J. Math. Computer Sci., 29 (2023), 1–11

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Some new exact solutions for a generalized variable coefficients KdV equation

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Abstract

In this paper, the variable coefficients KdV equation with general power nonlinearities is proposed. Firstly, it is transformed into a generalized KdV equation with constant coefficients using a point transformation. Then, the traveling wave transformation is utilized to transform the obtained constant coefficients generalized KdV equation into a generalized ordinary differential equation. We give a classification for the obtained generalized ordinary differential equation using a suitable integrating factor. Some new solutions are obtained for the generalized KdV equation with constant coefficients. All the obtained solutions in this paper for the variable coefficients KdV equation with general power nonlinearities are new.

Keywords: Exact solutions, generalized KdV equation, traveling wave, solitons.

2020 MSC: 35C05, 35C07, 35G20, 35Q53.

1. Introduction

Nonlinear evolution equations play a very vital role in modeling many engineering, physical, and biological phenomena [10, 13, 20, 23]. Many methods have been proposed in the literature for getting exact and approximate solutions to these equations. Examples of these methods are, (G′/G)-expansion method [10, 13], generalized F-expansion Method [23], Lie symmetry analysis method [7, 8, 11, 14, 19], generalized new auxiliary equation method [10-11], the invariant subspace method [3, 4, 15], homotopy perturbation methods [9, 18], and reproducing Kernel Hilbert Space Method [17].

In this paper, we investigate a type of generalized variable coefficients KdV equation with general power nonlinearities and variable coefficients which is given in the following form

$$u_t + f(t)\ u^n u_x + g(t)\ u^{2n}\ u_x + R(t)\left(u^k (u^m)_{xx}\right)_x = 0, \tag{1.1}$$

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doi: 10.22436/jmcs.029.01.01

Received: 2021-10-18 *Revised:* 2022-05-13 *Accepted:* 2022-06-01

where, k, n, m are constants and $f(t), g(t), R(t)$ are some smooth functions of the variable t. When $f(t) = \text{const}, g(t) = \text{const}, R(t) = 1$,Eq. (1.1) is studied in [20] and some compact solutions, periodic solutions and Jacobi elliptic functions solutions of Eq. (1.1) are obtained when $k + m = n + 1$.

In this paper, we first transform Eq. (1.1) into an equation with constant coefficients. Then we investigate the traveling wave solutions of the resultant equation to get exact traveling wave solutions of Eq. (1.1).

This paper has been organized as follows. In Section 2, we transform the variable coefficients KdV equation (1.1) into itself but with constant coefficients. In Section 3, we use the traveling wave transformation to obtain some new solutions of the variable coefficients KdV equation.

2. Transforming Eq. (1.1) into its constant coefficients equation

Theorem 2.1. *The generalized variable coefficients KdV equation (1.1) can be transformed into the generalized constant coefficients KdV equation*

$$\bar{u}_{\bar{t}} + a\bar{u}^n\bar{u}_{\bar{x}} + b\bar{u}^{2n}\bar{u}_{\bar{x}} + \left(\bar{u}^k(\bar{u}^m)_{\bar{x}\bar{x}}\right)_{\bar{x}} = 0, \tag{2.1}$$

under the transformation

$$\bar{t} = \int R(t)dt, \bar{x} = x, \qquad \bar{u} = u\,, \tag{2.2}$$

with the constraints:

$$f(t) = a\,R(t)\,,\ \ g(t) = b\,R(t)\,,$$

where a, b are arbitrary constants.

Proof. The transformation (2.2) leads to the change of variables

$$\frac{\partial}{\partial t} = R(t)\frac{\partial}{\partial \bar{t}}\,,\ \ \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}. \tag{2.3}$$

Applying the operators (2.3) to the function $\bar{u}\left(\bar{x}, \bar{y}, \bar{t}\right)$, we obtain

$$\frac{\partial \bar{u}}{\partial t} = R(t)\frac{\partial \bar{u}}{\partial \bar{t}}\,,\ \ \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \bar{x}}\ \ ,\ \ \frac{\partial^3 \bar{u}}{\partial x^2} = \frac{\partial^3 \bar{u}}{\partial \bar{x}^3}.$$

Using (2.2), we can get

$$u_t = R(t)\frac{\partial \bar{u}}{\partial \bar{t}}\,,\ \ u_x = \bar{u}_{\bar{x}}\,,\qquad u_{xx} = \bar{u}_{\bar{x}\bar{x}}. \tag{2.4}$$

From Eq. (2.4) and Eq. (1.1), we can get

$$\bar{u}_{\bar{t}} + \frac{f(t)}{R(t)}\bar{u}^n\bar{u}_{\bar{x}} + \frac{g(t)}{R(t)}\,\bar{u}^{2n}\bar{u}_{\bar{x}} + \left(\bar{u}^k(\bar{u}^m)_{\bar{x}\bar{x}}\right)_{\bar{x}} = 0. \tag{2.5}$$

Equation (2.5) is an equation with constant coefficients, if and only if, $f(t) = a\ R(t)$ and $g(t) = b\ R(t)$. Hence, Eq. (2.5) becomes

$$\bar{u}_{\bar{t}} + a\bar{u}^n\bar{u}_{\bar{x}} + b\ \bar{u}^{2n}\bar{u}_{\bar{x}} + \left(\bar{u}^k(\bar{u}^m)_{\bar{x}\bar{x}}\right)_{\bar{x}} = 0. \tag{2.6}$$

□

3. Solutions of the variable coefficients Eq. (1.1)

In this section, we obtain solutions of the variable coefficients Eq. (1.1) by solving the constant coefficients Eq. (2.6).

