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New fixed point theorems for θ-φ-contraction on b-metric spaces

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Abstract

In this paper, we define θ-φ-contraction on a b-metric space into itself by extending θ-φ-contraction introduced by Zheng et al. [D. W. Zheng, Z. Y. Cai, P. Wang, J. Nonlinear Sci. Appl., **10** (2017), 2662–2670] in metric space and also, we prove θ-type theorem in the setting of b-metric spaces as well as θ-φ-type theorem in the framework of b-rectangular metric spaces. Moreover, we give some applications to nonlinear integral equations. We also give illustrative examples to exhibit the utility of our results.

Keywords: Fixed point, rectangular b-metric space, θ-φ-contraction.

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1. Introduction

The Banach contraction principle is a fundamental result in fixed point theory [3]. Due to its importance, various mathematics studied many interesting extensions and generalizations, (see [4, 12, 16, 20]). In 2014, Jleli and Samet [11] analyzed a generalization of the Banach fixed point theorem on generalized metric spaces in a new type of contraction mappings called θ-contraction (or JS-contraction) and proved a fixed point result in generalized metric spaces. This direction has been studied and generalized in different spaces and various fixed point theorems have been developed (see [13–15]).

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, b-metric spaces were introduced by Bakhtin [2] and Czerwik [5], in such a way that triangle inequality is replaced by the b-triangle inequality: $d(x, y) \leqslant s\,(d(x, z) + d(z, y))$ for all pairwise distinct points x, y, z and $s \geqslant 1$. Any metric space is a b-metric space but in general, b-metric space might not be a metric space. Various fixed point results were established on such spaces. For more information on b-metric spaces and b-metric-like spaces, the readers can refer to (see [6–10, 17–19].

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Very recently, Zheng et al. [22] introduced a new concept of θ-ϕ-contraction and established some fixed point results for such mappings in complete metric space and generalized the results of Brower and Kannan.

In this paper, we introduce a new notion of generalized θ-ϕ-contraction and establish some results of fixed point for such mappings in complete b-metric space. The results presented in the paper extend the corresponding results of Kannan [12] and Reich [20] on b-rectangular metric space. Various examples are constructed to illustrate our results. As an application, we prove the existence and uniqueness of a solution for the nonlinear Fredholm integral equations. Also, we derive some useful corollaries of these results.

2. Preliminaries

Definition 2.1 ([5]). Let X be a nonempty set, $s \geqslant 1$ be a given real number, and let $d\colon X \times X \to [0, +\infty[$ be a function such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

1. $d(x, y) = 0$, if only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leqslant s[d(x, z) + d(z, y)]$, (b-rectangular inequality).

Then (X, d) is called a b-metric space.

Lemma 2.2 ([1]). *Let (X, d) be a b-metric space.*

(a) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, with $x \neq y$, $x_n \neq x$ and $y_n \neq y$ for all $n \in \mathbb{N}$. Then we have*

$$\frac{1}{s^2} d(x, y) \leqslant \lim_{n\to\infty} \inf d(x_n, y_n) \leqslant \lim_{n\to\infty} \sup d(x_n, y_n) \leqslant s^2 d(x, y).$$

(b) *In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have*

$$\frac{1}{s} d(x, z) \leqslant \lim_{n\to\infty} \inf d(x_n, z) \leqslant \lim_{n\to\infty} \sup d(x_n, z) \leqslant s d(x, z),$$

for all $x \in X$.

Lemma 2.3 ([21]). *Let (X, d) be a b-metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that

$$\begin{aligned}
\varepsilon &\leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leqslant s\varepsilon, \\
\varepsilon &\leqslant \lim_{k\to\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leqslant s\varepsilon, \\
\varepsilon &\leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leqslant s\varepsilon, \\
\frac{\varepsilon}{s} &\leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leqslant s^2\varepsilon.
\end{aligned}$$

The following definition was given by Jleli et al. in [11].

Definition 2.4 ([11]). Let Θ be the family of all functions $\theta :]0, +\infty[\to]1, +\infty[$ such that

(θ_1) θ is increasing,

(θ_2) for each sequence $(x_n) \subset]0,+\infty[$;

$$\lim_{n\to 0} x_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} \theta(x_n) = 1;$$

(θ_3) θ is continuous.

In [22], Zheng et al. presented the concept of θ-ϕ-contraction on metric spaces and proved the following nice result.

