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The spectrum maps of type the domain of general quantum difference in generalized Cesàro sequence space

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Abstract

We introduce the domain of general quantum difference in generalized Cesàro sequence space in this article. Some topological and geometric structures, the multiplication operators defined on it, and the eigenvalue distribution of operator ideals induced by this space and s-numbers have been presented.

Keywords: Cesàro sequence space, general quantum difference, multiplication mapping, s-numbers, pre-quasi ideal, minimum space.

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1. Introduction

The concept of variable exponent function spaces has continued to develop, as it is predicated on the boundedness of the Hardy-Littlewood maximal mapping. This section explores its applications in image processing, differential equations, and approximation theory. Recall that the closed operator ideals are certain to play an essential role in the Banach lattice principle. Some authors discussed geometric and topological structures of the quasi ideal generated by s-numbers and certain sequence spaces; see Pietsch [13, 14, 16], Makarov and Faried [11], and Yaying et al. [19]. Multiplication operators are used extensively in functional analysis, for example, in the eigenvalue distributions theorem, the geometric structure of Banach spaces, and the theory of fixed points. For different sequence spaces, some authors studied the properties of the multiplication operators, such as Komal et al. [10], İlkhan et al. [7], and Bakery and Mohammed [4].

Indicate the set of non-negative integers and the set of integers by $\mathbb{I}^+$ and $\mathbb{I}$, respectively. If $\mho = (\mho_x)$ is strictly increasing, where $\mho : \mathbb{I}^+ \to \mathbb{I}$, the general quantum difference $\nabla_\mho$ is defined in [6] by

$$\nabla_\mho \lambda_x = \begin{cases} \frac{\lambda_{\mho_x} - \lambda_{x-1}}{\mho_x - x + 1}, & \mho_x \neq x-1, \\ 0, & \text{otherwise.} \end{cases}$$

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Recall that if $\mho_x = x$, then $\nabla_{\mho}\lambda_x = \nabla\lambda_x = \lambda_x - \lambda_{x-1}$, where $\lambda_x = 0$ for $x < 0$, is the backward difference defined by Kizmaz [9]. For any two Banach spaces $\mathcal{G}$ and $\mathcal{Q}$, all through the article, we mark the space of all bounded linear, finite rank, approximable and compact operators from $\mathcal{G}$ into $\mathcal{Q}$ by $\mathcal{I}(\mathcal{G},\mathcal{Q})$, $\mathfrak{J}(\mathcal{G},\mathcal{Q})$, $\mathfrak{P}(\mathcal{G},\mathcal{Q})$, and $\mathfrak{T}(\mathcal{G},\mathcal{Q})$, respectively. When $\mathcal{G} = \mathcal{Q}$, we write $\mathcal{I}(\mathcal{G})$, $\mathfrak{J}(\mathcal{G})$, $\mathfrak{P}(\mathcal{G})$, and $\mathfrak{T}(\mathcal{G})$. The ideal of all bounded linear, finite rank, approximable and compact operators between any arbitrary Banach spaces, will be indicated by $\mathcal{I}$, $\mathfrak{J}$, $\mathfrak{P}$, and $\mathfrak{T}$.

Definition 1.1 ([17]). An operator $s : \mathcal{I}(\mathcal{G},\mathcal{Q}) \to [0,\infty)^{\mathbb{I}^+}$ is called an s-number, if the sequence $(s_x(H))_{x=0}^{\infty}$, for every $H \in \mathcal{I}(\mathcal{G},\mathcal{Q})$, holds the following conditions:

(a) $\|H\| = s_0(H) \geqslant s_1(H) \geqslant s_2(H) \geqslant \cdots \geqslant 0$, with $H \in \mathcal{I}(\mathcal{G},\mathcal{Q})$;
(b) $s_{x+y-1}(H_1 + H_2) \leqslant s_x(H_1) + s_y(H_2)$, with $H_1, H_2 \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ and $x, y \in \mathbb{I}^+$;
(c) $s_x(ZYH) \leqslant \|Z\| s_x(Y)\, \|H\|$, for every $H \in \mathcal{I}(\mathcal{G}_0,\mathcal{G})$, $Y \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ and $Z \in \mathcal{I}(\mathcal{Q},\mathcal{Q}_0)$, where $\mathcal{G}_0$ and $\mathcal{Q}_0$ are any two Banach spaces;
(d) if $G \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ and $\gamma \in \mathfrak{C}$, where $\mathfrak{C}$ is the space of all complex numbers, hence $s_x(\gamma G) = |\gamma| s_x(G)$;
(e) suppose $\mathrm{rank}(H) \leqslant x$, then $s_x(H) = 0$, for all $H \in \mathcal{I}(\mathcal{G},\mathcal{Q})$;
(f) $s_{y\geqslant x}(I_x) = 0$ or $s_{y<x}(I_x) = 1$, where I_x denots the identity mapping on the x-dimensional Hilbert space ℓ_2^x.

We give here a few examples of s-numbers as follows:

(1) The y-th Kolmogorov number, $d_y(H)$, where $d_y(H) = \inf_{\dim J \leqslant y} \sup_{\|\lambda\|\leqslant 1} \inf_{\beta\in J} \|H\lambda - \beta\|$.
(2) The y-th approximation number, $\alpha_y(H)$, where $\alpha_y(H) = \inf\{\|H - Z\| : Z \in \mathcal{I}(\mathcal{G},\mathcal{Q}) \text{ and } \mathrm{rank}(Z) \leqslant y\}$.

Notations 1.2 ([5]). Suppose $\mathcal{K}$ is a sequence space,

$$\begin{aligned}
\mathcal{I}^s_{\mathcal{K}} &:= \left\{\mathcal{I}^s_{\mathcal{K}}(\mathcal{G},\mathcal{Q})\right\}, \text{ where } \mathcal{I}^s_{\mathcal{K}}(\mathcal{G},\mathcal{Q}) := \left\{H \in \mathcal{I}(\mathcal{G},\mathcal{Q}) : ((s_x(H))_{x=0}^{\infty} \in \mathcal{K}\right\};\\
\mathcal{I}^\alpha_{\mathcal{K}} &:= \left\{\mathcal{I}^\alpha_{\mathcal{K}}(\mathcal{G},\mathcal{Q})\right\}, \text{ where } \mathcal{I}^\alpha_{\mathcal{K}}(\mathcal{G},\mathcal{Q}) := \left\{H \in \mathcal{I}(\mathcal{G},\mathcal{Q}) : ((\alpha_x(H))_{x=0}^{\infty} \in \mathcal{K}\right\};\\
\mathcal{I}^d_{\mathcal{K}} &:= \left\{\mathcal{I}^d_{\mathcal{K}}(\mathcal{G},\mathcal{Q})\right\}, \text{ where } \mathcal{I}^d_{\mathcal{K}}(\mathcal{G},\mathcal{Q}) := \left\{H \in \mathcal{I}(\mathcal{G},\mathcal{Q}) : ((d_x(H))_{x=0}^{\infty} \in \mathcal{K}\right\};\\
(\mathcal{I}^s_{\mathcal{K}})^\rho &:= \left\{(\mathcal{I}^s_{\mathcal{K}})^\rho(\mathcal{G},\mathcal{Q})\right\}, \text{ where}\\
(\mathcal{I}^s_{\mathcal{K}})^\rho(\mathcal{G},\mathcal{Q}) &:= \left\{Y \in \mathcal{I}(\mathcal{G},\mathcal{Q}) : ((\rho_x(Y))_{x=0}^{\infty} \in \mathcal{K} \text{ and } \|Y - \rho_x(Y)I\| \text{ is not invertible, for all } x \in \mathbb{I}^+\right\}.
\end{aligned}$$

Lemma 1.3 ([1]). *Assume $\sigma_x > 0$ and $\lambda_x, \beta_x \in \mathfrak{C}$, for all $x \in \mathbb{I}^+$, and $\hbar = \max\{1, \sup_x \sigma_x\}$, then*

$$|\lambda_x + \beta_x|^{\sigma_x} \leqslant 2^{\hbar-1}\left(|\lambda_x|^{\sigma_x} + |\beta_x|^{\sigma_x}\right).$$

The aim of this article is organized as follows. We define and discuss several inclusion relations for the domain of general quantum difference in generalized Cesàro sequence space, $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, equipped with the function $\top$ in Section 2. In Section 3, we investigate the sufficient conditions on $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ under definite function $\top$ to create pre-modular private sequence space ($\mathfrak{pss}$). This implies that it is a pre-quasi normed $\mathfrak{pss}$. The topological and geometric structures of the class $\mathcal{I}^s_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$, and the class $\left(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}\right)^\rho$ are given. In Section 4, we provide some topological and geometric behaviors of the multiplication mappings defined on this sequence space.

2. $(\mathbf{Ces}(\nabla_{\mho},\sigma))_{\top}$

The definition of the domain of general quantum difference in generalized Cesàro sequence space, $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, under the function $\top$, as well as several inclusion relations, are discussed in this section.

Definition 2.1. Suppose $(\sigma_y) \in (0,\infty)^{\mathbb{I}^+}$, where $(0,\infty)^{\mathbb{I}^+}$ is the space of all sequences of positive reals, and $\nabla_\mho$ is absolutely non-decreasing. The sequence space, $(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_1}$, is defined as:

$$(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_1} = \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top_1(\varepsilon\beta) < \infty, \text{ for some } \varepsilon > 0\Big\},$$

where $\top_1(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y}$.

Theorem 2.2. *If* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+}$, *then*

$$(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_1} \subset (\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_2},$$

where $\top_2(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y}$.