Consider the transformation given in [1, 6]:

$$\bar{u}\left(\bar{x}, \bar{t}\right) = y(z)\,,\ \ z = \alpha\bar{x} - \omega\,\bar{t}. \tag{3.1}$$

Accordingly, Eq. (2.6) becomes,

$$\left(-\omega + a\,\alpha y^{n} + b\alpha\, y^{2n}\right)\frac{dy}{dz} + \alpha^{3}\frac{d}{dz}\left(y^{k}\frac{d^{2}y^{m}}{dz^{2}}\right) = 0. \tag{3.2}$$

Integrating Eq. (3.2), we get,

$$-\omega\, y + \frac{a\alpha}{n+1}y^{n+1} + \frac{b\alpha}{2n+1}\, y^{2n+1} + \alpha^{3}y^{k}\frac{d^{2}y^{m}}{dz^{2}} = c_1, \tag{3.3}$$

where c_1 is a constant. Multiplying Eq. (3.3) by $my^{m-k-1}y'$, we obtain

$$-\omega\, m\, y^{m-k}y' + \frac{a\alpha}{n+1}m\, y^{m+n-k}y' + \frac{b\alpha}{2n+1}\, m\, y^{m+2n-k}y' + \alpha^{3}m\, y^{m-1}y'\frac{d^{2}y^{m}}{dz^{2}} = c_1 m\, y^{m-k-1}y'.$$

Integrating above equation, we obtain,

$$\begin{aligned} y'(z)^{2} = {} & \frac{2c_1}{\alpha^{3}m(m-k)}y^{-k-m+2} + \frac{2c_2}{\alpha^{3}m^{2}}y^{2-2m} + \frac{2}{\alpha^{3}m}y^{-k-m+3} \\ & \times\left(\frac{\omega}{-k+m+1} - \frac{a\alpha}{(n+1)\left(-k+m+n+1\right)}y^{n} - \frac{\alpha b}{(2n+1)\left(-k+m+2n+1\right)}y^{2n}\right), \end{aligned} \tag{3.4}$$

where c_2 is a constant. Equation (3.4) has the following solutions.

- For $k = 1-m,\ n = m$ and $c_1 = 0$ and letting

$$y = H\left(z\right)^{\frac{1}{m}}, \tag{3.5}$$

Eq. (3.4) becomes

$$\left(H'\right)^{2} = \frac{2c_2}{\alpha^{3}} + \frac{\omega}{\alpha^{3}}H^{2} - \frac{2a}{3\left(1+m\right)\alpha^{2}}H^{3} - \frac{b}{2\left(1+2m\right)\alpha^{2}}\, H^{4}. \tag{3.6}$$

Equation (3.6) has many solutions as mentioned in [2, 5, 16, 22]. One of the solutions is

$$H\left(z\right) = \frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^{2}-2}}\mathrm{cn}\left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha-4\alpha\gamma^{2}}}z,\gamma\right), \tag{3.7}$$

with

$$c_2 = \frac{-9\omega^{3}(1+m)^{2}}{32a^{2}\alpha^{2}(1-2\gamma^{2})^{2}}\ ,\ b = -\frac{2\alpha a^{2}\left(1+2m\right)}{9\omega(1+m)^{2}},$$

where cn is the Jacobi elliptic cosine function.

Substituting Eq. (3.7) into Eq. (3.5), we obtain

$$y = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^{2}-2}}\mathrm{cn}\left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha-4\alpha\gamma^{2}}}z,\gamma\right)\ \right)^{\frac{1}{m}}. \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.1), we obtain

$$\bar{u}\left(\bar{x},\bar{t}\right) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^{2}-2}}\mathrm{cn}\left(\frac{\sqrt{\omega}\ \ \left(\alpha\bar{x}-\omega\,\bar{t}\right)}{\alpha\sqrt{2\alpha-4\alpha\gamma^{2}}},\gamma\right)\ \right)^{\frac{1}{m}}.$$