Definition 2.5 ([22])**.** Let Φ be the family of all functions ϕ: $[1,+\infty[\to [1,+\infty[$, such that

(ϕ_1) ϕ is nondecreasing;
(ϕ_2) for each $t \in]1,+\infty[$, $\lim_{n\to\infty} \phi^n(t) = 1$;
(ϕ_3) ϕ is continuous.

Lemma 2.6 ([22])**.** *If $\phi \in \Phi$, then $\phi(1)$=1, and $\phi(t) < t$ for all $t \in]1,\infty[$.*

Definition 2.7 ([22])**.** Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x,y \in X$,

$$d(Tx,Ty) > 0 \Rightarrow \theta[d(Tx,Ty)] \leqslant \phi(\theta[N(x,y)]),$$

where

$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Theorem 2.8 ([22])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.*

3. Main results

In this paper, using the idea introduced by Zheng et al., we present the concept θ-ϕ-contraction in b-metric spaces and we prove some fixed point results for such spaces.

Definition 3.1. Let (X,d) be a b-metric space with parameter $s > 1$ space and $T : X \to X$ be a mapping.

(1) T is said to be a $\theta-$contraction if there exist $\theta \in \Theta$ and $r \in]0,1[$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta\left[s^3 d(Tx,Ty)\right] \leqslant \theta[M(x,y)]^r,$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(Tx,y)}{2s^2}\right\}.$$

(2) T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that

$$d(Tx,Ty) > 0 \Rightarrow \theta\left[s^3 d(Tx,Ty)\right] \leqslant \phi[\theta(M(x,y))],$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty) \frac{d(x,Ty)+d(Tx,y)}{2s^2}\right\}.$$

(3) T is said to be a θ-ϕ- Kannan-type contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x,y \in X$ with $d(Tx,Ty) > 0$, we have

$$\theta\left[s^3 d(Tx,Ty))\right] \leqslant \phi\left[\theta\left(\frac{d(x,Tx)+d(y,Ty)}{2}\right)\right].$$

(4) T is said to be a θ-ϕ-Reich-type contraction if there exist exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for all $x, y \in X$ with $d(Tx, Ty) > 0$, we have

$$\theta\left[s^3 d(Tx, Ty)\right] \leqslant \phi\left[\theta\left(\frac{d(x,y) + d(x,Tx) + d(y,Ty)}{3}\right)\right].$$

Theorem 3.2. *Let* (X, d) *be a complete* b*-metric space and* $T : X \to X$ *be a* θ*-contraction, i.e, there exist* $\theta \in \Theta$ *and* $r \in]0,1[$ *such that for any* $x, y \in X$*, we have*

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^3 d(Tx, Ty)\right] \leqslant \theta\left[M(x,y)\right]^r. \tag{3.1}$$

Then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Letting $x = x_{n-1}$ and $y = x_n$ in (3.1), we have

$$\theta\left[d(x_n, x_{n+1})\right] \leqslant \theta\left[s^3 d(x_n, x_{n+1})\right] \leqslant \left[\theta\left(M(x_{n-1}, x_n)\right)\right]^r, \forall n \in \mathbb{N}, \tag{3.2}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2s^2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s^2}\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2s^2} d(x_{n-1}, x_{n+1}) &\leqslant \frac{1}{2s^2}\left[s\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right)\right] \\ &= \frac{1}{2s}\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \\ &\leqslant \frac{1}{2}\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \leqslant \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \end{aligned}$$

we obtain

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (3.2), we have

$$\theta\left(d(x_n, x_{n+1})\right) \leqslant \left(\theta\left(d(x_n, x_{n+1})\right)\right)^r < \theta\left(d(x_n, x_{n+1})\right),$$

which is a contradiction. Hence $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus

$$\theta\left(d(x_n, x_{n+1})\right) \leqslant \left(\theta\left(d(x_{n-1}, x_n)\right)\right)^r. \tag{3.3}$$

Repeating this step, we conclude that

$$\theta\left(d(x_n, x_{n+1})\right) \leqslant \left(\theta\left(d(x_{n-1}, x_n)\right)\right)^r \leqslant \left(\theta\left(d(x_{n-2}, x_{n-1})\right)\right)^{r^2} \leqslant \cdots \leqslant \theta\left(d(x_0, x_1)\right)^{r^n}.$$