Proof. One gets

$$\begin{aligned}
&(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_1}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top_1(\varepsilon\beta) < \infty, \text{ for some } \varepsilon > 0\Big\}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\varepsilon\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\varepsilon\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y} < \infty, \text{ for some } \varepsilon > 0\Big\}\\
&\subset \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top_2(\varepsilon\beta) < \infty, \text{ for some } \varepsilon > 0\Big\} = (\mathrm{Ces}(\nabla_\mho,\sigma))_{\top_2}.
\end{aligned}$$

□

Theorem 2.3. *If* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+} \cap \ell_\infty$, *where* ℓ_∞ *is the space of bounded sequences of complex numbers, then*

$$(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top} = \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top(\varepsilon\beta) < \infty, \textit{ for any } \varepsilon > 0\Big\},$$

where $\top(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y}$.

Proof. As $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+} \cap \ell_\infty$, then

$$\begin{aligned}
(\mathrm{Ces}(\nabla_\mho,\sigma))_{\top} &= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top(\varepsilon\beta) < \infty, \text{ for some } \varepsilon > 0\Big\}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\varepsilon\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y} < \infty, \text{ for some } \varepsilon > 0\Big\}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \inf_y \varepsilon^{\sigma_y}\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y} < \infty, \text{ for some } \varepsilon > 0\Big\}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho\beta_x|}{y+1}\right)^{\sigma_y} < \infty\Big\}\\
&= \Big\{\beta = (\beta_y) \in \mathfrak{C}^{\mathbb{I}^+} : \top(\varepsilon\beta) < \infty, \text{ for any } \varepsilon > 0\Big\}.
\end{aligned}$$

□

Theorem 2.4. *Suppose* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+}$ *and* $\mho_x \neq x-1$, *for all* $x \in \mathbb{I}^+$, *then* $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ *is a non-absolute type, where* $\top(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho \beta_x|}{y+1}\right)^{\sigma_y}$.

Proof. Assume without loss of generality that $\mho_x = x-2$, for all $x \in \mathbb{I}^+$ and by taking $\beta = \Big(1,-1,0,0,0,\dots\Big)$, one has $|\beta| = \Big(1,1,0,0,0,\dots\Big)$. Also

$$\begin{aligned}\top(\beta) &= \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_{x-2}-\beta_{x-1}|}{y+1}\right)^{\sigma_y}\\ &= 2+(\frac{1}{2})^{\sigma_1}+\cdots \neq (\frac{1}{2})^{\sigma_1}+(\frac{1}{3})^{\sigma_2}+(\frac{1}{2})^{\sigma_3}+\cdots = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}||\beta|_{x-2}-|\beta|_{x-1}|}{y+1}\right)^{\sigma_y} = \top(|\beta|).\end{aligned}$$

For $\mho_x = x$, one obtains

$$\begin{aligned}\top(\beta) &= \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_{x}-\beta_{x-1}|}{y+1}\right)^{\sigma_y}\\ &= 1+(\frac{3}{2})^{\sigma_1}+(\frac{4}{3})^{\sigma_2}+\cdots \neq 1+(\frac{1}{2})^{\sigma_1}+(\frac{2}{3})^{\sigma_2}+\cdots = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}||\beta|_{x}-|\beta|_{x-1}|}{y+1}\right)^{\sigma_y} = \top(|\beta|).\end{aligned}$$

Hence, the sequence space $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ is a non-absolute type. □

Definition 2.5. Assume $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+}$. The sequence space $(\mathrm{Ces}(\nabla, \sigma))_\top$ is defined by:

$$(\mathrm{Ces}(\nabla,\sigma))_\top = \Big\{\beta = (\beta_x) \in \mathfrak{C}^{\mathbb{I}^+} : \top(\varepsilon\beta) < \infty, \text{ for some } \varepsilon > 0\Big\},$$

where $\top(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_x - \beta_{x-1}|}{y+1}\right)^{\sigma_y}$.

Theorem 2.6. *If* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+} \cap \ell_\infty$ *and* $\mho_x > x$, *for all* $x \in \mathbb{I}^+$, *one has*

$$(\mathrm{Ces}(\nabla,\sigma))_\top \subsetneqq (\mathrm{Ces}(\nabla_\mho,\sigma))_\top.$$

Proof. Let $\beta \in (\mathrm{Ces}(\nabla,\sigma))_\top$, since

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\frac{\beta_{\mho_x}-\beta_{x-1}}{\mho_x - x+1}\right|}{y+1}\right)^{\sigma_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_x - \beta_{x-1}|}{y+1}\right)^{\sigma_y} < \infty.$$

Hence, $\beta \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$. By taking $\beta = (1,0,1,0,\dots)$, so $\beta \notin (\mathrm{Ces}(\nabla,\sigma))_\top$ and $\beta \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$, where $\mho_x = x+1$. □

Theorem 2.7. *For* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+} \cap \ell_\infty$ *and* $\mho_x < x-1$, *for all* $x \in \mathbb{I}^+$, *we have*

$$(\mathrm{Ces}(\nabla,\sigma))_\top \subsetneqq (\mathrm{Ces}(\nabla_\mho,\sigma))_\top.$$

Proof. Assume $\beta \in (Ces(\nabla, \sigma))_{\top}$, since

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\frac{\beta_{\mho_x}-\beta_{x-1}}{\mho_x - x+1}\right|}{y+1}\right)^{\sigma_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_{\mho_{x+1}}-\beta_{\mho_x}|}{y+1}\right)^{\sigma_y}$$
$$= \sum_{y=0}^{\infty}\left(\frac{\sum_{m=0}^{\mho_y}|\beta_m-\beta_{m-1}|}{y+1}\right)^{\sigma_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_x-\beta_{x-1}|}{y+1}\right)^{\sigma_y} < \infty.$$

We have $\beta \in (Ces(\nabla_{\mho}, \sigma))_{\top}$. □

Theorem 2.8. *Let* $(\sigma_x) \in (0,\infty)^{\mathbb{I}^+} \cap \ell_{\infty}$, *we get*

$$(Ces(\sigma))_{\top} \subsetneqq (Ces(\nabla, \sigma))_{\top}.$$

Proof. Assume $\beta \in (Ces(\sigma))_{\top}$, since

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_x-\beta_{x-1}|}{y+1}\right)^{\sigma_y} \leqslant 2^{\hbar}\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\beta_x|}{y+1}\right)^{\sigma_y} < \infty.$$

So, $\beta \in (Ces(\nabla, \sigma))_{\top}$. □

3. Operator ideals of type-$(Ces(\nabla_{\mho}, \sigma))_{\top}$ spaces

In this section, we discuss the pre-modularity of $(Ces(\nabla_{\mho}, \sigma))_{\top}$ under the function $\top$, where $\top(\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}|\beta_x||}{y+1}\right)^{\sigma_y}$, for all $\beta \in (Ces(\nabla_{\mho}, \sigma))_{\top}$. The topological and geometric structures of the class $\mathfrak{I}^s_{(Ces(\nabla_{\mho},\sigma))_{\top}}$, and the class $\left(\mathfrak{I}^s_{(Ces(\nabla_{\mho},\sigma))_{\top}}\right)^{\rho}$ are presented.

Indicate the linear space of sequences by $\mathcal{D}$, $e_y = (0, 0, \ldots, 1, 0, 0, \ldots)$, where 1 lies at the y^{th} coordinate, $[x]$ is the integral part of x, $\mathfrak{C}^{\mathbb{I}^+}$ is the space of all sequences of complex numbers, and F is the space of all sequences with finite non-zero coordinates.

Definition 3.1 ([3]). The space $\mathcal{D}$ is said to be a private sequence space ($\mathfrak{pss}$), if the following conditions are verified:

(1) $e_y \in \mathcal{D}$, for $y \in \mathbb{I}^+$;
(2) $\mathcal{D}$ is solid, i.e., if $a = (a_y) \in \mathfrak{C}^{\mathbb{I}^+}$, where $|b| = (|b_y|) \in \mathcal{D}$ and $|a_y| \leqslant |b_y|$, with $y \in \mathbb{I}^+$, then $|a| \in \mathcal{D}$;
(3) let $(|a_y|)_{y=0}^{\infty} \in \mathcal{D}$, then $\left(\left|a_{[\frac{y}{2}]}\right|\right)_{y=0}^{\infty} \in \mathcal{D}$.

Definition 3.2 ([3]). A subspace of the $\mathfrak{pss}$-$\mathcal{D}$ is called a pre-modular $\mathfrak{pss}$, if there exists a mapping $\top : \mathcal{D} \to [0,\infty)$ that verifies the following conditions:

(i) if $a \in \mathcal{D}$, $a = \theta \Longleftrightarrow \top(|a|) = 0$, and $\top(a) \geqslant 0$, with θ is the zero vector of $\mathcal{D}$;
(ii) assume $a \in \mathcal{D}$ and $\varpi \in \mathfrak{C}$, we have $E_0 \geqslant 1$ with $\top(\varpi a) \leqslant |\varpi| E_0 \top(a)$;
(iii) $\top(a+b) \leqslant G_0(\top(a) + \top(b))$ verifies for some $G_0 \geqslant 1$, for all $a, b \in \mathcal{D}$;
(iv) suppose $y \in \mathbb{I}^+$, $|a_y| \leqslant |b_y|$, one has $\top((|a_y|)) \leqslant \top((|b_y|))$;
(v) the inequality, $\top((|a_y|)) \leqslant \top((|a_{[\frac{y}{2}]}|)) \leqslant D_0\top((|a_y|))$ holds, for $D_0 \geqslant 1$;
(vi) $\overline{F} = \mathcal{D}_{\top}$;
(vii) we have $\mu > 0$ with $\top(a, 0, 0, 0, \ldots) \geqslant \mu |a| \top(1, 0, 0, 0, \ldots)$, where $a \in \mathfrak{C}$.