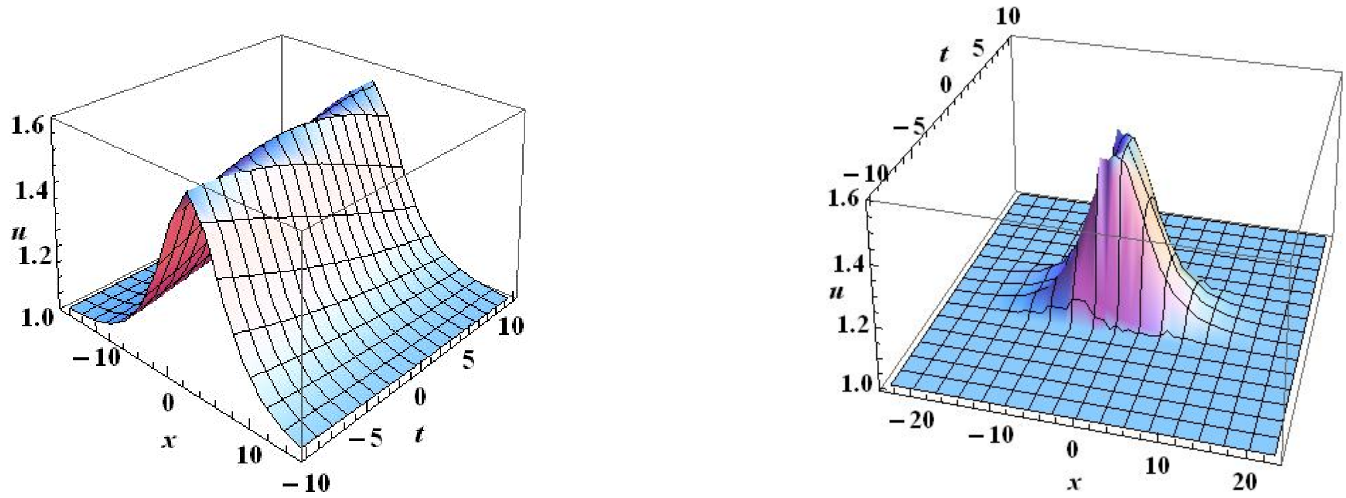
Eventually, the solution of Eq. (1.1) can be expressed as

$$u(x,t)=\left(\frac{3\omega(1+m)}{2a\alpha}+\frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^2-2}}\mathrm{cn}\left(\frac{\sqrt{\omega}\ \left(\alpha x-\omega\int R(t)dt\right)}{\alpha\sqrt{2\alpha-4\alpha\gamma^2}},\gamma\right)\right)^{\frac{1}{m}}. \tag{3.9}$$

When $\gamma=1$, solution (3.9) degenerates to

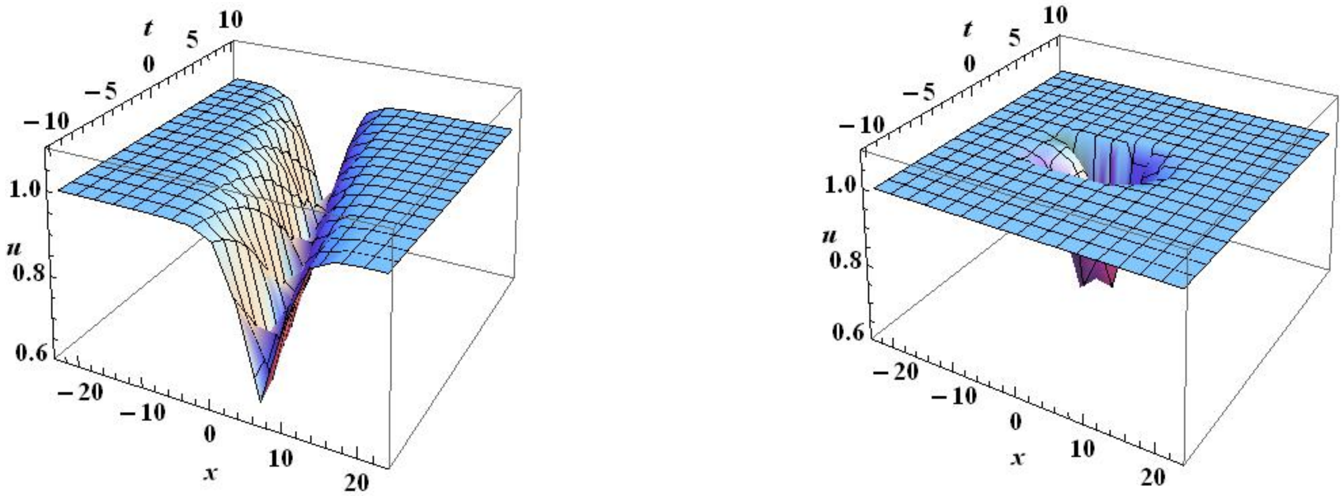
$$u(x,t)=\left(\frac{3\omega(1+m)}{2a\alpha}+\frac{3\omega(1+m)}{a\alpha\sqrt{2}}\mathrm{sech}\left(\frac{\sqrt{\omega}\ \left(\alpha x-\omega\int R(t)dt\right)}{\alpha\sqrt{-2\alpha}}\right)\right)^{\frac{1}{m}}. \tag{3.10}$$

For $a<0$ and $m>0$, we obtain bright soliton solutions. For $a>0$ and $m<0$, we obtain dark soliton solutions.



(a) $R(t)=1$ (b) $R(t)=te^{t^2}$

Figure 1: Solution (3.10) when $\omega=1, \alpha=-\frac{9}{2}, m=2, a=-1$, and different forms of $R(t)$.



(a) $R(t)=1$ (b) $R(t)=te^{t^2}$

Figure 2: Solution (3.10) when $\omega=1, \alpha=-\frac{3}{2}, a=1, m=-2$, and different forms of $R(t)$.

From Figures 1 and 2, it is clear that we can retrieve bright solitons, bright rogue waves, dark solitons, and dark rouge waves from solution (3.10) according to the values of ω, α, a, m, and $R(t)$.

