{1+kU^*} + \mu S^*\right) - \frac{\beta SU}{1+kU} - \mu S\right] \\ &\quad + \left(\frac{U-U^*}{U}\right)\left[\frac{\beta SU}{1+kU} - \left(\frac{\beta S^*}{1+kU^*} + \eta\frac{Q^*}{U^*}\right)U + \eta Q\right] + \frac{\eta Q^*}{A_2 U^*}\left(\frac{Q-Q^*}{Q}\right)\left[A_2 U - \left(A_2\frac{U^*}{Q^*}\right)Q\right] \\ &= -(1+kU^*)\left(\frac{S-S^*}{S}\right)\left\{\beta\left[\frac{(SU-S^*U^*)+kUU^*(S-S^*)}{(1+kU)(1+kU^*)}\right] + \mu(S-S^*)\right\} + \left(\frac{U-U^*}{U}\right) \\ &\quad \times\left\{\beta U[\frac{(S-S^*)+k(SU^*-S^*U)}{(1+kU)(1+kU^*)}] + \eta\left(Q - Q^*\frac{U}{U^*}\right)\right\} + \frac{\eta Q^*}{U^*}\left(\frac{Q-Q^*}{Q}\right)\left(U - U^*\frac{Q}{Q^*}\right) \\ &= -(1+kU^*)\left(\frac{S-S^*}{S}\right)\left\{\beta\left[\frac{S(U-U^*)+U^*(S-S^*)+kUU^*(S-S^*)}{(1+kU)(1+kU^*)}\right] + \mu(S-S^*)\right\} + \left(\frac{U-U^*}{U}\right) \\ &\quad \times\left\{\beta U\left[\frac{(S-S^*)+k[U^*(S-S^*)-S^*(U-U^*)]}{(1+kU)(1+kU^*)}\right] + \eta\left(Q - Q^*\frac{U}{U^*}\right)\right\} + \frac{\eta Q^*}{U^*}\left(\frac{Q-Q^*}{Q}\right)\left(U - U^*\frac{Q}{Q^*}\right) \\ &= -(S-S^*)\beta\left(\frac{U-U^*}{1+kU}\right) - \frac{(S-S^*)^2}{S}[\beta U^* + \mu(1+kU^*)] \\ &\quad + (U-U^*)\left\{\beta\left[\frac{S-S^*}{1+kU} - \frac{kS^*(U-U^*)}{(1+kU)(1+kU^*)}\right] + \eta\left(\frac{Q}{U} - \frac{Q^*}{U^*}\right)\right\} + \eta Q^*\left(1-\frac{Q^*}{Q}\right)\left(\frac{U}{U^*} - \frac{Q}{Q^*}\right) \\ &= -(S-S^*)\beta\left(\frac{U-U^*}{1+kU}\right) - \frac{(S-S^*)^2}{S}[\beta U^* + \mu(1+kU^*)] + (U-U^*)\beta\left(\frac{S-S^*}{1+kU}\right) - \frac{\beta kS^*(U-U^*)^2}{(1+kU)(1+kU^*)} \\ &\quad + \eta Q^*\left(\frac{Q}{Q^*} - \frac{U}{U^*} - \frac{U^*Q}{UQ^*} + 1\right) + \eta Q^*\left(\frac{U}{U^*} - \frac{Q}{Q^*} - \frac{UQ^*}{U^*Q} + 1\right) \\ &= -\frac{(S-S^*)^2}{S}[\beta U^* + \mu(1+kU^*)] - \frac{\beta kS^*(U-U^*)^2}{(1+kU)(1+kU^*)} - \eta Q^*\left(\frac{U^*Q}{UQ^*} + \frac{UQ^*}{U^*Q} - 2\right) \\ &= -\frac{[\beta U^* + \mu(1+kU^*)](S-S^*)^2}{S} - \frac{\beta kS^*(U-U^*)^2}{(1+kU)(1+kU^*)} - \eta Q^*\left(\sqrt{\frac{U^*Q}{UQ^*}} - \sqrt{\frac{UQ^*}{U^*Q}}\right)^2.\end{aligned}$$

Hence, we have $\hat{V}' \leqslant 0$ when $R_0 > 1$ ($U^* > 0$). Note that $\hat{V}' = 0$ if and only if $S = S^*$, $U = U^*$, and $Q = Q^*$. Accordingly, if $R_0 > 1$, then $\hat{V}' < 0$ for $S \neq S^*$, $U \neq U^*$, and $Q \neq Q^*$. Thus, the drug persistent equilibrium $\mathbf{E}^* = (S^*, U^*, Q^*)$ is globally asymptotically stable when $R_0 > 1$. As a result, if $R_0 > 1$, then every solution $\mathbf{x} = (S, U, Q)$ of model (2.1)-(2.3) which starts in region ω will approach $\mathbf{E}^* = (S^*, U^*, Q^*)$ as $t \to \infty$. □