Definition 3.3 ([3])**.** If $\top$ holds the conditions (i)-(iii) of Definition 3.2, then the $\mathfrak{pss}$ $\mathcal{D}_\top$ is called a pre-quasi normed $\mathfrak{pss}$. When the space $\mathcal{D}$ is complete with $\top$, then $\mathcal{D}_\top$ is said to be a pre-quasi Banach $\mathfrak{pss}$.

Theorem 3.4 ([3])**.** *Every pre-modular* $\mathfrak{pss}$ $\mathcal{D}_\top$ *is a pre-quasi normed* $\mathfrak{pss}$.

We mark the space of all monotonic increasing sequences of positive reals by $\mathfrak{I}_\nearrow$.

Theorem 3.5.

(f1) *If* $(\sigma_y) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ *with* $\sigma_0 > 1$.
(f2) *The inequality* $|\mho(y) - y + 1| \geqslant 1$ *holds, for all* $y \in \mathbb{I}^+$.
(f3) *Suppose* $|\lambda_y| \leqslant |\beta_y|$, *with* $y \in \mathbb{I}^+$, *then* $|\nabla_\mho|\lambda_y|| \leqslant |\nabla_\mho|\beta_y||$.

Then the space $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ *is a* $\mathfrak{pss}$.

Proof.

(1-i) Suppose $w, u \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$. We have

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x+u_x||}{y+1}\right)^{\sigma_y} \leqslant 2^{\hbar-1}\left(\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|u_x||}{y+1}\right)^{\sigma_y}\right)<\infty,$$

then, $w + u \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$.

(1-ii) Assume $\varpi \in \mathfrak{C}$, $w \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ and since $(\sigma_x) \in \mathfrak{I}_\nearrow \cap \ell_\infty$, one obtains

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\varpi w_x||}{y+1}\right)^{\sigma_y} \leqslant \sup_y |\varpi|^{\sigma_y}\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}<\infty.$$

Hence, $\varpi w \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$. By Parts (1-i) and (1-ii), one has $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ is a linear space. Since $(\sigma_y) \in \mathfrak{I}_\nearrow \cap \ell_\infty$, $\sigma_0 > 1$, $e_y \in (\mathrm{Ces}(\sigma))_\top$ with $y \in \mathbb{I}^+$, and

$$(\mathrm{Ces}(\sigma))_\top \subsetneqq (\mathrm{Ces}(\nabla, \sigma))_\top \subsetneqq (\mathrm{Ces}(\nabla_\mho, \sigma))_\top\,.$$

So, $e_y \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$, for every $y \in \mathbb{I}^+$.

(2) Assume $|w_y| \leqslant |u_y|$, with $y \in \mathbb{I}^+$ and $|u| \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$. We get

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|u_x||}{y+1}\right)^{\sigma_y}<\infty,$$

hence $|w| \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$.

(3) Suppose $(|w_y|) \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$, with $(\sigma_y) \in \mathfrak{I}_\nearrow \cap \ell_\infty$, one obtains

$$\begin{aligned}
&\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_\mho|w_{[\frac{x}{2}]}|\right|}{y+1}\right)^{\sigma_y}\\
&=\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{2y}\left|\nabla_\mho|w_{[\frac{x}{2}]}|\right|}{2y+1}\right)^{\sigma_{2y}}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{2y+1}\left|\nabla_\mho|w_{[\frac{x}{2}]}|\right|}{2y+2}\right)^{\sigma_{2y+1}}\\
&\leqslant \sum_{y=0}^{\infty}\left(\frac{|\nabla_\mho|w_y|+\sum_{x=0}^{y}2\,|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}2\,|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}\\
&\leqslant 2^{\hbar-1}\left(\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}2\,|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}\right)+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}2\,|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}\\
&\leqslant (2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar})\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|w_x||}{y+1}\right)^{\sigma_y}<\infty,
\end{aligned}$$

then $(|w_{[\frac{y}{2}]}|) \in (\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$. □

Theorem 3.6. *If the conditions of Theorem 3.5 are satisfied, then the space* $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ *is a pre-modular* $\mathfrak{pss}$.

Proof.

(i) It is clear that, $\top(w) \geqslant 0$ and $\top(|w|) = 0 \Leftrightarrow w = \theta$.

(ii) We have $E_0 = \max\left\{1, \sup_x |\varpi|^{\sigma_x - 1}\right\} \geqslant 1$ with $\top(\varpi w) \leqslant E_0|\varpi|\top(w)$, for all $w \in \mathrm{Ces}(\nabla_{\mho},\sigma)$ and $\varpi \in \mathfrak{C}$.

(iii) The inequality $\top(w+u) \leqslant 2^{\hbar-1}(\top(w)+\top(u))$ holds, with $w, u \in \mathrm{Ces}(\nabla_{\mho},\sigma)$.

(iv) It follows from the proof of part (2) of Theorem 3.5.

(v) It follows from the proof of part (3) of Theorem 3.5, that $D_0 = 2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar} \geqslant 1$.

(vi) Obviously, $\overline{F} = \mathrm{Ces}(\nabla_{\mho},\sigma)$.

(vii) There are $0 < \mu \leqslant \sup_x |w|^{\sigma_x - 1}$ with $\top(w,0,0,0,\dots) \geqslant \mu|w|\top(1,0,0,0,\dots)$, for all $w \neq 0$ and $\mu > 0$, if $w = 0$. □

Theorem 3.7. *Assume the conditions of Theorem 3.5 are satisfied, then the space* $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ *is a pre-quasi Banach* $\mathfrak{pss}$.

Proof. By Theorem 3.6, the space $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-modular $\mathfrak{pss}$. From Theorem 3.4, the space $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-quasi normed $\mathfrak{pss}$. To prove that $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-quasi Banach $\mathfrak{pss}$, let $\beta^k = (\beta^k_y)_{y=0}^{\infty}$ be a Cauchy sequence in $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, we have for every $\varepsilon \in (0,1)$, that $k_0 \in \mathbb{I}^+$ with $k, l \geqslant k_0$, we get

$$\top(\beta^k - \beta^l) = \sum_{y=0}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_{\mho}|\beta^k_x - \beta^l_x||}{y+1} \right)^{\sigma_y} < \varepsilon^{\hbar}.$$

Then, for $k, l \geqslant k_0$ and $y \in \mathbb{I}^+$, one has $\left|\nabla_{\mho}|\beta^k_y - \beta^l_y|\right| < \varepsilon$. Hence, $(\nabla_{\mho}|\beta^l_y|)$ is a Cauchy sequence in $\mathfrak{C}$, for constant $y \in \mathbb{I}^+$, which implies $\lim_{l\to\infty} \nabla_{\mho}|\beta^l_y| = \nabla_{\mho}|\beta^0_y|$, for constant $y \in \mathbb{I}^+$. Hence, $\top(\beta^k - \beta^0) < \varepsilon^{\hbar}$, for every $k \geqslant k_0$. To prove $\beta^0 \in (\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, we obtain $\top(\beta^0) \leqslant 2^{\hbar-1}(\top(\beta^k - \beta^0) + \top(\beta^k)) < \infty$, then $\beta^0 \in (\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, which yields that $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-quasi Banach $\mathfrak{pss}$. □

We recall here the basic concepts of operator ideals.

Definition 3.8 ([8])**.** A class $\mathbb{E} \subseteq \mathcal{J}$ is called an operator ideal when every element $\mathbb{E}(\mathcal{G},\mathcal{Q}) = \mathbb{E} \cap \mathcal{J}(\mathcal{G},\mathcal{Q})$ holds the following setups:

(i) $I_{\Lambda} \in \mathbb{E}$, if Λ marks a Banach space of one dimension.

(ii) $\mathbb{E}(\mathcal{G},\mathcal{Q})$ is a linear space on $\mathfrak{C}$.

(iii) Assume $V_1 \in \mathcal{J}(\mathcal{G}_0,\mathcal{G})$, $V_2 \in \mathbb{E}(\mathcal{G},\mathcal{Q})$ and $V_3 \in \mathcal{J}(\mathcal{Q},\mathcal{Q}_0)$, then $V_3V_2V_1 \in \mathbb{E}(\mathcal{G}_0,\mathcal{Q}_0)$, where $\mathcal{G}_0$ and $\mathcal{Q}_0$ are normed spaces.

Definition 3.9 ([5])**.** A mapping $\Omega : \mathbb{E} \to [0,\infty)$ is said to be a pre-quasi norm on the mapping ideal $\mathbb{E}$ if it holds the following settings:

(1) if $V \in \mathbb{E}(\mathcal{G},\mathcal{Q})$, $\Omega(V) \geqslant 0$ and $\Omega(V) = 0 \Longleftrightarrow V = 0$;

(2) there are $E_0 \geqslant 1$ so that $\Omega(\zeta V) \leqslant E_0|\zeta|\Omega(V)$, with $V \in \mathbb{E}(\mathcal{G},\mathcal{Q})$ and $\zeta \in \mathfrak{C}$;

(3) there are $G_0 \geqslant 1$ so that $\Omega(Z_1 + V_2) \leqslant G_0[\Omega(V_1) + \Omega(V_2)]$, for all $V_1, V_2 \in \mathbb{E}(\mathcal{G},\mathcal{Q})$;

(4) there are $D_0 \geqslant 1$, if $V_1 \in \mathcal{J}(\mathcal{G}_0,\mathcal{G})$, $V_2 \in \mathbb{E}(\mathcal{G},\mathcal{Q})$ and $V_3 \in \mathcal{J}(\mathcal{Q},\mathcal{Q}_0)$, then

$$\Omega(V_3V_2V_1) \leqslant D_0 \,\|V_3\|\Omega(V_2)\,\|V_1\|.$$

Theorem 3.10 ([3]). *If $(\mathcal{D})_{\top}$ is a pre-modular $\mathfrak{pss}$, then the function Ω is a pre-quasi norm on $\mathcal{I}^{s}_{(\mathcal{D})_{\top}}$, where $\Omega(V) = \top(s_y(V))_{y=0}^{\infty}$, for every $V \in \mathcal{I}^{s}_{(\mathcal{D})_{\top}}(\mathcal{G}, \mathcal{Q})$.*

Theorem 3.11 ([5]). *Every quasi norm on $\mathbb{E}$ is a pre-quasi norm on the ideal $\mathbb{E}$.*

Definition 3.12 ([16]). A Banach space $\mathcal{Y}$ is called simple, if the space $\mathcal{I}(\mathcal{Y})$ contains a unique non-trivial closed ideal.