Also, Eq. (3.6) has the following solution

$$H(z) = \frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{2\gamma^2+2}} \mathrm{sn}\left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha+2\alpha\gamma^2}} z, \gamma\right), \tag{3.11}$$

with

$$c_2 = \frac{-9\omega^3(1+m)^2(1-\gamma^2)^2}{32a^2\alpha^2(1+\gamma^2)^2}, \quad b = -\frac{2\alpha a^2(1+2m)}{9\omega(1+m)^2},$$

where sn is the Jacobi elliptic sine function.

Substituting Eq. (3.11) into Eq. (3.5), we get

$$y = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{2\gamma^2+2}} \mathrm{sn}\left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha+2\alpha\gamma^2}} z, \gamma\right)\right)^{\frac{1}{m}}. \tag{3.12}$$

From Eqs. (3.12) and (3.1), we can get

$$\bar{u}(\bar{x}, \bar{t}) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{2\gamma^2+2}} \mathrm{sn}\left(\frac{\sqrt{\omega}\,(\alpha\bar{x} - \omega\,\bar{t})}{\alpha\sqrt{2\alpha+2\alpha\gamma^2}}, \gamma\right)\right)^{\frac{1}{m}}.$$

Eventually, the solution of Eq. (1.1) can be expressed as

$$u(x,t) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{2\gamma^2+2}} \mathrm{sn}\left(\frac{\sqrt{\omega}\,\left(\alpha x - \omega \int R(t)dt\right)}{\alpha\sqrt{2\alpha+2\alpha\gamma^2}}, \gamma\right)\right)^{\frac{1}{m}}. \tag{3.13}$$

When $\gamma = 1$, solution (3.13) degenerates to

$$u(x,t) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega(1+m)}{2a\alpha} \tanh\left(\frac{\sqrt{\omega}\,\left(\alpha x - \omega \int R(t)dt\right)}{2\alpha\sqrt{\alpha}}\right)\right)^{\frac{1}{m}}. \tag{3.14}$$

• For $k = 1-m$, $n = 2m$, and $c_1 = 0$ and letting

$$y = G(z)^{\frac{-1}{2m}}, \tag{3.15}$$

substituting Eq. (3.15) into Eq. (3.4), we get

$$G'(z)^2 = -\frac{4b}{3\alpha^2(4m+1)} - \frac{2a}{\alpha^2(2m+1)} G(z) + \frac{4\omega}{\alpha^3} G(z)^2 + \frac{8c_2}{\alpha^3} G(z)^3. \tag{3.16}$$

Many solutions of Eq. (3.16) are mentioned in [2, 5, 16, 22]. One of them is

$$G(z) = \frac{a\alpha(\omega + \alpha^3(1-2\gamma^2)\beta_3^2)}{2(1+2m)\left(\omega^2 - \alpha^6(1-\gamma^2+\gamma^4)\beta_3^4\right)} + \beta_2\,\mathrm{cn}^2(\beta_3 z, \gamma), \tag{3.17}$$

with

$$\beta_2 = \frac{3a\alpha^4\gamma^2\beta_3^2}{2(1+2m)\left(\omega^2 - \alpha^6(1-\gamma^2+\gamma^4)\beta_3^4\right)},$$

$$c_2 = \frac{(1+2m)(-\omega^2+\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)}{3a\alpha},$$
$$b = \frac{a^2\,(1+4m)\,\alpha\left(-\omega^3+3\alpha^6\left(1-\gamma^2+\gamma^4\right)\omega\beta_3^4+\alpha^9\left(2-3\gamma^2-3\gamma^4+2\gamma^6\right)\beta_3^6\right)}{4(1+2m)^2(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)^2},$$

where β_3 is an arbitrary constant. Substituting Eq. (3.17) into Eq. (3.15), we get

$$y = \left(\frac{a\alpha(\omega+\alpha^3(1-2\gamma^2)\beta_3^2)}{2\,(1+2m)\left(\omega^2-\alpha^6\left(1-\gamma^2+\gamma^4\right)\beta_3^4\right)}+\ \beta_2\,\mathrm{cn}^2\left(\beta_3\,z,\gamma\right)\right)^{\frac{-1}{2m}}. \tag{3.18}$$