7. Sensitivity analysis

In this section, the differential sensitivity analysis [4] is adopted for determining the sensitivity of the basic reproduction number R_0 to the model parameters. Note that the outcomes of differential sensitivity

analysis are called the normalized forward sensitivity indices (NFSIs) when the quotients which normalize the sensitivity indices are introduced to remove the effects of units. The NFSI of a variable to a parameter is the ratio of the relative change in that variable to the relative change in that parameter [2]. If a variable is the differentiable function of a parameter, then the NFSI can be found by partial differentiation. Note that R_0 is chosen as the variable in this study as it is directly related to the initial spread of drug use. Therefore, we can determine the parameter to be targeted and so some effective control measures can be devised to control or eradicate the drug epidemics. The NFSIs of R_0 to the model parameters are listed in Table 2 where the computations are shown in **Appendix 2**.

Table 2: NFSIs of R_0 to the model parameters.

No.	Parameter	Sign of NFSI	|NFSI|
1	μ	-	>1
2	Λ	+	1
	β	+	1
4	α	-	<1
	γ	-	<1
	σ	-	<1
	η	+	<1

Based on Table 2, the three parameters with a positive sign are Λ, β, and η. Thus, the increase (decrease) in these parameters will lead to the increase (decrease) in the value of R_0. In contrast, the four parameters with a negative sign are μ, α, γ, and σ. Hence, the increase (decrease) in these parameters will lead to the decrease (increase) in the value of R_0. It is worth noting that the NFSI of R_0 to the measure of psychological or inhibitory effect k is zero, without involving any computation. This is because the expression of R_0 is independent of the parameter k. Recall that that the magnitude of NFSI indicates the relative change of R_0 when a parameter varies. For instance, R_0 will increase 30% when the parameter Λ increases 30% and R_0 will increase less than 50% when parameter η increases 50%. According to Table 2, we can conclude that the most sensitive parameter on R_0 is the natural death rate μ, which has an absolute value of NFSI greater than unity. However, it is unethical and impractical to increase the natural death rate μ in practice. Consequently, we should put our attention on the recruitment rate Λ and effective contact rate $\beta = c\hat{\beta}$. Similar to parameter μ, the former is also unethical and impractical to be increased its value. Therefore, the effective contact rate $\beta = c\hat{\beta}$ turns out to be the most significant parameter on R_0. As a result, the average number of contacts per unit time c and the probability that a contact results in a new drug user $\hat{\beta}$ should be targeted to reduce the value of β. In the last section, several effective control measures will be suggested and discussed.

8. Numerical simulations

The numerical simulations will be provided here for illustrating the analytical results in Section 6. With the help of MATLAB, the fourth-order Runge-Kutta method is adopted to obtain the time-series plots and phase portraits. Three examples which represent the cases of $R_0 < 1$, $R_0 = 1$, and $R_0 > 1$ are provided in the following.

Example 8.1. Given $\Lambda = 2$, $\mu = 0.02$, $\beta = 0.003$, $\sigma = 0.28$, $\alpha = 0.15$, $\gamma = 0.65$, $\eta = 0.3$, $k = 0.1$, and initial conditions: $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$, hence, we have $R_0 \approx 0.8571 < 1$ and $E_0 = (100, 0, 0)$. According to Theorem 6.1, drug-free equilibrium $E_0 = (100, 0, 0)$ of model (2.1)-(2.3) is globally asymptotically stable. The corresponding time series plots and phase portrait are illustrated in Figures 3 and 4, respectively.