From (3.3) and using (θ_1) we get

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Therefore, $d(x_n,x_{n+1})_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geqslant 0$ such that

$$\lim_{n\to\infty} d(x_{n+1},x_n) = \alpha.$$

Now, we claim that $\alpha = 0$. Arguing by contraction, we assume that $\alpha > 0$. Since $d(x_n,x_{n+1})_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$d(x_n,x_{n+1}) \geqslant \alpha, \quad \forall n \in \mathbb{N}.$$

By the property of θ, we get

$$1 < \theta(\alpha) \leqslant \theta(d(x_0,x_1))^{r^n}. \tag{3.4}$$

Letting $n \to \infty$ in (3.4), we obtain

$$1 < \theta(\alpha) \leqslant 1.$$

This is a contradiction. Therefore,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.5}$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d(x_n,x_m) = 0$. Suppose to the contrary. By Lemma 2.3, there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\{n_{(k)}\}$ and $\{m_{(k)}\}$ such that

i) $\varepsilon \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leqslant s\varepsilon$;

ii) $\frac{\varepsilon}{s} \leqslant \lim_{k\to\infty} \inf d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \leqslant s^2\varepsilon$;

iii) $\frac{\varepsilon}{s} \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right) \leqslant s^2\varepsilon$;

vi) $\frac{\varepsilon}{s^2} \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right) \leqslant s^3\varepsilon$.

From (3.1) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ we have

$$\begin{aligned} M\left(x_{m_{(k)}},x_{n_{(k)}}\right) = \max\Big\{ & d\left(x_{m_{(k)}},x_{n_{(k)}}\right), d\left(x_{m_{(k)}},x_{m(k)+1}\right), d\left(x_{n_{(k)}},x_{n_{(k)+1}}\right), \frac{1}{2s^2}\Big(d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) \\ & + d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\Big)\Big\}. \end{aligned}$$

Taking the limit as $k \to \infty$ and using (3.5) and Lemma 2.3, we have

$$\begin{aligned} \lim_{k\to\infty} M\left(x_{m_{(k)}},x_{n_{(k)}}\right) &= \lim_{k\to\infty} \max\Big\{ d\left(x_{m_{(k)}},x_{n_{(k)}}\right), d\left(x_{m_{(k)}},x_{m(k)+1}\right), d\left(x_{n_{(k)}},x_{n_{(k)+1}}\right), \\ & \qquad \frac{1}{2s^2}\left(d\left(x_{n_{(k)}},x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)}},x_{n_{(k)+1}}\right)\right)\Big\} \\ &\leqslant \max\{s\varepsilon, 0, 0, \frac{1}{2s^2}(s^2\varepsilon + s^2\varepsilon)\} = s\varepsilon. \end{aligned}$$

So we have

$$\lim_{k\to\infty} M\left(x_{m_{(k)}},x_{n_{(k)}}\right) \leqslant s\varepsilon. \tag{3.6}$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.1), we obtain

$$\theta\left[s^3 d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right] \leqslant \left[\theta\left(M\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r.$$

Letting $k \to \infty$ the above inequality, applying the continuity of θ and using (3.6), we obtain

$$\theta\left(\frac{\varepsilon}{s^2}s^3\right) = \theta(\varepsilon s) \leqslant \theta\left(s^3 \lim_{k\to\infty} d\left(x_{m_{(k)+1}},x_{n_{(k)+1}}\right)\right) \leqslant \left[\theta\left(\lim_{k\to\infty} M\left(x_{m_{(k)}},x_{n_{(k)}}\right)\right)\right]^r.$$

Therefore,

$$\theta(s\varepsilon) \leqslant [\theta(s\varepsilon)]^r < \theta(s\varepsilon).$$

Since θ is increasing, we get

$$s\varepsilon < s\varepsilon,$$

which is a contradiction. Thus

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Now, we show that $d(Tz, z) = 0$ by contradiction. Assume that

$$d(Tz, z) > 0.$$

Since $x_n \to z$ as $n \to \infty$, from Lemma 2.2, we conclude that

$$\frac{1}{s^2} d(z, Tz) \leqslant \lim_{n\to\infty} \sup d(Tx_n, Tz) \leqslant s^2 d(z, Tz).$$

Now, letting $x = x_n$ and $y = z$ in (3.1), we have

$$\theta\left(s^3 d(Tx_n, Tz)\right) \leqslant [\theta(M(x_n, z))]^r, \ \forall n \in \mathbb{N},$$

where

$$M(x_n, z) = \max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2}(d(z, Tx_n) + d(x_n, Tz))\right\}.$$