From Eq. (3.18) and Eq. (3.1), we can get

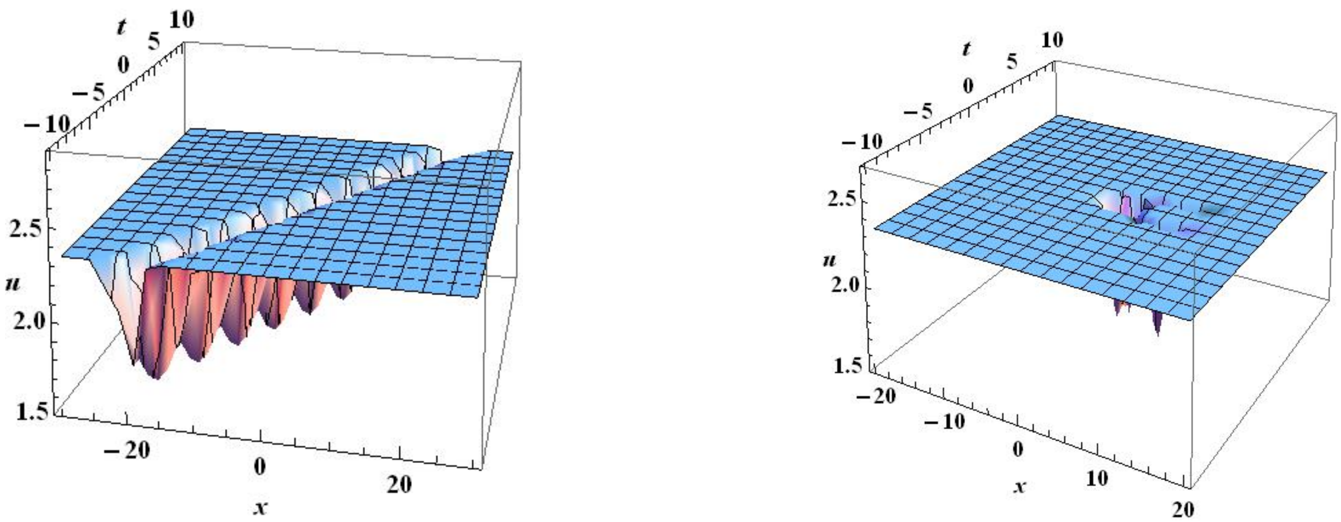
$$\bar{u}\left(\bar{x},\bar{t}\right) = \left(\frac{a\alpha(\omega+\alpha^3(1-2\gamma^2)\beta_3^2)}{2\,(1+2m)\left(\omega^2-\alpha^6\left(1-\gamma^2+\gamma^4\right)\beta_3^4\right)}+\ \beta_2\,\mathrm{cn}^2\left(\beta_3\ \left(\alpha\bar{x}-\omega\,\bar{t}\right),\gamma\right)\right)^{\frac{-1}{2m}}.$$

Using Eq. (2.2), we obtain

$$u\left(x,t\right) = \left(\frac{a\alpha\left(\omega+\alpha^3\left(1-2\gamma^2\right)\beta_3^2\right)}{2\,(1+2m)\left(\omega^2-\alpha^6\left(1-\gamma^2+\gamma^4\right)\beta_3^4\right)}+\ \beta_2\,\mathrm{cn}^2\left(\beta_3\ \left(\alpha x-\omega\int R\left(t\right)\,dt\right),\gamma\right)\right)^{\frac{-1}{2m}}. \tag{3.19}$$

When $\gamma=1$, the solution (3.19) degenerates to

$$u\left(x,t\right) = \left(\frac{a\mathrm{ff}}{2\,(2m+1)\left(\alpha^3\beta_3^2+\omega\right)}+\frac{3a\alpha^4\beta_3^2}{2\,(1+2m)\left(\omega^2-\alpha^6\beta_3^4\right)}\,\mathrm{sech}^2\left(\beta_3\ \left(\alpha x-\omega\int R\left(t\right)\,dt\right)\right)\right)^{\frac{-1}{2m}}. \tag{3.20}$$

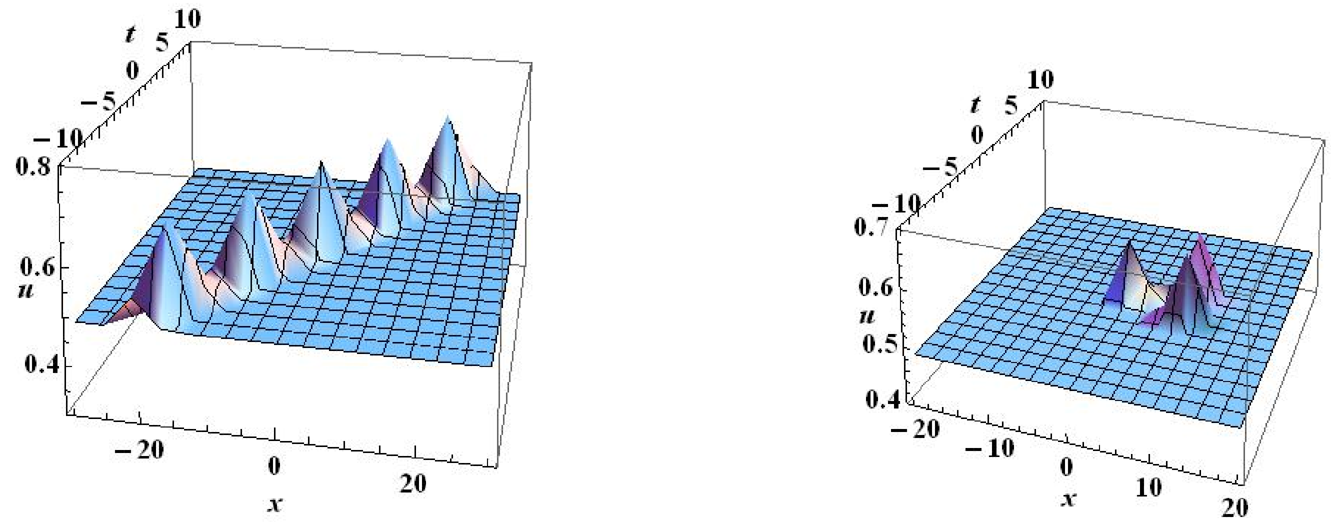


(a) $R(t)=1$ (b) $R(t)=te^{t^2}$

Figure 3: Solution (3.20) when $\beta_3=\alpha=a=1, m=\omega=2$, and different forms of $R\left(t\right)$.