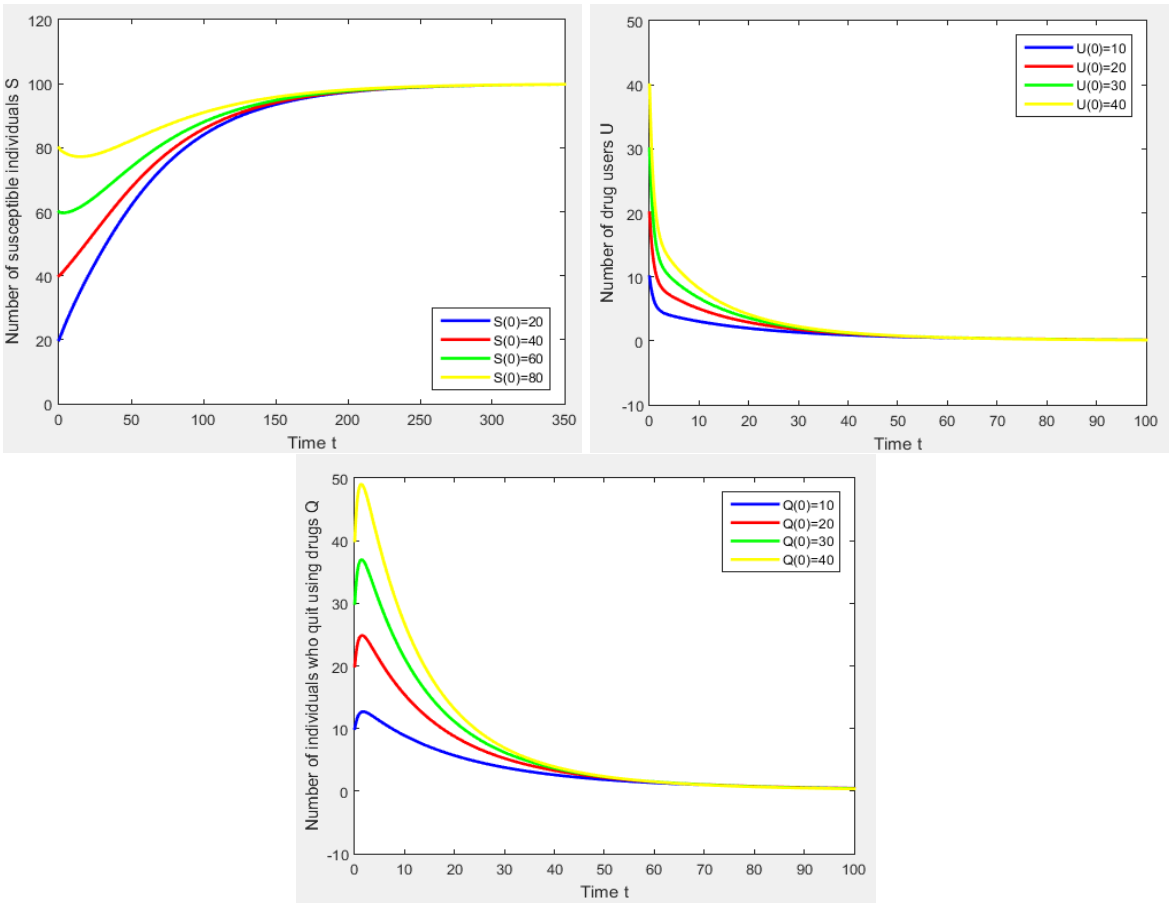

Figure 3: Time Series Plots of Variable S, U, and Q, for different initial conditions when $R_0 < 1$.

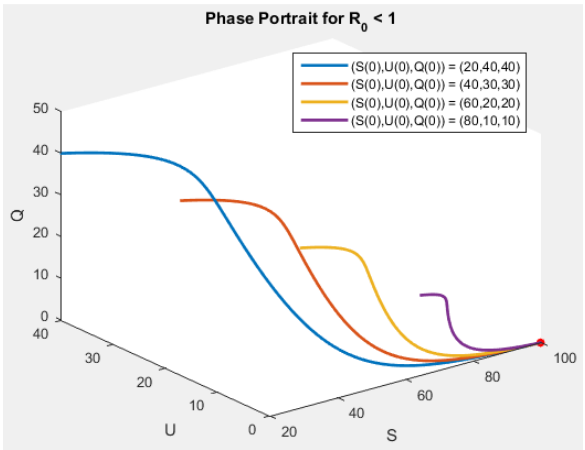

Figure 4: Phase Portrait for $R_0 < 1$.

From Figure 3, we can observe that the variable S with initial values $S(0) = 20, 40, 60, 80$, approaches $\Lambda/\mu = 100$ as $t \to \infty$. Besides, the variable U with initial values $U(0) = 10, 20, 30, 40$, approaches $U_0 = 0$ as $t \to \infty$ and variable Q with initial values $Q(0) = 10, 20, 30, 40$, approaches $Q_0 = 0$ as $t \to \infty$. On the other hand, the phase portrait in Figure 4 shows that the trajectories which start at $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$ will approach the drug-free equilibrium $E_0 = (100, 0, 0)$ as $t \to \infty$. Thus, if $R_0 < 1$, then the solutions of model (2.1)-(2.3) with any non-negative initial conditions $(S(0), U(0), Q(0))$ will approach $E_0 = (\Lambda/\mu, 0, 0)$ as $t \to \infty$. As a result, the drug-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$ of model

(2.1)-(2.3) is said to be globally asymptotically stable when $R_0 < 1$.