Theorem 3.13 ([16]). *If $\mathcal{Y}$ is a Banach space with $\dim(\mathcal{Y}) = \infty$, then*

$$\mathfrak{I}(\mathcal{Y}) \subsetneqq \mathfrak{P}(\mathcal{Y}) \subsetneqq \mathfrak{T}(\mathcal{Y}) \subsetneqq \mathcal{I}(\mathcal{Y}).$$

Theorem 3.14 ([2]). *Assume that R is the set of real numbers. Suppose that s-type $\mathcal{D}_{\top} := \Big\{\lambda = (s_x(A)) \in R^{\mathbb{I}^+} : A \in \mathcal{I}(\mathcal{G}, \mathcal{Q}) \text{ and } \top(\lambda) < \infty\Big\}$. If $\mathcal{I}^{s}_{\mathcal{D}_{\top}}$ is a mapping ideal, then one has the following.*

1. *$F \subset$ s-type $\mathcal{D}_{\top}$.*
2. *Suppose that $(s_y(A_1))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$ and $(s_y(A_2))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$, then $(s_y(A_1 + A_2))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$.*
3. *Assume that $\varepsilon \in \mathfrak{C}$ and $(s_y(A))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$, then $|\varepsilon|\,(s_y(A))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$.*
4. *The sequence space $\mathcal{D}_{\top}$ is solid, i.e., when $(s_y(B))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$ and $s_y(A) \leqslant s_y(B)$, for all $y \in \mathbb{I}^+$ and $A, B \in \mathcal{I}(\mathcal{G}, \mathcal{Q})$, then $(s_y(A))_{y=0}^{\infty} \in$ s-type $\mathcal{D}_{\top}$.*

From Theorem 3.14, we get the following properties of the s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$.

Theorem 3.15. *Suppose that s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top} := \Big\{\lambda = (s_y(A)) \in R^{\mathbb{I}^+} : A \in \mathcal{I}(\mathcal{G}, \mathcal{Q}) \text{ and } \top(\lambda) < \infty\Big\}$. When $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}$ is a mapping ideal, then the following conditions are satisfied.*

1. *We have s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top} \supset F$.*
2. *Assume that $(s_y(A_1))_{y=0}^{\infty} \in$ s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$ and $(s_y(A_2))_{y=0}^{\infty} \in$ s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$, hence $(s_y(A_1 + A_2))_{y=0}^{\infty} \in$ s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$.*
3. *For all $g \in \mathfrak{C}$ and $(s_y(A))_{y=0}^{\infty} \in$ s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$, then $|g|\,(s_y(A))_{y=0}^{\infty} \in$ s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$.*
4. *The s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$ is solid.*

Theorem 3.16. *If the conditions (f1) and (f2) of Theorem 3.5 are satisfied, we have $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}$ is not operator ideal.*

Proof. Suppose we choose $\sigma_x = 2$, $\top(x) = x$, for every $x \in \mathbb{I}^+$, $w = (1, 1, 1, \ldots)$ and $\nu = (1, 0, 1, 0, \ldots)$. Evidently, $|\nu_x| \leqslant |w_x|$, for every $x \in \mathbb{I}^+$ and $w \in$s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$. But $\nu \notin$s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$. Then the s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$ is not solid. By Theorem 3.14, we have that $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}$ is not a mapping ideal. □

Theorem 3.17 ([3]). *Assume $\mathcal{D}$ is a $\mathfrak{pss}$, then $\mathcal{I}^{s}_{\mathcal{D}}$ is an operator ideal.*

According to Theorem 3.17, we deduce the following Theorem.

Theorem 3.18. *If the conditions of Theorem 3.5 are satisfied, then $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}$ is an operator ideal.*

In this part, we present the sufficient conditions (not necessary) on $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$ so that $\overline{\mathfrak{I}} = \mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}$. This implies an answer about the non-linearity of s-type $(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}$ spaces (see Rhoades [18]).

Theorem 3.19. $\mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho}, \sigma))_{\top}}(\mathcal{G}, \mathcal{Q}) = \overline{\mathfrak{I}(\mathcal{G}, \mathcal{Q})}$, *if the conditions of Theorem 3.5 are satisfied. But the converse is not necessarily true.*

Proof. To prove that $\overline{\mathfrak{J}(\mathcal{G},\mathcal{Q})} \subseteq \mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$, since $e_y \in (\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, for all $y \in \mathbb{I}^+$ and $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a linear space, assume that $A \in \mathfrak{J}(\mathcal{G},\mathcal{Q})$, we have $(\alpha_y(A))_{y=0}^{\infty} \in F$. To show that $\mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q}) \subseteq \overline{\mathfrak{J}(\mathcal{G},\mathcal{Q})}$, suppose $A \in \mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$, one gets $(\alpha_x(A))_{x=0}^{\infty} \in (\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$. As $\top(\alpha_y(A))_{y=0}^{\infty} < \infty$, suppose $\kappa \in (0,1)$, one has $y_0 \in \mathbb{I}^+ \setminus \{0\}$ such that $\top((\alpha_y(A))_{y=y_0}^{\infty}) < \frac{\kappa}{2^{2\hbar+3}\xi d}$, for some $b \geqslant 1$, where $\xi = \max\left\{1, \sum_{y=y_0}^{\infty}\left(\frac{1}{y+1}\right)^{\sigma_l}\right\}$. Since $\alpha_y(A)$ is decreasing, we have

$$\sum_{x=x_0+1}^{2x_0}(|\nabla_{\mho}\alpha_{2x_0}(A)|)^{\sigma_x} \leqslant \sum_{x=x_0+1}^{2x_0}\left(\frac{\sum_{y=0}^{x}|\nabla_{\mho}\alpha_y(A)|}{x+1}\right)^{\sigma_x} \leqslant \sum_{x=x_0}^{\infty}\left(\frac{\sum_{y=0}^{x}|\nabla_{\mho}\alpha_y(A)|}{x+1}\right)^{\sigma_x} < \frac{\kappa}{2^{2\hbar+3}\xi b}.$$

Therefore, we have $B \in \mathfrak{J}_{2x_0}(\mathcal{G},\mathcal{Q})$ such that $\mathrm{rank}(B) \leqslant 2x_0$ and

$$\sum_{x=2x_0+1}^{3x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} \leqslant \sum_{x=x_0+1}^{2x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} < \frac{\kappa}{2^{2\hbar+3}\xi b},$$

as $(\sigma_x) \in \mathfrak{I}_{\nearrow} \cap \ell_{\infty}$, we can take

$$\sum_{x=0}^{x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} < \frac{\kappa}{2^{2\hbar+3}\xi b}.$$

By using inequalities (1)-(4), we have

$$\begin{aligned}
d(A,B) &= \top\left(\alpha_x(A-B)\right)_{x=0}^{\infty}\\
&= \sum_{x=0}^{3x_0-1}\left(\frac{\sum_{y=0}^{x}|\nabla_{\mho}\alpha_y(A-B)|}{x+1}\right)^{\sigma_x} + \sum_{x=3x_0}^{\infty}\left(\frac{\sum_{y=0}^{x}|\nabla_{\mho}\alpha_y(A-B)|}{x+1}\right)^{\sigma_x}\\
&\leqslant \sum_{x=0}^{3x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} + \sum_{x=x_0}^{\infty}\left(\frac{\sum_{y=0}^{x+2x_0}|\nabla_{\mho}\alpha_y(A-B)|}{x+2x_0+1}\right)^{\sigma_{x+2x_0}}\\
&\leqslant \sum_{x=0}^{3x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} + \sum_{x=x_0}^{\infty}\left(\frac{\sum_{y=0}^{2x_0-1}|\nabla_{\mho}\alpha_y(A-B)| + \sum_{y=2x_0}^{x+2x_0}|\nabla_{\mho}\alpha_y(A-B)|}{x+1}\right)^{\sigma_x}\\
&\leqslant 3\sum_{x=0}^{x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} + \sum_{x=x_0}^{\infty}\left(\frac{\sum_{y=0}^{2x_0-1}|\nabla_{\mho}\alpha_y(A-B)| + \sum_{y=0}^{x}|\nabla_{\mho}\alpha_{y+2x_0}(A-B)|}{x+1}\right)^{\sigma_x}\\
&\leqslant 3\sum_{x=0}^{x_0}\left(\Big|\nabla_{\mho}\|A-B\|\Big|\right)^{\sigma_x} + 2^{2\hbar}\xi\sup_x\left(\sum_{y=0}^{x_0}|\nabla_{\mho}\|A-B\||\right)^{\sigma_x} + 2^{\hbar}\sum_{x=x_0}^{\infty}\left(\frac{\sum_{y=0}^{x}|\nabla_{\mho}\alpha_y(A)|}{x+1}\right)^{\sigma_x} < \kappa.
\end{aligned}$$

Contrarily, we have a counterexample as $I_4 \in \mathcal{I}^{\alpha}_{(\mathrm{Ces}(\nabla,(0,2,2,\ldots)))_{\top}}(\mathcal{G},\mathcal{Q})$, but $\sigma_0 > 1$ is not satisfied. □

We present here the following question, for which conditions on $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, are $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$ complete and closed?