Taking the limit as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} \sup M(x_n, z) &= \lim_{n\to\infty} \sup \max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2}(d(z, Tx_n) + d(x_n, Tz))\right\} \\ &= d(z, Tz).\end{aligned}$$

Therefore,

$$\theta\left(s^3 d(Tx_n, Tz)\right) \leqslant \left[\theta\left(\max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2}(d(z, Tx_n) + d(x_n, Tz))\right\}\right)\right]^r. \tag{3.7}$$

Taking $n \to \infty$ in (3.7) and using (3.5) and θ_3, we obtain

$$\theta\left[s^3 \frac{1}{s} d(z, Tz)\right] = \theta[s d(z, Tz)] \leqslant \theta\left[s^3 \lim_{n\to\infty} d(Tx_n, Tz)\right] \leqslant [\theta(d(z, Tz))]^r < \theta(d(z, Tz)).$$

By (θ_1), we get

$$s d(z, Tz) < d(z, Tz).$$

This implies that

$$d(z, Tz)(s-1) < 0 \Rightarrow s < 1,$$

which is a contradiction. Hence $Tz = z$.

Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Then we have

$$d(z, u) = d(Tz, Tu) > 0.$$

Letting $x = z$ and $y = u$ in (3.1), we have

$$\theta\left(d\left(z,u\right)\right) = \theta\left(d\left(Tu,Tz\right)\right) \leqslant \theta\left(s^{3}d\left(Tu,Tz\right)\right) \leqslant \left[\theta\left(M\left(z,u\right)\right)\right]^{r},$$

where

$$M\left(z,u\right) = \max\left\{d\left(z,u\right), d\left(z,Tz\right), d\left(u,Tu\right), \frac{1}{2s^{2}}\left(d\left(u,Tz\right)+d\left(z,Tu\right)\right)\right\} = d\left(z,u\right).$$

Therefore, we have

$$\theta\left(d\left(z,u\right)\right) \leqslant \left[\theta\left(d\left(z,u\right)\right)\right]^{r} < \theta\left(d\left(z,u\right)\right),$$

which implies that

$$d\left(z,u\right) < d\left(z,u\right),$$

which is a contradiction. Therefore $u = z$. □

Corollary 3.3. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in]0, 1[$ such that for any $x, y \in X$, we have*

$$d\left(Tx,Ty\right) > 0 \Rightarrow \theta\left[s^{3}d\left(Tx,Ty\right)\right] \leqslant \left[\theta\left(d\left(x,y\right)\right)\right]^{k}.$$

Then T has a unique fixed point.

Example 3.4. Let $X = [1, +\infty[$. Define $d : X \times X \to [0, +\infty[$ by $d\left(x,y\right) = |x-y|^{2}$. Then (X, d) is a b-metric space with coefficient $s = 2$. Define a mapping $T : X \to X$ by

$$T(x) = x^{\frac{1}{4}}.$$

Evidently, $T(x) \in X$. Let $\theta\left(t\right) = e^{\sqrt{t}}$, $r = \frac{1}{\sqrt{2}}$. It is obvious that $\theta \in \Theta$ and $r \in]0, 1[$. Consider the following possibilities:

1. $x, y \in [1, +\infty[$, $y < x$. Then

$$T(x) = x^{\frac{1}{4}},\ T(y) = y^{\frac{1}{4}},\ d\left(Tx,Ty\right) = (x^{\frac{1}{4}} - y^{\frac{1}{4}})^{2}.$$

On the other hand

$$\theta\left[s^{3}d\left(Tx,Ty\right)\right] = e^{\sqrt{8}(x^{\frac{1}{4}}-y^{\frac{1}{4}})}$$

and

$$\begin{aligned} M(x,y) &= \max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right), \frac{1}{2s^{2}}\left(d\left(y,Tx\right)+d\left(x,Ty\right)\right)\right\} \\ &\geqslant d(x,y) = (x-y)^{2} = \left[(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{1}{4}}+y^{\frac{1}{4}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})\right]^{2} \geqslant \left[4(x^{\frac{1}{4}}-y^{\frac{1}{4}})\right]^{2}. \end{aligned}$$