Example 8.2. Given $\Lambda = 2$, $\mu = 0.02$, $\beta = 0.0105$, $\sigma = 0.48$, $\alpha = 0.4$, $\gamma = 0.7$, $\eta = 0.02$, $k = 0.1$, and initial conditions: $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$, So, we have $R_0 = 1$ and $E_0 = (100, 0, 0)$. Based on Theorem 6.1, the drug-free equilibrium $E_0 = (100, 0, 0)$ of model (2.1)-(2.3) is globally asymptotically stable. The corresponding time series plots and phase portrait are illustrated in Figures 5 and 6, respectively.

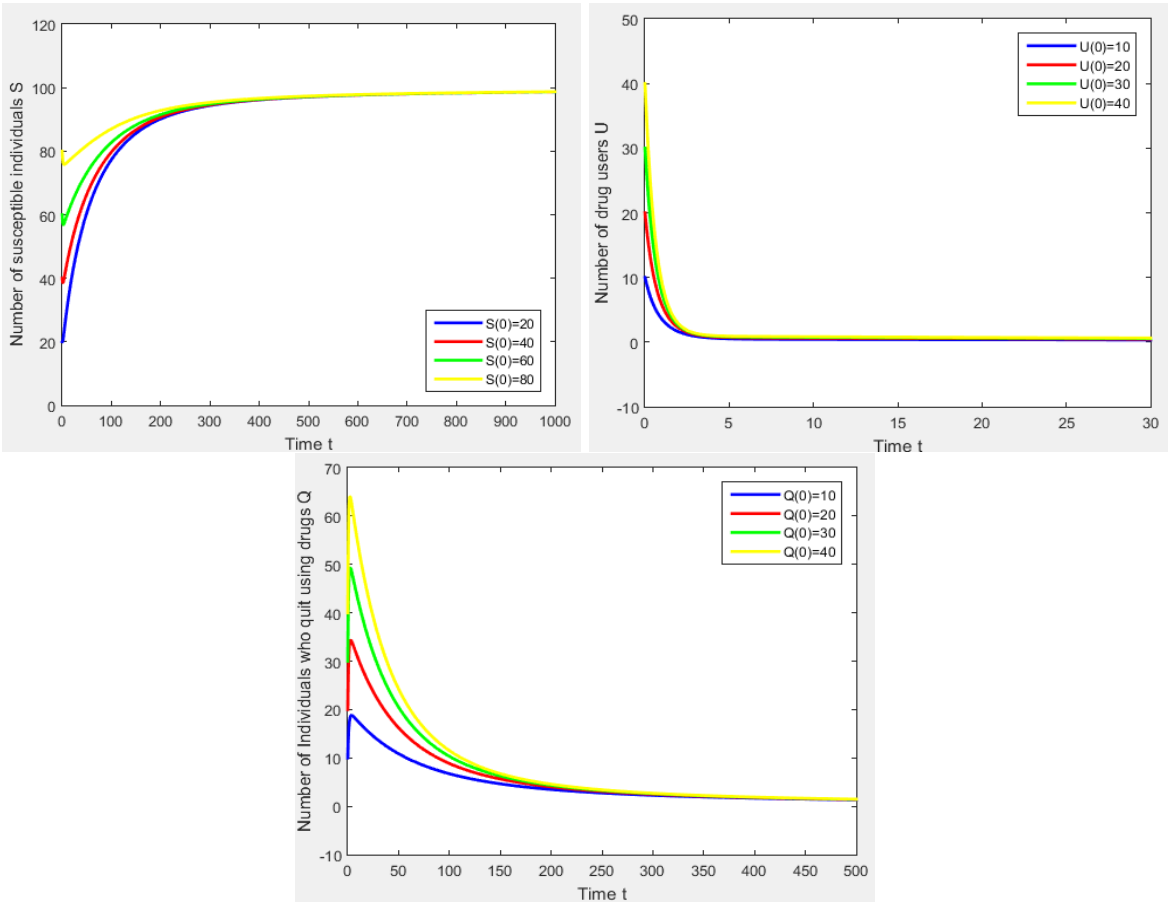

Figure 5: Time Series Plots of Variable S, U and Q, for different initial conditions when $R_0 = 1$.