Theorem 3.20. *The subclass $\left(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}, \Omega\right)$ is a pre-quasi Banach ideal, where $\Omega(V) = \top\left((s_y(V))_{y=0}^{\infty}\right)$, if the conditions of Theorem 3.5 are verified.*

Proof. As $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-modular $\mathfrak{pss}$, hence from Theorem 3.10, Ω is a pre-quasi norm on $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$. Suppose $(U_f)_{f\in\mathbb{I}^+}$ is a Cauchy sequence in $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$. As $\mathcal{I}(\mathcal{G},\mathcal{Q}) \supseteq \mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$, we have

$$\Omega(U_f - U_g) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(U_f-U_g)|}{y+1}\right)^{\sigma_y} \geqslant \left(\Big|\nabla_{\mho}\,\|U_f - U_g\|\Big|\right)^{\sigma_0},$$

so $(\mathcal{U}_g)_{g\in\mathbb{I}^+}$ is a Cauchy sequence in $\mathfrak{I}(\mathcal{G},\mathcal{Q})$. Since $\mathfrak{I}(\mathcal{G},\mathcal{Q})$ is a Banach space, then there is $\mathcal{U}\in\mathfrak{I}(\mathcal{G},\mathcal{Q})$ with $\lim_{g\to\infty}\|\mathcal{U}_g-\mathcal{U}\|=0$. As $(s_y(\mathcal{U}_g))_{y=0}^{\infty}\in(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, for all $g\in\mathbb{I}^+$. From Definition 3.2, conditions (ii), (iii) and (v), one gets

$$\begin{aligned}\Omega(\mathcal{U})&=\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma_y}\\&\leqslant 2^{\hbar-1}\left[\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_{\mho}s_{[\frac{x}{2}]}(\mathcal{U}-\mathcal{U}_g)\right|}{y+1}\right)^{\sigma_y}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_{\mho}s_{[\frac{x}{2}]}(\mathcal{U}_g)\right|}{y+1}\right)^{\sigma_y}\right]\\&\leqslant 2^{\hbar-1}\sum_{y=0}^{\infty}\left(\left|\nabla_{\mho}\|\mathcal{U}-\mathcal{U}_g\|\right|\right)^{\sigma_y}+2^{\hbar-1}D_0\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(\mathcal{U}_g)\|}{y+1}\right)^{\sigma_y}<\infty.\end{aligned}$$

Hence, $(s_f(\mathcal{U}))_{f=0}^{\infty}\in(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, then $\mathcal{U}\in\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$. □

Theorem 3.21. *If $\mathcal{G}$ and $\mathcal{Q}$ are normed spaces, and having Theorem 3.5 confirmed, then $\left(\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}},\Omega\right)$ is a pre-quasi closed ideal, where $\Omega(\mathcal{U})=\top\left((s_f(\mathcal{U}))_{f=0}^{\infty}\right)$.*

Proof. As $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ is a pre-modular $\mathfrak{pss}$, from Theorem 3.10, then Ω is a pre-quasi norm on $\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$. Assume $\mathcal{U}_g\in\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$, for all $g\in\mathbb{I}^+$ and $\lim_{g\to\infty}\Omega(\mathcal{U}_g-\mathcal{U})=0$. Since $\mathfrak{I}(\mathcal{G},\mathcal{Q})\supseteq\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$, we have

$$\Omega(\mathcal{U}-\mathcal{U}_g)=\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(\mathcal{U}-\mathcal{U}_g)|}{y+1}\right)^{\sigma_y}\geqslant\left(\left|\nabla_{\mho}\|\mathcal{U}-\mathcal{U}_g\|\right|\right)^{\sigma_0},$$

so $(\mathcal{U}_g)_{g\in\mathbb{I}^+}$ is a convergent sequence in $\mathfrak{I}(\mathcal{G},\mathcal{Q})$. Since $(s_y(\mathcal{U}_g))_{y=0}^{\infty}\in(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, for all $g\in\mathbb{I}^+$. By Definition 3.2, conditions (ii), (iii) and (v), one obtains

$$\begin{aligned}\Omega(\mathcal{U})&=\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma_y}\\&\leqslant 2^{\hbar-1}\left[\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_{\mho}s_{[\frac{x}{2}]}(\mathcal{U}-\mathcal{U}_g)\right|}{y+1}\right)^{\sigma_y}+\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_{\mho}s_{[\frac{x}{2}]}(\mathcal{U}_g)\right|}{y+1}\right)^{\sigma_y}\right]\\&\leqslant 2^{\hbar-1}\sum_{y=0}^{\infty}\left(\left|\nabla_{\mho}\|\mathcal{U}-\mathcal{U}_g\|\right|\right)^{\sigma_y}+2^{\hbar-1}D_0\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho}s_x(\mathcal{U}_g)\|}{y+1}\right)^{\sigma_y}<\infty.\end{aligned}$$

Then $(s_y(\mathcal{U}))_{y=0}^{\infty}\in(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$, therefore $\mathcal{U}\in\mathfrak{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}(\mathcal{G},\mathcal{Q})$. □

We introduce in this part the sufficient conditions on $(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}$ so that $\mathfrak{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$ is strictly contained for different $\top$ and powers, and $\mathfrak{I}^{\alpha}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$ is minimum.

Theorem 3.22. *If $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{G})=\dim(\mathcal{Q})=\infty$, and the conditions of Theorem 3.5 are satisfied with $\mho_2(y)\geqslant\mho_1(y)$ and $1<\sigma_y^{(1)}<\sigma_y^{(2)}$, for every $y\in\mathbb{I}^+$, then*

$$\mathfrak{I}^{s}_{\left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)}))\right)_{\top}}(\mathcal{G},\mathcal{Q})\subsetneqq\mathfrak{I}^{s}_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)}))\right)_{\top}}(\mathcal{G},\mathcal{Q})\subsetneqq\mathfrak{I}(\mathcal{G},\mathcal{Q}).$$

Proof. Assume that $\mathcal{U} \in \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma^{(1)}_y))\right)_\top}(\mathcal{G},\mathcal{Q})$, then $(s_y(\mathcal{U})) \in \left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma^{(1)}_y))\right)_\top$. We have

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_2}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma^{(2)}_y} < \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_1}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma^{(1)}_y} < \infty,$$

then $\mathcal{U} \in \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma^{(2)}_y))\right)_\top}(\mathcal{G},\mathcal{Q})$. Next, if we take $(s_y(\mathcal{U}))_{y=0}^\infty$ with $\sum_{x=0}^{y}\left|\nabla_{\mho_1}s_x(\mathcal{U})\right| = \frac{y+1}{\sigma^{(1)}_y\sqrt{y+1}}$, we get $\mathcal{U} \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ so that

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_1}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma^{(1)}_y} = \sum_{y=0}^{\infty}\frac{1}{y+1} = \infty,$$

and

$$\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_2}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma^{(2)}_y} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_1}s_x(\mathcal{U})\|}{y+1}\right)^{\sigma^{(2)}_y} = \sum_{y=0}^{\infty}\left(\frac{1}{y+1}\right)^{\frac{\sigma^{(2)}_y}{\sigma^{(1)}_y}} < \infty.$$

Hence $\mathcal{U} \notin \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma^{(1)}_y))\right)_\top}(\mathcal{G},\mathcal{Q})$ and $\mathcal{U} \in \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma^{(2)}_y))\right)_\top}(\mathcal{G},\mathcal{Q})$. Evidently, $\mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma^{(2)}_y))\right)_\top}(\mathcal{G},\mathcal{Q}) \subset \mathcal{I}(\mathcal{G},\mathcal{Q})$. Next, if we put $(s_y(\mathcal{U}))_{y=0}^\infty$ with $\sum_{x=0}^{y}\left|\nabla_{\mho_2}s_x(\mathcal{U})\right| = \frac{y+1}{\sigma^{(2)}_y\sqrt{y+1}}$, then we obtain $\mathcal{U} \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ so that $\mathcal{U} \notin \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma^{(2)}_y))\right)_\top}(\mathcal{G},\mathcal{Q})$. □

Theorem 3.23. *If $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, *and the conditions of Theorem 3.5 are satisfied, hence $\mathcal{I}^\alpha_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}$ is minimum.*