From Figures 3 and 4, it is clear that we can retrieve bright solitons, bright rogue waves, dark solitons, and dark rouge waves from solution (3.20) according to the values of $\omega, \beta_3, \alpha, a, m$, and $R\left(t\right)$. Also, Eq. (3.16) has the following solution

$$G\left(z\right) = \frac{a\alpha(\omega+\alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)}+\ \beta_2\,\mathrm{sn}^2\left(\beta_3\,z,\gamma\right), \tag{3.21}$$



(a) $R(t) = 1$ (b) $R(t) = te^{t^2}$

Figure 4: Solution (3.20) when $\beta_3 = \alpha = 1, a = -1, m = -2, \omega = 2$, and different forms of $R(t)$.

with

$$\beta_2 = \frac{3a\alpha^4\gamma^2\beta_3^2}{2(1+2m)(-\omega^2+\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)},$$
$$c_2 = \frac{(1+2m)(-\omega^2+\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)}{3a\alpha},$$
$$b = \frac{a^2(1+4m)\alpha(-\omega^3+3\alpha^6(1-\gamma^2+\gamma^4)\omega\beta_3^4+\alpha^9(2-3\gamma^2-3\gamma^4+2\gamma^6)\beta_3^6)}{4(1+2m)^2(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)^2},$$

where β_3 is an arbitrary constant. Substituting Eq. (3.21) into Eq. (3.15), we get

$$y = \left(\frac{a\alpha(\omega+\alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)} + \beta_2 \operatorname{sn}^2(\beta_3 z, \gamma)\right)^{\frac{-1}{2m}}. \tag{3.22}$$

From Eq. (3.22) and Eq. (3.1), we can get

$$\bar{u}(\bar{x},\bar{t}) = \left(\frac{a\alpha(\omega+\alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)} + \beta_2 \operatorname{sn}^2(\beta_3(\alpha\bar{x}-\omega\bar{t}), \gamma)\right)^{\frac{-1}{2m}}.$$

Using Eq. (2.2), we obtain

$$u(x,t) = \left(\frac{a\alpha(\omega+\alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)} + \beta_2 \operatorname{sn}^2\left(\beta_3\left(\alpha x-\omega\int R(t)\,dt\right), \gamma\right)\right)^{\frac{-1}{2m}}. \tag{3.23}$$

When $\gamma = 1$, the solution (3.23) degenerates to

$$u(x,t) = \left(\frac{a\alpha(\omega+\alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2-\alpha^6(1-\gamma^2+\gamma^4)\beta_3^4)} + \beta_2 \tanh^2\left(\beta_3\left(\alpha x-\omega\int R(t)\,dt\right)\right)\right)^{\frac{-1}{2m}}. \tag{3.24}$$

• For $k = 1 - m,\ c_1 = c_2 = 0$, Eq. (3.4) becomes

$$(y')^2 = \frac{-b}{\alpha^2 m(2n+1)(m+n)}\, y^2 \left(\frac{a(2n+1)(m+n)}{b(n+1)(2m+n)} + y^n \right)^2, \tag{3.25}$$

with the constraint

$$\omega = -\frac{a^2 m(m+n)(1+2n)\alpha}{b(1+n)^2(2m+n)^2},$$

(3.25) has the solution

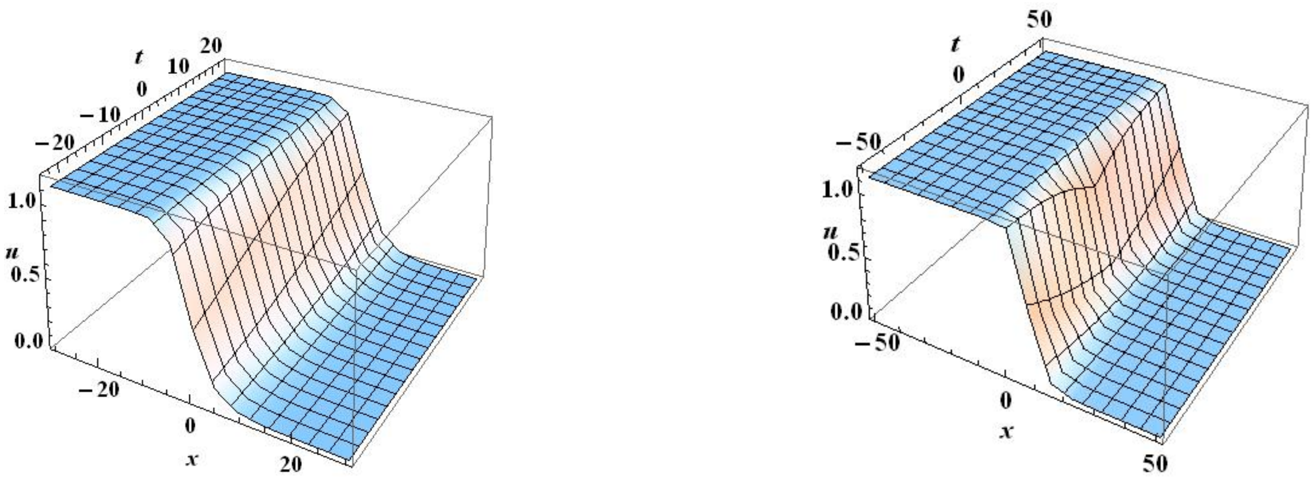
$$y = \left(\frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \tanh\left(\frac{an\sqrt{2n+1}\sqrt{m+n}}{2\alpha(n+1)\sqrt{-bm}(2m+n)} z \right) - \frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \right)^{\frac{1}{n}}. \tag{3.26}$$