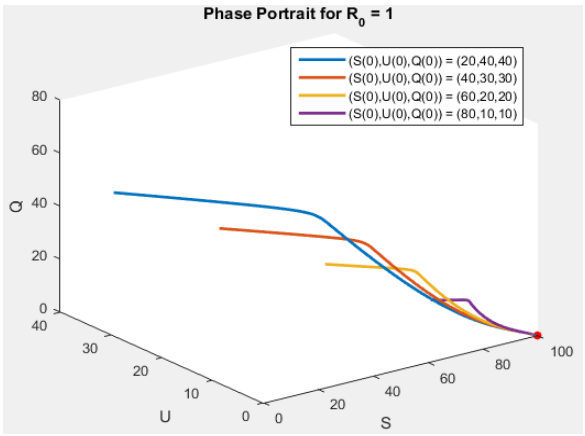

Figure 6: Phase Portrait for $R_0 = 1$.

From Figure 5, we can observe that variable S with initial values $S(0) = 10, 30, 50, 70$ approaches $\Lambda/\mu = 100$ as $t \to \infty$. Additionally, the variable U with initial values $U(0) = 10, 30, 50, 70$ approaches $U_0 = 0$ as $t \to \infty$ and variable Q with initial values $Q(0) = 10, 30, 50, 70$ approaches $Q_0 = 0$ as $t \to \infty$. On the other hand, the phase portrait in Figure 6 shows that the trajectories which start at $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$ will approach the drug-free equilibrium $E_0 = (100, 0, 0)$ as $t \to \infty$. Therefore, the drug-free equilibrium $E_0 = (\Lambda/\mu, 0, 0)$ of model (2.1)-(2.3) is said to be globally asymptotically stable if $R_0 = 1$. Note that $R_0 = 1$ indicates a drug user will influence a susceptible individual to start taking drugs in a fully susceptible population. So intuitively, the drug epidemic will eventually become endemic. However, the analytical result and numerical simulations give a counter-intuitive result where a drug epidemic will die out even though $R_0 = 1$. This is probably owing to the natural and drug-related death rates, and recovery rates through abstinence and treatment.

Example 8.3. Given $\Lambda = 2$, $\mu = 0.02$, $\beta = 0.01$, $\eta = 0.3$, $\sigma = 0.28$, $\alpha = 0.15$, $\gamma = 0.65$, $k = 0.1$, and initial conditions: $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$, hence, we have $R_0 \approx 2.8571 > 1$ and $\mathbf{E}^* \approx (46, 3, 8)$. According to Theorem 6.2, the drug persistent equilibrium $\mathbf{E}^* \approx (46, 3, 8)$ of model (2.1)-(2.3) is globally asymptotically stable. The corresponding time series plots and phase portrait are illustrated in Figures 7 and 8, respectively.