Proof. Let the sufficient conditions be verified. Hence $(\mathcal{I}^\alpha_{\mathrm{Ces}(\nabla_\mho,\sigma)},\Omega)$, where

$$\Omega(Z) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho\alpha_x(Z)\|}{y+1}\right)^{\sigma_y},$$

is a pre-quasi Banach ideal. Assume that $\mathcal{I}^\alpha_{\mathrm{Ces}(\nabla_\mho,\sigma)}(\mathcal{G},\mathcal{Q}) = \mathcal{I}(\mathcal{G},\mathcal{Q})$, one has $\sigma > 0$ with $\Omega(Z) \leqslant \sigma\|Z\|$, for all $Z \in \mathcal{I}(\mathcal{G},\mathcal{Q})$. From Dvoretzky's theorem [15], for all $b \in \mathbb{I}^+$, one gets the quotient spaces $\mathcal{G}/Y_b$ and subspaces M_b of $\mathcal{Q}$, which can be transformed onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leqslant 2$ and $\|X_b\|\|X_b^{-1}\| \leqslant 2$. Presume that I_b is the identity operator on ℓ_2^b, T_b is the quotient operator from $\mathcal{G}$ onto $\mathcal{G}/Y_b$ and J_b is the natural embedding operator from M_b into $\mathcal{Q}$. If m_z is the Bernstein numbers [13], one gets

$$\begin{aligned}1 = m_z(I_b) = m_z(X_bX_b^{-1}I_bV_bV_b^{-1}) &\leqslant \|X_b\|m_z(X_b^{-1}I_bV_b)\|V_b^{-1}\|\\ &= \|X_b\|m_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\|\\ &\leqslant \|X_b\|d_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\|\\ &= \|X_b\|d_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\| \leqslant \|X_b\|\alpha_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\|,\end{aligned}$$

for $0 \leqslant x \leqslant b$. Assume that l is the greatest integer with $\mho(l) = 0$. Hence we obtain

$$\begin{aligned}&\sum_{x=0}^{y}\frac{x+1}{|1-l|} \leqslant \|X_b\|\sum_{x=0}^{y}\left|\nabla_\mho\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|\|V_b^{-1}\|\\ &\Rightarrow \left(\frac{\sum_{x=0}^{y}\frac{x+1}{|1-l|}}{y+1}\right)^{\sigma_y} \leqslant (\|X_b\|\|V_b^{-1}\|)^{\sigma_y}\left(\frac{\sum_{x=0}^{y}\left|\nabla_\mho\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|}{y+1}\right)^{\sigma_y}.\end{aligned}$$

Therefore, for some $\rho \geqslant 1$, we have

$$\begin{aligned}\sum_{y=0}^{b}\left(\frac{\sum_{x=0}^{y}\frac{x+1}{|1-l|}}{y+1}\right)^{\sigma_y} &\leqslant \rho\|X_b\|\|V_b^{-1}\|\sum_{y=0}^{b}\left(\frac{\sum_{x=0}^{y}\left|\nabla_{\mho}\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|}{y+1}\right)^{\sigma_y}\\ &\Rightarrow \sum_{y=0}^{b}\left(\frac{\sum_{x=0}^{y}\frac{x+1}{|1-l|}}{y+1}\right)^{\sigma_y} \leqslant \rho\|X_b\|\|V_b^{-1}\|\Omega(J_bX_b^{-1}I_bV_bT_b)\\ &\Rightarrow \sum_{y=0}^{b}\left(\frac{\sum_{x=0}^{y}\frac{x+1}{|1-l|}}{y+1}\right)^{\sigma_y} \leqslant \rho\sigma\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}I_bV_bT_b\|\\ &\Rightarrow \sum_{y=0}^{b}\left(\frac{\sum_{x=0}^{y}\frac{x+1}{|1-l|}}{y+1}\right)^{\sigma_y} \leqslant \rho\sigma\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}\|\|I_b\|\|V_bT_b\|\\ &= \rho\sigma\|X_b\|\|V_b^{-1}\|\|X_b^{-1}\|\|I_b\|\|V_b\| \leqslant 4\rho\sigma.\end{aligned}$$

This implies a contradiction, when $b \to \infty$. Hence, $\mathcal{G}$ and $\mathcal{Q}$ both cannot be infinite dimensional when $\mathcal{I}^{\alpha}_{\mathrm{Ces}(\nabla_{\mho},\sigma)}(\mathcal{G},\mathcal{Q}) = \mathcal{I}(\mathcal{G},\mathcal{Q})$. □

Clearly, as Theorem 3.23, we can easily show the following theorem.

Theorem 3.24. *If $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, and the conditions of Theorem 3.5 are satisfied, then $\mathcal{I}^{d}_{\mathrm{Ces}(\nabla_{\mho},\sigma)}$ is minimum.*

We discuss here the conditions such that the class $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho},\sigma))_{\top}}$ is simple.

Lemma 3.25 ([16]). *Suppose $C \in \mathcal{I}(\mathcal{G},\mathcal{Q})$ and $C \notin \mathfrak{P}(\mathcal{G},\mathcal{Q})$, then $A \in \mathcal{I}(\mathcal{G})$ and $B \in \mathcal{I}(\mathcal{Q})$ with $BCXe_y = e_y$, for every $y \in \mathbb{I}^{+}$.*

Theorem 3.26. *Assume $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, and the conditions of Theorem 3.5 are satisfied with $\mho_2(y) \geqslant \mho_1(y)$ and $1 < \sigma_y^{(1)} < \sigma_y^{(2)}$, for every $y \in \mathbb{I}^{+}$, then*

$$\mathcal{I}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q}),\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big) = \mathfrak{P}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q}),\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big).$$

Proof. If $A \in \mathcal{I}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q}),\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big)$ and $A \notin \mathfrak{P}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q}),$ $\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big)$, in view of Lemma 3.25, we have $B \in \mathcal{I}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big)$ and $C \in \mathcal{I}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big)$ with $CABI_g = I_g$. Therefore, for all $g \in \mathbb{I}^{+}$, we obtain

$$\begin{aligned}\|I_g\|_{\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})} &= \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_1}s_x(I_g)|}{y+1}\right)^{\sigma_y^{(1)}}\\ &\leqslant \|CAB\|\|I_g\|_{\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q})} \leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_{\mho_2}s_x(I_g)|}{y+1}\right)^{\sigma_y^{(2)}}.\end{aligned}$$

This contradicts Theorem 3.22. Hence $A \in \mathfrak{P}\Big(\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)})))_{\top}}(\mathcal{G},\mathcal{Q}),\mathcal{I}^{s}_{(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)})))_{\top}}(\mathcal{G},\mathcal{Q})\Big)$. □

Corollary 3.27. *If $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, *and the conditions of Theorem 3.5 are verified with* $\mho_2(y) \geqslant \mho_1(y)$ *and* $1 < \sigma_y^{(1)} < \sigma_y^{(2)}$, *for every* $y \in \mathbb{I}^+$, *then*

$$\mathcal{J}\left(\mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)}))\right)_\top}(\mathcal{G},\mathcal{Q}), \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)}))\right)_\top}(\mathcal{G},\mathcal{Q})\right) = \mathfrak{T}\left(\mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_2},(\sigma_y^{(2)}))\right)_\top}(\mathcal{G},\mathcal{Q}), \mathcal{I}^s_{\left(\mathrm{Ces}(\nabla_{\mho_1},(\sigma_y^{(1)}))\right)_\top}(\mathcal{G},\mathcal{Q})\right).$$

Proof. Clearly, since $\mathfrak{P} \subset \mathfrak{T}$. □

Theorem 3.28. *Assume $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, *and the conditions of Theorem 3.5 are confirmed, then* $\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}$ *is simple.*

Proof. Suppose that the closed ideal $\mathfrak{T}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}))$ contains an operator $A \notin \mathfrak{P}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}))$. By using Lemma 3.25, there are $B, C \in \mathcal{J}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}))$ with $CABI_g = I_g$. This gives that $I_{\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q})} \in \mathfrak{T}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}))$. Hence $\mathcal{J}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q})) = \mathfrak{T}(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}))$. Therefore, $\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}$ is a simple Banach space. □

We offer here the sufficient conditions on $(\mathrm{Ces}(\nabla_\mho,\sigma))_\top$ so that the class $\mathcal{I}$ with the sequence of eigenvalues in $(\mathrm{Ces}(\nabla_\mho,\sigma))_\top$ equals $\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}$.

Theorem 3.29. *If $\mathcal{G}$ and $\mathcal{Q}$ are Banach spaces with* $\dim(\mathcal{G}) = \dim(\mathcal{Q}) = \infty$, *and the conditions of Theorem 3.5 are satisfied, and $\nabla_\mho^{-1}$ exists and is bounded linear, then*

$$\left(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}\right)^\rho(\mathcal{G},\mathcal{Q}) = \mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}).$$

Proof. Assume that $U \in \left(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}\right)^\rho(\mathcal{G},\mathcal{Q})$, then $(\rho_y(U))_{y=0}^\infty \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$ and $\|U - \rho_y(U)I\| = 0$, for every $y \in \mathbb{I}^+$. One has $U = \rho_y(U)I$, for every $y \in \mathbb{I}^+$, hence $s_y(U) = s_y(\rho_y(U)I) = |\rho_y(U)|$, for all $y \in \mathbb{I}^+$. So, $(s_y(U))_{y=0}^\infty \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$, then $U \in \mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q})$. Next, suppose that $U \in \mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q})$. Hence $(s_y(U))_{y=0}^\infty \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$. Therefore, we get

$$\sum_{y=0}^\infty (|\nabla_\mho s_y(U)|)^{\sigma_y} \leqslant \sum_{y=0}^\infty \left(\frac{\sum_{x=0}^y |\nabla_\mho s_x(U)|}{y+1}\right)^{\sigma_y} < \infty.$$

So $\lim_{y\to\infty} \nabla_\mho s_y(U) = 0$. As $\nabla_\mho^{-1}$ exists and is bounded linear, hence $\lim_{y\to\infty} s_y(U) = 0$. Suppose $\|U - s_y(U)I\|^{-1}$ exists for every $y \in \mathbb{I}^+$. Therefore, $\|U - s_y(U)I\|^{-1}$ exists and bounded for every $y \in \mathbb{I}^+$. Hence, $\lim_{y\to\infty} \|U - s_y(U)I\|^{-1} = \|U\|^{-1}$ exists and is bounded. As $\left(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}, \Omega\right)$ is a pre-quasi operator ideal, we have

$$I = UU^{-1} \in \mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}(\mathcal{G},\mathcal{Q}) \Rightarrow (s_y(I))_{y=0}^\infty \in \mathrm{Ces}(\nabla_\mho,\sigma) \Rightarrow \lim_{y\to\infty} s_y(I) = 0.$$

We have a contradiction, as $\lim_{y\to\infty} s_y(I) = 1$. Then $\|U - s_y(U)I\| = 0$, for all $y \in \mathbb{I}^+$, which gives $U \in \left(\mathcal{I}^s_{(\mathrm{Ces}(\nabla_\mho,\sigma))_\top}\right)^\rho(\mathcal{G},\mathcal{Q})$. □

4. Multiplication operators on $(\mathrm{Ces}(\nabla_\mho,\sigma))_\top$

Some geometric and topological properties of the multiplication operators defined on the space $(\mathrm{Ces}(\nabla_\mho,\sigma))_\top$ with the function $\top$ have been introduced, where $\top(\beta) = \sum_{y=0}^\infty \left(\frac{\sum_{x=0}^y |\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y}$, for every $\beta \in (\mathrm{Ces}(\nabla_\mho,\sigma))_\top$.