From Eq. (3.26) and Eq. (3.1), we can get

$$\bar{u}(\bar{x}, \bar{t}) = \left(\frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \tanh\left(\frac{an\sqrt{2n+1}\sqrt{m+n}}{2\alpha(n+1)\sqrt{-bm}(2m+n)} (\alpha\bar{x} - \omega\bar{t}) \right) - \frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \right)^{\frac{1}{n}}.$$

Using Eq. (2.2), we obtain

$$\begin{aligned} u(x,t) = &\left(\frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \tanh\left(\frac{an\sqrt{2n+1}\sqrt{m+n}}{2(n+1)\sqrt{-bm}(2m+n)} \right.\right. \\ &\left.\left. \times \left(x + \frac{a^2 m(m+n)(1+2n)}{b(1+n)^2(2m+n)^2} \int R(t)\,dt \right) \right) - \frac{a(2n+1)(m+n)}{2b(n+1)(2m+n)} \right)^{\frac{1}{n}}. \end{aligned} \tag{3.27}$$



(a) $R(t) = 1$ (b) $R(t) = \tanh t$

Figure 5: Solution (3.27), when $n = 2,\ m = 1,\ b = -1,\ a = 1$ and different forms of $R(t)$.

Figure 5 represents the kink wave solution (3.27) .

• For $m + k = 2n + 1,\ c_1 = c_2 = 0$, letting

$$y = H(z)^{\frac{-1}{n}}, \tag{3.28}$$

and substituting Eq. (3.28) in Eq. (3.4), we get

$$H'(z)^2 = \frac{-bn^2}{\alpha^2 m^2 (2n+1)} H(z)^2 - \frac{2an^2}{\alpha^2 m(n+1)(2m-n)} H(z)^3 + \frac{n^2\omega}{\alpha^3 m(m-n)} H(z)^4. \tag{3.29}$$

Equation (3.29) can be solved easily to get

$$H(z)=\frac{c^2(1+n)\left(2m^2-mn\right)\alpha^2}{an^2}\frac{1}{1+Be^{cz}}, \tag{3.30}$$

when

$$\omega=-\frac{a^2n^2(-m+n)}{c^2m(1+n)^2(-2m+n)^2\alpha},\quad b=-\frac{c^2m^2(1+2n)\alpha^2}{n^2},$$

where B, c are constants. Substituting Eq. (3.30) into Eq. (3.28), we obtain

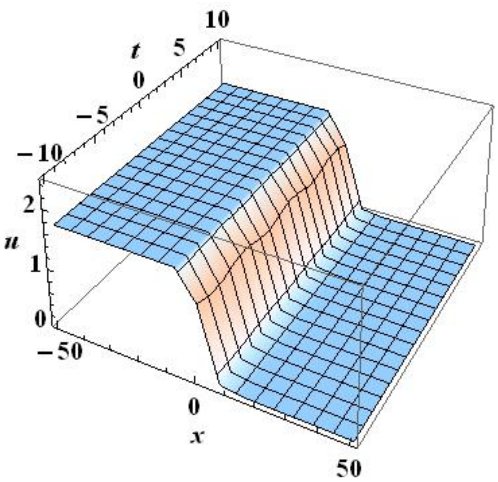
$$y=\left(\frac{c^2(1+n)\left(2m^2-mn\right)\alpha^2}{an^2}\frac{1}{1+Be^{cz}}\right)^{\frac{-1}{n}}. \tag{3.31}$$

From Eqs. (3.31) and (3.1), we can get

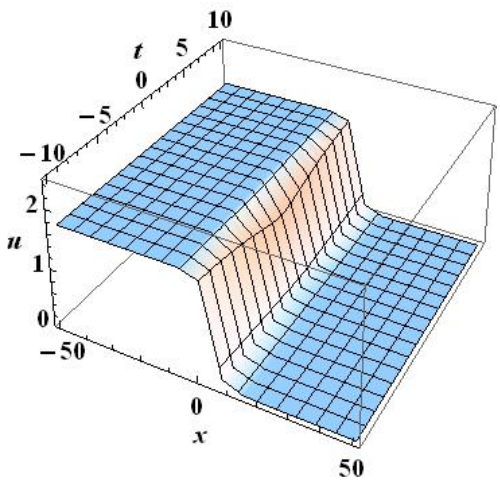
$$\bar{u}(\bar{x},\bar{t})=\left(\frac{c^2(1+n)\left(2m^2-mn\right)\alpha^2}{an^2}\frac{1}{1+Be^{c(\alpha\bar{x}-\omega\,\bar{t})}}\right)^{\frac{-1}{n}}.$$

Using Eq. (2.2), we obtain

$$u(x,t)=\left(\frac{c^2(1+n)\left(2m^2-mn\right)\alpha^2}{an^2}\frac{1}{1+B\exp\left(c\left(\alpha x+\frac{a^2n^2(-m+n)}{c^2m(1+n)^2(-2m+n)^2\alpha}\int R(t)\,dt\right)\right)}\right)^{\frac{-1}{n}}. \tag{3.32}$$



(a) $R(t)=\cos t$



(b) $R(t)=\tanh t$

Figure 6: Solution (3.32) when $\alpha=B=c=1, a=-1, m=2, n=-2$ and different forms of $R(t)$.