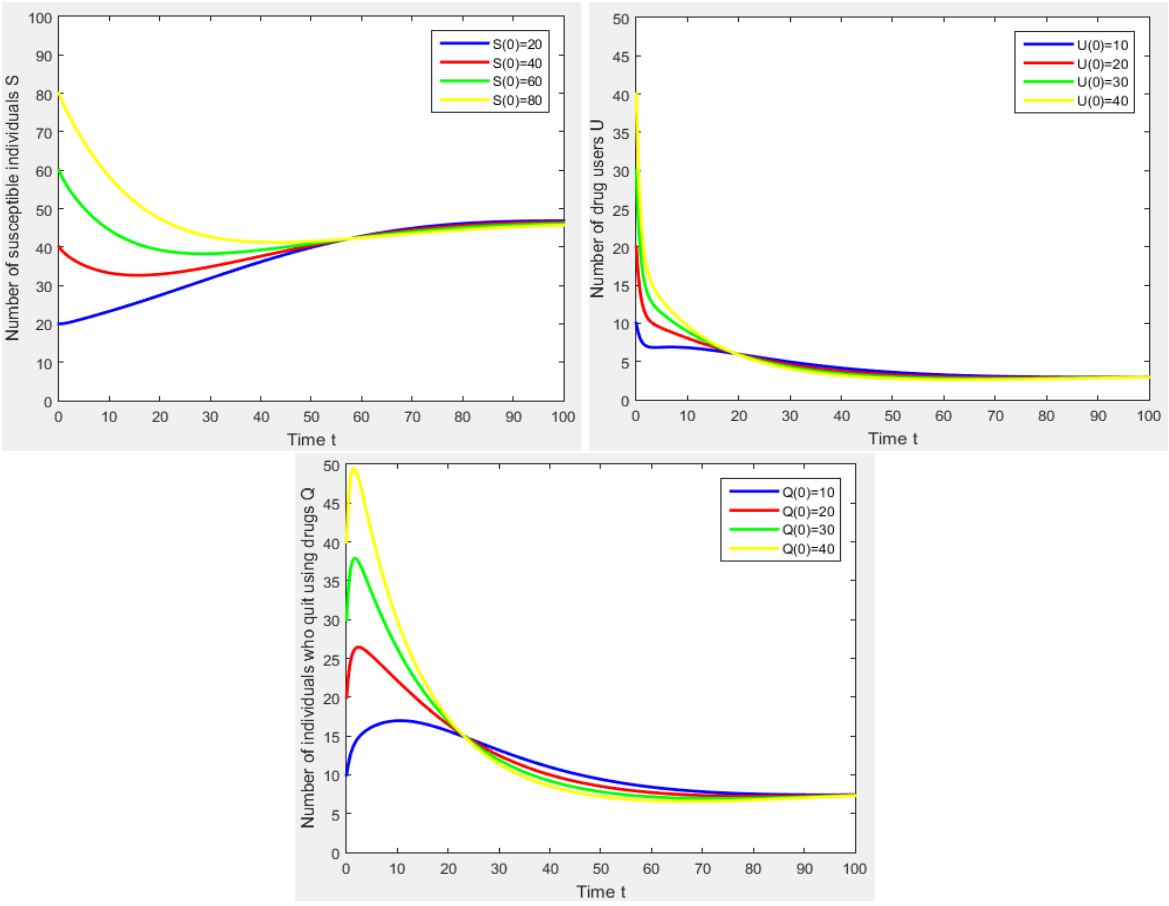

Figure 7: Time series plots of variable S, U, and Q, for different initial conditions when $R_0 > 1$.

From Figure 7, we can observe that variable S with initial value $S(0) = 20, 40, 60, 80$ approaches $S^* = 46$ as $t \to \infty$. Moreover, variables U and Q with initial values $U(0) = Q(0) = 20, 40, 60, 80$ approach $U^* = 3$ and $Q^* = 8$, respectively, as $t \to \infty$. On the other hand, the phase portrait in Figure 8 shows that the trajectories which start at $(20, 40, 40)$, $(40, 30, 30)$, $(60, 20, 20)$, and $(80, 10, 10)$ will approach the drug persistent equilibrium $E^* = (46, 3, 8)$ as $t \to \infty$. Thus, if $R_0 > 1$, then every solution (S, U, Q) of model

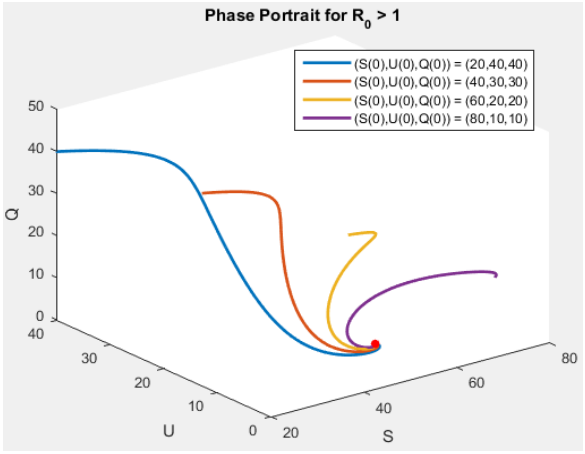

Figure 8: Phase portrait for $R_0 > 1$.

(2.1)-(2.3) with any non-negative initial condition $(S(0), U(0), Q(0))$ will approach the drug persistent equilibrium $E^* = (S^*, U^*, Q^*)$ as $t \to \infty$. As a result, the drug persistent equilibrium $E^* = (S^*, U^*, Q^*)$ of model (2.1)-(2.3) is said to be globally asymptotically stable when $R_0 > 1$.

From equation (4.3), we found that the number of drug users at drug persistent state U^* is inversely proportional to the measure of psychological or inhibitory effect k. However, no conclusion can be drawn on the number of drug users U for $t > 0$, when parameter k is varied. Thus, the numerical simulations are performed to investigate the relationship between the measure of psychological or inhibitory effect k and the number of drug users U.

Example 8.4. Given $\Lambda = 50$, $\mu = 0.02$, $\beta = 0.2$, $\sigma = 0.48$, $\alpha = 0.35$, $\gamma = 0.6$, $\eta = 0.02$, and initial condition $U(0) = 5$, the solution curves of variable U for $k = 0.05$, 0.5, 5, and 50 are illustrated in Figure 9.

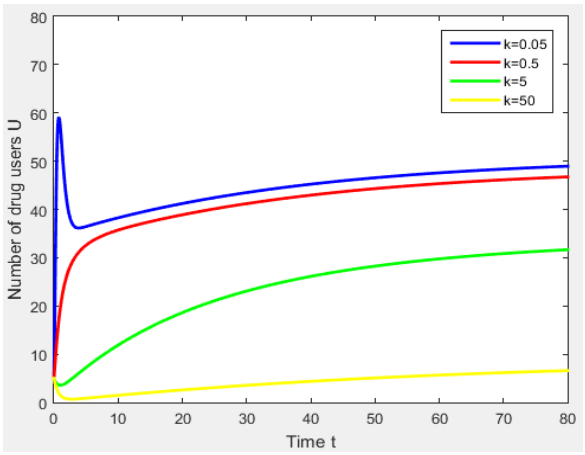

Figure 9: Solution curves of variable U for different values of k.