Hence

$$\begin{aligned} \left[\theta\left(d(x,y)\right)\right]^{\frac{1}{\sqrt{2}}} &= \left[e^{\left[(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{1}{4}}+y^{\frac{1}{4}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})\right]}\right]^{\frac{1}{\sqrt{2}}} \\ &\geqslant \left[e^{\left[4(x^{\frac{1}{4}}-y^{\frac{1}{4}})\right]}\right]^{\frac{1}{\sqrt{2}}} = \left[e^{\left[x^{\frac{1}{4}}-y^{\frac{1}{4}}\right]}\right]^{\frac{4}{\sqrt{2}}} = \left[e^{\sqrt{8}\left[x^{\frac{1}{4}}-y^{\frac{1}{4}}\right]}\right]. \end{aligned}$$

This implies that

$$\theta(s^{3}d(Tx,Ty) \leqslant \phi\left[\theta(d(x,Tx))\right]^{\frac{1}{\sqrt{2}}} \leqslant \left[\theta(\max\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\}, d\left(y,Tx\right))\right]^{\frac{1}{\sqrt{2}}}.$$

2. $x < y$ with $x, y \in [1, +\infty[$. By a similar method, we conclude that

$$\theta(s^3 d(Tx, Ty) \leqslant [\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))]^{\frac{1}{\sqrt{2}}}.$$

Hence, the condition (3.1) is satisfied. Therefore, T has a unique fixed point $z = 1$.

Theorem 3.5. *Let* (X, d) *be a complete* b*-metric space and* $T : X \to X$ *be a mapping. Suppose that there exist* $\theta \in \Theta$ *and* $\phi \in \Phi$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta\left[s^3 d(Tx, Ty)\right] \leqslant \phi\left[\theta(M(x,y))\right] \tag{3.8}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s^2} d(y,Tx)\right\}.$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X and define a sequence $\{x_n\}$ by

$$x_{n+1} = Tx_n = T^{n+1}x_0,$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$, then the proof is finished.

We can suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Letting $x = x_{n-1}$ and $y = x_n$ in (3.8), we have

$$\theta\left[d(x_n, x_{n+1})\right] \leqslant \theta\left[s^3 d(x_n, x_{n+1})\right] \leqslant \phi\left[\theta(M(x_{n-1}, x_n))\right], \forall n \in \mathbb{N}, \tag{3.9}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2s^2}\right\} \\ &= \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2s^2}\right\}. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{2s^2} d(x_{n-1}, x_{n+1}) &\leqslant \frac{1}{2s^2}\left[s\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right)\right] \\ &= \frac{1}{2s}\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \\ &\leqslant \frac{1}{2}\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) \leqslant \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}, \end{aligned}$$

we obtain

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then by (3.9), we have

$$\theta(d(x_n, x_{n+1})) \leqslant \phi(\theta(d(x_n, x_{n+1}))) < \theta(d(x_n, x_{n+1})),$$

which is a contradiction. Hence $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$. Thus

$$\theta(d(x_n, x_{n+1})) \leqslant \phi(\theta(d(x_{n-1}, x_n))).$$

Repeating this step, we conclude that

$$\theta(d(x_n, x_{n+1})) \leqslant \phi(\theta(d(x_{n-1}, x_n))) \leqslant \phi^2(\theta(d(x_{n-2}, x_{n-1}))) \leqslant \cdots \leqslant \phi^n \theta(d(x_0, x_1)).$$

From (3.3) and using Lemma 2.6 and (θ_1), we get

$$d\left(x_n, x_{n+1}\right) < d\left(x_{n-1}, x_n\right).$$

Therefore, $d\left(x_n,x_{n+1}\right)_{n\in\mathbb{N}}$ is a monotone strictly decreasing sequence of nonnegative real numbers. Consequently, there exists $\alpha \geqslant 0$ such that

$$\lim_{n\to\infty} d\left(x_{n+1},x_n\right) = \alpha.$$

Now, we claim that $\alpha = 0$. Arguing by contraction, we assume that $\alpha > 0$. Since $d\left(x_n,x_{n+1}\right)_{n\in\mathbb{N}}$ is a nonnegative decreasing sequence, we have

$$d\left(x_n,x_{n+1}\right) \geqslant \alpha, \quad \forall n \in \mathbb{N}.$$

This implies that

$$1 < \theta\left(\alpha\right) \leqslant \theta\left(d\left(x_{n+1}, x_n\right)\right) \leqslant \phi\left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right] \leqslant \cdots \leqslant \phi^n\theta\left(d\left(x_0, x_1\right)\right).$$