Definition 4.1 ([3]). Suppose $\mathcal{D}_\top$ is a pre-quasi normed $\mathfrak{pss}$ and $\wp = (\wp_y) \in \mathfrak{C}^{\mathbb{I}^+}$. The operator $M_\wp : \mathcal{D}_\top \to \mathcal{D}_\top$ is called a multiplication on $\mathcal{D}_\top$, when $M_\wp \beta = \left(\wp_y \beta_y\right) \in \mathcal{D}_\top$, with $\beta \in \mathcal{D}_\top$. If $M_\wp \in \mathcal{I}(\mathcal{D}_\top)$, then the multiplication operator is said to be generated by $\wp$.

Theorem 4.2. *If $\wp \in \mathfrak{C}^{\mathbb{I}^+}$ and the conditions of Theorem 3.5 are verified, then*

$$\wp \in \ell_\infty \Longleftrightarrow M_\wp \in \mathcal{I}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top).$$

Proof. Assume that $\wp \in \ell_\infty$. Then, we have $\xi > 0$ with $|\wp_x| \leqslant \xi$, for all $x \in \mathbb{I}^+$. If $\beta \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$, we obtain

$$\begin{aligned}\top(M_\wp\beta) = \top(\wp\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\wp_x\beta_x||}{y+1}\right)^{\sigma_y} &\leqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\xi\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y}\\ &\leqslant \sup_y \xi^{\sigma_y}\sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y} = \sup_y \xi^{\sigma_y}\top(\beta).\end{aligned}$$

Hence, $M_\wp \in \mathcal{I}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$.

Next, suppose $M_\wp \in \mathcal{I}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$ and $\wp \notin \ell_\infty$. Then for every $m \in \mathbb{I}^+$, one has $\xi \in \mathbb{I}^+$ so that $|\wp_m| > \xi$. Hence

$$\top(M_\wp e_x) = \top(\wp e_x) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\wp_x(e_m)_x||}{y+1}\right)^{\sigma_y} > \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\xi\nabla_\mho|(e_m)_x||}{y+1}\right)^{\sigma_y} > \xi^{\sigma_0}\top(e_m).$$

So, $M_\wp \notin \mathcal{I}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$. So $\wp \in \ell_\infty$. □

Theorem 4.3. *If $\wp \in \mathfrak{C}^{\mathbb{I}^+}$ and $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$ is a pre-quasi normed $\mathfrak{pss}$, then $|\wp_y| = 1$, for all $y \in \mathbb{I}^+$, if and only if $M_\wp$ is an isometry.*

Proof. Suppose the sufficient condition is satisfied. We have

$$\top(M_\wp\beta) = \top(\wp\beta) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\wp_x\beta_x||}{y+1}\right)^{\sigma_y} = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\beta_x||}{y+1}\right)^{\sigma_y},$$

for all $\beta \in (\mathrm{Ces}(\nabla_\mho, \sigma))_\top$. So, $M_\wp$ is an isometry. Assume $M_\wp$ is an isometry and $|\wp_b| < 1$, for some $b = b_0$. We have

$$\top(M_\wp e_{b_0}) = \top(\wp e_{b_0}) = \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|\wp_x(e_{b_0})_x||}{y+1}\right)^{\sigma_y} < \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}|\nabla_\mho|(e_{b_0})_x||}{y+1}\right)^{\sigma_y} = \top(e_{b_0}).$$

Also if $|\wp_{b_0}| > 1$, clearly, $\top(M_\wp e_{b_0}) > \top(e_{b_0})$. We have a contradiction for the two cases. Hence, $|\wp_b| = 1$, for every $b \in \mathbb{I}^+$. □

Theorem 4.4. *If $\wp \in \mathfrak{C}^{\mathbb{I}^+}$ and the conditions of Theorem 3.5 are verified, then $M_\wp \in \mathfrak{P}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$, if and only if $(\wp_b)_{b=0}^{\infty} \in c_0$, where c_0 is the space of null sequences of complex numbers.*

Proof. Assume that $M_\wp \in \mathfrak{P}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$, then $M_\wp \in \mathfrak{T}((\mathrm{Ces}(\nabla_\mho, \sigma))_\top)$. Suppose $\lim_{y\to\infty}\wp_y \neq 0$. Then, we have $\eta > 0$ so that $U_\eta = \{y \in \mathbb{I}^+ : |\wp_y| \geqslant \eta\} \nsubseteq \mathfrak{I}$, where $\mathfrak{I}$ is the space of all sets with finite number of elements. If $\{\xi_y\}_{y\in\mathbb{I}^+} \subset U_\eta$, then $\{e_{\xi_y} : \xi_y \in U_\eta\} \in \ell_\infty$ is an infinite set in $(\mathrm{Ces}(\nabla_\mho, \sigma))_\top$. As

$$\begin{aligned}\top(M_\wp e_{\xi_f} - M_\wp e_{\xi_g}) = \top(\wp e_{\xi_f} - \wp e_{\xi_g}) &= \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\nabla_\mho|\wp_x((e_{\xi_f})_x - (e_{\xi_g})_x)|\right|}{y+1}\right)^{\sigma_y}\\ &\geqslant \sum_{y=0}^{\infty}\left(\frac{\sum_{x=0}^{y}\left|\eta\nabla_\mho|((e_{\xi_f})_x - (e_{\xi_g})_x)|\right|}{y+1}\right)^{\sigma_y} \geqslant \inf_k \eta^{\sigma_k}\top(e_{\xi_f} - e_{\xi_g}),\end{aligned}$$

for all $\xi_f, \xi_g \in U_\eta$. So, $\{e_{\xi_g} : \xi_g \in U_\eta\} \in \ell_\infty$, which cannot have a convergent subsequence under $M_\wp$. Hence $M_\wp \notin \mathfrak{T}((Ces(\nabla_\mho, \sigma))_\top)$. This implies $M_\wp \notin \mathfrak{P}((Ces(\nabla_\mho, \sigma))_\top)$, this gives a contradiction. Therefore, $\lim_{y\to\infty} \wp_y = 0$. Next, if $\lim_{y\to\infty} \wp_y = 0$, hence for every $\eta > 0$, we have $U_\eta = \{y \in \mathbb{I}^+ : |\wp_y| \geqslant \eta\} \subset \mathfrak{I}$. Therefore, for all $\eta > 0$, we get $\dim\left(\left((Ces(\nabla_\mho, \sigma))_\top\right)_{U_\eta}\right) = \dim\left(\mathfrak{C}^{U_\eta}\right) < \infty$. So $M_\wp \in \mathfrak{I}\left(\left((Ces(\nabla_\mho, \sigma))_\top\right)_{U_\eta}\right)$. Let $\wp_f \in \mathfrak{C}^{\mathbb{I}^+}$, for every $f \in \mathbb{I}^+$, where

$$(\wp_f)_g = \begin{cases} \wp_g, & g \in U_{\frac{1}{f+1}},\\ 0, & \text{otherwise.}\end{cases}$$

Indeed, $M_{\wp_f} \in \mathfrak{I}\left(\left((Ces(\nabla_\mho, \sigma))_\top\right)_{U_{\frac{1}{f+1}}}\right)$ such as $\dim\left(\left((Ces(\nabla_\mho, \sigma))_\top\right)_{U_{\frac{1}{f+1}}}\right) < \infty$, for every $f \in \mathbb{I}^+$. As $(\sigma_y) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ with $\sigma_0 > 1$, we have

$$\begin{aligned}
\top((M_\wp - M_{\wp_f})\beta) &= \top\left(\left((\wp_g - (\wp_f)_g)\beta_g\right)_{g=0}^\infty\right)\\
&= \sum_{g=0}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|(\wp_x - (\wp_f)_x)\beta_x||}{g+1}\right)^{\sigma_g}\\
&= \sum_{g=0, g\in U_{\frac{1}{f+1}}}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|(\wp_x - (\wp_f)_x)\beta_x||}{g+1}\right)^{\sigma_g}\\
&\quad + \sum_{g=0, g\notin U_{\frac{1}{f+1}}}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|(\wp_x - (\wp_f)_x)\beta_x||}{g+1}\right)^{\sigma_g}\\
&= \sum_{g=0, g\notin U_{\frac{1}{f+1}}}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|\wp_x\beta_x||}{g+1}\right)^{\sigma_g}\\
&\leqslant \frac{1}{(f+1)^{\sigma_0}} \sum_{g=0, g\notin U_{\frac{1}{f+1}}}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|\beta_x||}{g+1}\right)^{\sigma_g}\\
&< \frac{1}{(f+1)^{\sigma_0}} \sum_{g=0}^\infty \left(\frac{\sum_{x=0}^g |\nabla_\mho|\beta_x||}{g+1}\right)^{\sigma_g} = \frac{1}{(f+1)^{\sigma_0}}\top(\beta).
\end{aligned}$$

So, $\|M_\wp - M_{\wp_f}\| \leqslant \frac{1}{(f+1)^{\sigma_0}}$. This implies $M_\wp$ is a limit of finite rank mappings. Hence, $M_\wp \in \mathfrak{P}((Ces(\nabla_\mho, \sigma))_\top)$. □