From Figure 9, we can observe that the higher the measure of psychological or inhibitory effect k, the lower the number of drug users U, given the same initial condition $U(0) = 5$. Thus, the prevalence of drug use is lower when the value of k is higher. Accordingly, a drug epidemic is easier to be controlled when the measure of psychological or inhibitory effect k, is higher. Based on Figure 9, it is evident that the difference in the curves of the number of drug users U for $k = 0.05$ and $k = 0.5$ is inconsiderable. However, it is worth noting that the curve of the number of drug users U for $k = 0.05$ initially soars to a peak, that is, about 60 drug users. Then, the number of drug users U starts lessening and then approaches the steady-state value $U^* \approx 50$. The drastic increase in the number of drug users U should be avoided

as it may cause the prisons and rehabilitation centres to become overcrowded. Accordingly, the polices and rehabilitation team will be stressed to cope with this catastrophic situation. Therefore, it is extremely crucial to have a higher measure of psychological or inhibitory effect k, as this measure plays some role in the occurrence of an outbreak of drug use, besides the prevalence of drug use.

9. Conclusion

In this paper, an illicit drugs model with the saturated incidence rate and relapse of individuals who quit using drugs is proposed. It is crucial to note that the basic reproduction number R_0 of the proposed model is independent of the measure of psychological or inhibitory effect k, and hence the psychological or inhibitory effect has no impact on the initial spread of drug use. Besides, it is found that the proposed model does not have multiple drug persistent equilibria even though it has complex dynamics. Furthermore, $R_0 \leqslant 1$ is found to be a sufficient condition for eradicating the drug epidemics. Moreover, the results of sensitivity analysis show that some effective control measures should be put in place to diminish the average number of contacts per unit time c, or the probability that a contact results in a new drug user $\hat{\beta}$. For the former, the policymakers and anti-drug agencies should urge the public not to have any contact with relatives or friends who use drugs. Rather, the public should send them to rehabilitation centres for receiving treatment. As a result, the susceptible individuals will have a lower chance to contact the drug users per unit time. For the latter, more anti-drug campaigns and advertisements should be introduced and stricter laws on drug use are also should be imposed. Thus, the public is aware of the adverse effect of drug use, and afraid of the punishment for drug use. Consequently, the probability that a contact results in a new drug user will become lower. Finally, the numerical simulations show that it is crucial to have a higher measure of psychological or inhibitory effect k, so that the drug epidemic is easier to be controlled.

Note that the proposed illicit drugs model is not comprehensive, since several ideal assumptions have been made. For future work, the proposed model may be extended to contain more subpopulations for describing the dynamics of illicit drugs use better. Additionally, the population can be structured in terms of age or gender, since individuals with different characteristics may have different drug use patterns. Notwithstanding the imperfection, our study provides some useful insights into the dynamics of drug epidemics. In addition, our study also gives a conclusion on the control measures that should be devised to control or eradicate the drug epidemics effectively.

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Appendices

Appendix 1: Derivation of basic reproduction number R_0

Since the compartments of model (2.1)-(2.3) involve drug use are the classes of drug users U and individuals who quit using drugs Q, model (2.1)-(2.3) is first rearranged as

$$\frac{dU}{dt} = \frac{\beta SU}{1+kU} - A_1U + \eta Q, \quad \frac{dQ}{dt} = A_2U - A_3Q, \quad \frac{dS}{dt} = \Lambda - \frac{\beta SU}{1+kU} - \mu S.$$

Let $\mathbf{x} = (S, U, Q)$, then the matrices for the new drug use terms and the other transition terms are

$$\mathcal{F}(\mathbf{x}) = \begin{pmatrix} \frac{\beta SU}{1+kU} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Upsilon(\mathbf{x}) = \begin{pmatrix} A_1U - \eta Q \\ -A_2U + A_3Q \\ -\Lambda + \frac{\beta SU}{1+kU} + \mu S \end{pmatrix},$$

respectively. Hence, model (2.1)-(2.3) can be denoted as

$$\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) - \Upsilon(\mathbf{x}).$$