Figure 6 represents the kink wave solution (3.32).

- For $c_1=c_2=0$, Eq. (3.4) becomes

$$(y')^2=\frac{-2b}{\alpha^2m(2n+1)(-k+m+2n+1)}y^{-k-m+3}\left(\frac{a(2n+1)(-k+m+2n+1)}{2b(n+1)(-k+m+n+1)}+y^n\right)^2, \tag{3.33}$$

with the constraint

$$\omega=\frac{a^2(1-k+m)(-1+k-m-2n)(1+2n)\alpha}{4b(1+n)^2(1-k+m+n)^2}.$$

Equation (3.33) has the implicit solution:

$$\begin{aligned} c_3+z=&\frac{2\sqrt{2}\sqrt{-bm}(1+n)(1-k+m+n)\alpha}{a(-1+k+m)\sqrt{(1+2n)(1-k+m+2n)}}y^{\frac{1}{2}(k+m-1)}\\ &\times\ {}_2F_1\left(1,\frac{k+m-1}{2n};\frac{k+m-1}{2n}+1;\frac{2b\,(n+1)\,(-k+m+n+1)\,y^n}{a\,(k-m-2n-1)\,(2n+1)}\right). \end{aligned}\tag{3.34}$$

From Eqs. (3.34) and (3.1), we can get

$$\begin{aligned} c_3+\alpha\bar{x}-\omega\,\bar{t}=&\frac{2\sqrt{2}\sqrt{-bm}\,(1+n)\,(1-k+m+n)\,\alpha}{a\,(-1+k+m)\,\sqrt{(1+2n)\,(1-k+m+2n)}}\bar{u}^{\frac{1}{2}(k+m-1)}\\ &\times\ {}_2F_1\left(1,\frac{k+m-1}{2n};\frac{k+m-1}{2n}+1;\frac{2b\,(n+1)\,(-k+m+n+1)\,\bar{u}^n}{a\,(k-m-2n-1)\,(2n+1)}\right). \end{aligned}$$

Using Eq. (2.2), we can retrieve

$$\begin{aligned} &c_3+\alpha x-\frac{a^2\,(1-k+m)\,(-1+k-m-2n)\,(1+2n)\,\alpha}{4b(1+n)^2(1-k+m+n)^2}\int R\,(t)\,dt\\ &=\frac{2\sqrt{2}\sqrt{-bm}\,(1+n)\,(1-k+m+n)\,\alpha}{a\,(-1+k+m)\,\sqrt{(1+2n)\,(1-k+m+2n)}}u^{\frac{1}{2}(k+m-1)}\\ &\times\ {}_2F_1\left(1,\frac{k+m-1}{2n};\frac{k+m-1}{2n}+1;\frac{2b\,(n+1)\,(-k+m+n+1)\,u^n}{a\,(k-m-2n-1)\,(2n+1)}\right). \end{aligned}\tag{3.35}$$

In case of $m+k=n+1$, the implicit solution (3.35) becomes

$$\begin{aligned} u\,(x,t)=&\left(\frac{a\,(2n+1)\,(-2m-n)}{4bm\,(n+1)}\tanh^2\left(\frac{\sqrt{a}n}{2\sqrt{2}\alpha m\sqrt{n+1}}\right.\right.\\ &\left.\left.\times\left(-\frac{\alpha a^2\,(2n+1)\,(2m-n)\,(2m+n)}{16bm^2(n+1)^2}\int R\,(t)\,dt+c_3+\alpha x\right)\right)\right)^{\frac{1}{n}}. \end{aligned}\tag{3.36}$$

4. Conclusions

In this paper, using the transformation (2.2), the variable coefficients generalized KdV equation (1.1) is transformed into the constant coefficients KdV equation (2.1). Then we used the traveling wave transformation (3.1) to investigate the exact solutions of the KdV equation (2.1). We have obtained some new solutions for Eq. (2.1) that are not reported in [20]. The new obtained solutions are given by Eq. (3.10) and Eq. (3.14) when $k=1-m,\ n=m$, Eq. (3.20) and Eq. (3.24) when $k=1-m,\ n=2m$, Eq. (3.27), when $k=1-m,\ c_1=c_2=0$, Eq. (3.32) $k+m=2n+1,\ c_1=c_2=0$ and Eq. (3.35) for arbitrary $n,\ m$ and k. Using the function $R(t)$ we can get many types of solutions for Eq. (1.1). Some of these types are bright solitons (Figure 1 (a) and Figure 4 (a)), dark solitons (Figure 2 (a) and Figure 3 (a)), bright rogue wave solution (Figure 1 (b) and Figure 4 (b)), dark rogue wave solution (Figure 2 (b) and Figure 3 (b)), and kink solution (Figures 5 and 6). To our knowledge, the obtained solutions in this paper for Eq. (1.1) are new.

Acknowledgments

The research is supported by the Deanship of Scientific Research, Prince Sattam Bin Abdulaziz University, Alkharj, Saudi Arabia.

The authors are thankful to the unanimous reviewers for their valuable comments and the Editorial team of JMCS for their support in bringing this article in the present form.

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