Letting $n \to \infty$ and using the properties of ϕ and θ, we get

$$1 < \theta\left(\alpha\right) \leqslant \lim_{n\to\infty} \phi^n\theta\left(d\left(x_0, x_1\right)\right) = 1,$$

which is a contradictions. Thus $\alpha = 0$ and so we have

$$\lim_{n\to\infty} d\left(x_n,x_{n+1}\right) = 0.$$

Next, we shall prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., $\lim_{n,m\to\infty} d\left(x_n,x_m\right) = 0$. Suppose to the contrary. By Lemma 2.3, there is an $\varepsilon > 0$ such that for an integer k there exist two sequences $\left\{n_{(k)}\right\}$ and $\left\{m_{(k)}\right\}$ such that

i) $\varepsilon \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leqslant s\varepsilon$;

ii) $\frac{\varepsilon}{s} \leqslant \lim_{k\to\infty} \inf d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \leqslant s^2\varepsilon$;

iii) $\frac{\varepsilon}{s} \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \leqslant s^2\varepsilon$;

vi) $\frac{\varepsilon}{s^2} \leqslant \lim_{k\to\infty} \inf d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leqslant \lim_{k\to\infty} \sup d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \leqslant s^3\varepsilon$.

From (3.8) and by setting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ we have:

$$\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) \leqslant s\varepsilon. \tag{3.10}$$

Now, letting $x = x_{m_{(k)}}$ and $y = x_{n_{(k)}}$ in (3.8), we obtain

$$\theta\left[s^3 d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right] \leqslant \phi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

Letting $k \to \infty$ in the above inequality and applying the continuity of θ and ϕ and using (3.10), we obtain

$$\theta\left(\frac{\varepsilon}{s^2}s^3\right) = \theta\left(\varepsilon s\right) \leqslant \theta\left(s^3 \lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leqslant \phi\left[\theta\left(\lim_{k\to\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right].$$

By Lemma 2.6, we get

$$\theta(s\varepsilon) \leqslant \phi\left[\theta(s\varepsilon)\right] < \theta(s\varepsilon).$$

Since θ is increasing, we get

$$s\varepsilon < s\varepsilon,$$

which is a contradiction. Thus

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness of (X, d), there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = 0.$$

Now, we show that $d(Tz, z) = 0$ by contradiction, we assume that

$$d(Tz, z) > 0.$$

Since $x_n \to z$ as $n \to \infty$, from Lemma 2.2, we conclude that

$$\frac{1}{s^2} d(z, Tz) \leqslant \lim_{n\to\infty} \sup d(Tx_n, Tz) \leqslant s^2 d(z, Tz).$$

Now, letting $x = x_n$ and $y = z$ in (3.8), we have

$$\theta\left(s^3 d(Tx_n, Tz)\right) \leqslant [\theta(M(x_n, z))]^r, \ \forall n \in \mathbb{N},$$

where

$$M(x_n, z) = \max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2}(d(z, Tx_n) + d(x_n, Tz))\right\}.$$

As in the proof of Theorem 3.2, we have

$$\lim_{n\to\infty} \sup M(x_n, z) = d(z, Tz).$$

Therefore,

$$\theta\left(s^3 d(Tx_n, Tz)\right) \leqslant \phi\left[\theta\left(\max\left\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), \frac{1}{2s^2}(d(z, Tx_n) + d(x_n, Tz))\right\}\right)\right]. \tag{3.11}$$

Letting $n \to \infty$ in (3.11) and using the properties of ϕ and θ, we obtain

$$\theta\left[s^3 \frac{1}{s} d(z, Tz)\right] = \theta[s d(z, Tz)] \leqslant \theta\left[s^3 \lim_{n\to\infty} d(Tx_n, Tz)\right] \leqslant \phi[\theta(d(z, Tz))] < \theta(d(z, Tz)).$$

By (θ_1), we get

$$s d(z, Tz) < d(z, Tz).$$

This implies that

$$d(z, Tz)(s-1) < 0 \Rightarrow s < 1,$$

which is a contradiction. Hence $Tz = z$.