We then differentiate the first and second rows of matrices $\mathcal{F}(\mathbf{x})$ and $\Upsilon(\mathbf{x})$ with respect to U and Q. Accordingly, the Jacobian of matrices $\mathcal{F}(\mathbf{x})$ and $\Upsilon(\mathbf{x})$ that evaluated at equation (3.1) are

$$\mathbf{F}(\mathbf{E_0}) = \begin{pmatrix} \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{V}(\mathbf{E_0}) = \begin{pmatrix} A_1 & -\eta \\ -A_2 & A_3 \end{pmatrix},$$

respectively. Thus, the next-generation matrix is found to be

$$\mathbf{FV^{-1}} = \frac{\beta\Lambda}{\mu(A_1A_3 - \eta A_2)} \begin{pmatrix} A_3 & \eta \\ 0 & 0 \end{pmatrix}.$$

As a result, the basic reproduction number is given by

$$R_0 = \rho(\mathbf{FV^{-1}}) = \frac{\beta\Lambda A_3}{\mu(A_1A_3 - \eta A_2)},$$

where $\rho(\mathbf{FV^{-1}})$ is the spectral radius of next-generation matrix $\mathbf{FV^{-1}}$.

Appendix 2: Computations of sensitivity analysis

1. β :

$$\Upsilon_\beta^{R_0} = \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = \frac{\Lambda(\eta+\mu)}{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]} \times \frac{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\Lambda(\eta+\mu)} = +1 > 0,$$
$$\left|\Upsilon_\beta^{R_0}\right| = 1.$$

2. Λ :

$$\Upsilon_\Lambda^{R_0} = \frac{\partial R_0}{\partial \Lambda} \times \frac{\Lambda}{R_0} = \frac{\beta(\eta+\mu)}{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]} \times \frac{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta(\eta+\mu)} = +1 > 0,$$
$$\left|\Upsilon_\Lambda^{R_0}\right| = 1.$$

3. η :

$$\Upsilon_\eta^{R_0} = \frac{\partial R_0}{\partial \eta} \times \frac{\eta}{R_0} = \frac{\beta\Lambda\mu(\alpha+\gamma)}{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]^2} \times \frac{\eta\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta\Lambda(\eta+\mu)}$$
$$= \frac{\eta\mu(\alpha+\gamma)}{(\eta+\mu)[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]} > 0,$$
$$\left|\Upsilon_\eta^{R_0}\right| < 1.$$

4. μ :

$$\Upsilon_\mu^{R_0} = \frac{\partial R_0}{\partial \mu} \times \frac{\mu}{R_0} = -\beta\Lambda \frac{\mu(\alpha+\gamma)(2\eta+\mu)+(\eta+\mu)^2(\sigma+2\mu)}{\{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]\}^2} \times \frac{\mu^2[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta\Lambda(\eta+\mu)}$$
$$= -\frac{\mu(\alpha+\gamma)(2\eta+\mu)+(\eta+\mu)^2(\sigma+2\mu)}{\mu(\alpha+\gamma)(\eta+\mu)+(\eta+\mu)^2(\sigma+\mu)} < 0,$$
$$\left|\Upsilon_\mu^{R_0}\right| > 1.$$

5. α :

$$\Upsilon_\alpha^{R_0} = \frac{\partial R_0}{\partial \alpha} \times \frac{\alpha}{R_0} = -\frac{\beta\Lambda(\eta+\mu)}{[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]^2} \times \frac{\alpha\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta\Lambda(\eta+\mu)}$$
$$= -\frac{\alpha\mu}{\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)} < 0,$$
$$\left|\Upsilon_\alpha^{R_0}\right| < 1.$$

6. γ :

$$\begin{aligned}\Upsilon_{\gamma}^{R_0} = \frac{\partial R_0}{\partial \gamma} \times \frac{\gamma}{R_0} &= -\frac{\beta\Lambda(\eta+\mu)}{[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]^2} \times \frac{\gamma\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta\Lambda(\eta+\mu)} \\ &= -\frac{\gamma\mu}{\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)} < 0, \\ \left|\Upsilon_{\gamma}^{R_0}\right| &< 1.\end{aligned}$$

7. σ :

$$\begin{aligned}\Upsilon_{\sigma}^{R_0} = \frac{\partial R_0}{\partial \sigma} \times \frac{\sigma}{R_0} &= -\frac{\beta\Lambda(\eta+\mu)^2}{\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]^2} \times \frac{\sigma\mu[\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)]}{\beta\Lambda(\eta+\mu)} \\ &= -\frac{(\eta+\mu)\sigma}{\mu(\alpha+\gamma)+(\eta+\mu)(\sigma+\mu)} < 0, \\ \left|\Upsilon_{\sigma}^{R_0}\right| &< 1.\end{aligned}$$