Theorem 4.5. *If $\wp \in \mathfrak{C}^{\mathbb{I}^+}$ and the settings of Theorem 3.5 are confirmed, then $M_\wp \in \mathfrak{T}((Ces(\nabla_\mho, \sigma))_\top)$, if and only if $(\wp_y)_{y=0}^\infty \in c_0$.*

Proof. Evidently, since $\mathfrak{P}((Ces(\nabla_\mho, \sigma))_\top) \subsetneqq \mathfrak{T}((Ces(\nabla_\mho, \sigma))_\top)$. □

Corollary 4.6. *Suppose that the setups of Theorem 3.5 are satisfied, then $\mathfrak{T}((Ces(\nabla_\mho, \sigma))_\top) \subsetneqq \mathcal{I}((Ces(\nabla_\mho, \sigma))_\top)$.*

Proof. As the multiplication mapping I on $(Ces(\nabla_\mho, \sigma))_\top$ is generated by $\wp = (1, 1, \ldots)$, this implies $I \notin \mathfrak{T}((Ces(\nabla_\mho, \sigma))_\top)$ and $I \in \mathcal{I}((Ces(\nabla_\mho, \sigma))_\top)$. □

Theorem 4.7. *Suppose that $M_\wp \in \mathcal{I}((Ces(\nabla_\mho, \sigma))_\top)$, where the space $(Ces(\nabla_\mho, \sigma))_\top$ is a pre-quasi Banach $\mathfrak{pss}$. Then there exist $r > 0$ and $t > 0$ so that $r < |\wp_y| < t$, with $y \in (\ker(\wp))^c$, if and only if, $\mathrm{Range}(M_\wp)$ is closed.*

Proof. Let the settings be satisfied. Then, we have $\eta > 0$ with $|\wp_y| \geqslant \eta$, for every $y \in (\ker(\wp))^c$. To show the space $Range(M_\wp)$ is closed, suppose l is a limit point of $Range(M_\wp)$. Hence $M_\wp \beta_y \in (Ces(\nabla_\mho, \sigma))_\top$, for all $y \in \mathbb{I}^+$ such that $\lim_{y\to\infty} M_\wp \beta_y = l$. Clearly, the sequence $M_\wp \beta_y$ is a Cauchy sequence. As $(\sigma_y) \in \mathfrak{I}_\nearrow \cap \ell_\infty$ with $\sigma_0 > 1$, we have

$$\begin{aligned}
\top(M_\wp \beta_f - M_\wp \beta_g) &= \sum_{y=0}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| (\wp_x(\beta_f)_x - \wp_x(\beta_g)_x)||}{y+1} \right)^{\sigma_y} \\
&= \sum_{y=0, y\in(\ker(\wp))^c}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| (\wp_x(\beta_f)_x - \wp_x(\beta_g)_x)||}{y+1} \right)^{\sigma_y} \\
&\quad + \sum_{y=0, y\notin(\ker(\wp))^c}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| (\wp_x(\beta_f)_x - \wp_x(\beta_g)_x)||}{y+1} \right)^{\sigma_y} \\
&\geqslant \sum_{y=0, y\in(\ker(\wp))^c}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| (\wp_x(\beta_f)_x - \wp_x(\beta_g)_x)||}{y+1} \right)^{\sigma_y} \\
&= \sum_{y=0}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| (\wp_x(m_f)_x - \wp_x(m_g)_x)||}{y+1} \right)^{\sigma_y} \\
&> \sum_{y=0}^{\infty} \left(\frac{\sum_{x=0}^{y} |\nabla_\mho| \eta((m_f)_x - (m_g)_x)||}{y+1} \right)^{\sigma_y} \geqslant \inf_y \eta^{\sigma_y} \top\Big(m_f - m_g\Big),
\end{aligned}$$

where

$$(m_f)_y = \begin{cases} (\beta_f)_y, & y \in (\ker(\wp))^c, \\ 0, & y \notin (\ker(\wp))^c. \end{cases}$$

then, $\{m_f\}$ is a Cauchy sequence in $(Ces(\nabla_\mho, \sigma))_\top$. Since $(Ces(\nabla_\mho, \sigma))_\top$ is complete, we have $\beta \in (Ces(\nabla_\mho, \sigma))_\top$ with $\lim_{y\to\infty} m_y = \beta$. As $M_\wp \in \mathfrak{I}((Ces(\nabla_\mho, \sigma))_\top)$, we get $\lim_{y\to\infty} M_\wp m_y = M_\wp \beta$. But $\lim_{y\to\infty} M_\wp m_y = \lim_{y\to\infty} M_\wp \beta_y = l$. So $M_\wp \beta = l$. Hence $l \in Range(M_\wp)$. Hence $Range(M_\wp)$ is closed. Next, assume that the necessity condition is verified. So, we have $\eta > 0$ such that $\top(M_\wp \beta) \geqslant \eta \top(\beta)$, with $\beta \in \Big((Ces(\nabla_\mho, \sigma))_\top \Big)_{(\ker(\wp))^c}$. Suppose $U = \Big\{ y \in (\ker(\wp))^c : |\wp_y| < \eta \Big\} \neq \emptyset$, then for $f_0 \in U$, we have

$$\begin{aligned}
\top(M_\wp e_{f_0}) = \top\Big(\Big(\wp_g (e_{f_0})_g\Big)_{g=0}^{\infty} \Big) &= \sum_{g=0}^{\infty} \left(\frac{\sum_{x=0}^{g} |\nabla_\mho| \wp_x (e_{f_0})_x||}{g+1} \right)^{\sigma_g} \\
&< \sum_{g=0}^{\infty} \left(\frac{\sum_{x=0}^{g} |\nabla_\mho| \eta (e_{f_0})_x||}{g+1} \right)^{\sigma_g} \leqslant \sup_g \eta^{\sigma_g} \top(e_{f_0}),
\end{aligned}$$

this explains a contradiction. So $U = \emptyset$, one has $|\wp_y| \geqslant \eta$, with $y \in (\ker(\wp))^c$. □

Theorem 4.8. *If $(Ces(\nabla_\mho, \sigma))_\top$ is a pre-quasi Banach $\mathfrak{pss}$ and $\wp \in \mathfrak{C}^{\mathbb{I}^+}$, then, one has $r > 0$ and $t > 0$ with $r < |\wp_y| < t$, for every $y \in \mathbb{I}^+$, if and only if, $M_\wp \in \mathfrak{I}((Ces(\nabla_\mho, \sigma))_\top)$ is invertible.*

Proof. Let the sufficient condition be confirmed. Assume that $\eta \in \mathfrak{C}^{\mathbb{I}^+}$ with $\eta_x = \frac{1}{\wp_x}$. From Theorem 4.2, the operators $M_\wp$ and M_η are bounded linear. Then $M_\wp . M_\eta = M_\eta . M_\wp = I$. Hence $M_\eta = M_\wp^{-1}$. Next, if $M_\wp$ is invertible, therefore, $Range(M_\wp) = \Big((Ces(\nabla_\mho, \sigma))_\top \Big)_{\mathbb{I}^+}$. So, $Range(M_\wp)$ is closed. By using Theorem 4.7, one has $r > 0$ such that $|\wp_y| \geqslant r$, for all $y \in (\ker(\wp))^c$. One has $\ker(\wp) = \emptyset$, if $\wp_{y_0} = 0$, with $y_0 \in \mathbb{I}^+$, this gives $e_{y_0} \in \ker(M_\wp)$, so we have a contradiction since $\ker(M_\wp)$ is trivial. Hence, $|\wp_y| \geqslant r$, for all $y \in \mathbb{I}^+$, as $M_\wp \in \ell_\infty$. In view of Theorem 4.2, one has $t > 0$ such that $|\wp_y| \leqslant t$, for all $y \in \mathbb{I}^+$. Hence, we get $r \leqslant |\wp_y| \leqslant t$, with $y \in \mathbb{I}^+$. □

Definition 4.9 ([12]). Assume the space $(\text{Range}(G))^c$ is the complement of $\text{Range}(G)$. An operator $G \in \mathfrak{I}(\mathcal{D})$ is called Fredholm, when $\dim(\text{Range}(G))^c < \infty$, $\dim(\ker(G)) < \infty$ and $\text{Range}(G)$ is closed.

Theorem 4.10. *Suppose $M_\wp \in \mathfrak{I}((\text{Ces}(\nabla_\mho, \sigma))_\top)$, where $(\text{Ces}(\nabla_\mho, \sigma))_\top$ is a pre-quasi Banach $\mathfrak{pss}$. Then $M_\wp$ is Fredholm mapping, if and only if*

(g) $\ker(\wp) \subsetneqq \mathbb{I}^+$ *is finite;*
(h) $|\wp_y| \geqslant \rho$, *with* $y \in (\ker(\wp))^c$.

Proof. Assume the conditions (g) and (h) are satisfied. From Theorem 4.7, the condition (h) explains that $\text{Range}(M_\wp)$ is closed. The condition (g) gives that $\dim(\ker(M_\wp)) < \infty$ and $\dim((\text{Range}(M_\wp))^c) < \infty$. Hence, $M_\wp$ is Fredholm. If $M_\wp$ is the Fredholm operator, suppose $\ker(\wp) \subsetneqq \mathbb{I}^+$ is infinite, hence $e_x \in \ker(M_\wp)$, for every $x \in \ker(\wp)$. Since e_x's are linearly independent, one has $\dim(\ker(M_\wp)) = \infty$, which implies a contradiction. So, $\ker(\wp) \subsetneqq \mathbb{I}^+$ must be finite. The condition (h) follows from Theorem 4.7. □

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