Now, suppose that $z, u \in X$ are two fixed points of T such that $u \neq z$. Therefore, we have

$$d(z, u) = d(Tz, Tu) > 0.$$

Letting $x = z$ and $y = u$ in (3.8), we have

$$\theta(d(z, u)) = \theta(d(Tu, Tz)) \leqslant \theta\left(s^3 d(Tu, Tz)\right) \leqslant \phi[\theta(M(z, u))],$$

where

$$M(z, u) = \max\left\{d(z, u), d(z, Tz), d(u, Tu), \frac{1}{2s^2}(d(u, Tz) + d(z, Tu))\right\} = d(z, u).$$

Therefore, we have

$$\theta\left(d\left(z,u\right)\right)\leqslant\phi\left[\theta\left(d\left(z,u\right)\right)\right]<\theta\left(d\left(z,u\right)\right),$$

which implies that

$$d\left(z,u\right)<d\left(z,u\right).$$

This is a contradiction. Therefore $u=z$. □

It follows from Theorem 3.5 that we obtain the followed fixed point theorems for θ-ϕ-Kannan-type contraction and θ-ϕ-Reich-type contraction. The results presented in the paper improve and extend the corresponding results due to Kannan-type contraction and Reich-type contraction on rectangular b-metric space.

Theorem 3.6. *Let* (X,d) *be a complete* b*-metric space and* $T:X\to X$ *be a Kannan-type contraction. Then* T *has a unique fixed point.*

Proof. Since T is a Kannan-type contraction, there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that

$$\begin{aligned}\theta\left[s^3d\left(Tx,Ty\right)\right]&\leqslant\phi\left[\theta\left(\frac{d\left(Tx,x\right)+d\left(Ty,y\right)}{2}\right)\right]\\&\leqslant\phi\left[\theta\left(\max\{d\left(x,Tx\right),d\left(y,Ty\right)\}\right)\right]\\&\leqslant\phi\left[\theta\left(\max\left\{d(x,y),d\left(x,Tx\right),d\left(y,Ty\right),\frac{1}{2s^2}\left(d\left(y,Tx\right)+d\left(x,Ty\right)\right)\right\}\right)\right].\end{aligned}$$

Therefore, T is a θ-ϕ-contraction. As in the proof of Theorem 3.4, we conclude that T has a unique fixed point. □

Theorem 3.7. *Let* (X,d) *be a complete* b*-metric space and* $T:X\to X$ *be a Reich-type contraction. Then* T *has a unique fixed point.*

Proof. Since T is a Reich-type contraction, there exist $\theta\in\Theta$ and $\phi\in\Phi$ such that

$$\begin{aligned}\theta\left[s^3d\left(Tx,Ty\right)\right]&\leqslant\phi\left[\theta\left(\frac{d\left(x,y\right)+d\left(Tx,x\right)+d\left(Ty,y\right)}{3}\right)\right]\\&\leqslant\phi\left[\theta\left(\max\left\{d(x,y),d\left(x,Tx\right),d\left(y,Ty\right),\frac{1}{2s^2}\left(d\left(y,Tx\right)+d\left(x,Ty\right)\right)\right\}\right)\right].\end{aligned}$$

Therefore, T is a θ-ϕ-contraction. As in the proof of Theorem 3.5, we conclude that T has a unique fixed point. □

Corollary 3.8. *Let* (X,d) *be a complete b-rectangular metric space and* $T:X\to X$ *be a Kannan type mapping, i.e., there exists* $\alpha\in\left]0,\frac{1}{2}\right[$ *such that for all* $x,y\in X$,

$$d\left(Tx,Ty\right)>0\Rightarrow s^3d\left(Tx,Ty\right)\leqslant\alpha\left[\left(d\left(Tx,x\right)+d\left(Ty,y\right)\right)\right].$$

Then T *has a unique fixed point.*

Proof. Let $\theta(t)=e^t$ for all $t\in]0,+\infty[$, and $\phi\left(t\right)=t^{2\alpha}$ for all $t\in[1,+\infty[$. Clearly $\phi\in\Phi$ and $\theta\in\Theta$. We prove that T is a θ-ϕ-Kannan-type contraction. Indeed,

$$\begin{aligned}\theta\left(s^3d\left(Tx,Ty\right)\right)=e^{s^3d\left(Tx,Ty\right)}&\leqslant e^{\alpha\left(d\left(Tx,x\right)+d\left(Ty,y\right)\right)}\\&=e^{2\alpha\left(\frac{d\left(Tx,x\right)+d\left(Ty,y\right)}{2}\right)}\end{aligned}$$

$$= \left[e^{\left(\frac{d(Tx,x)+d(Ty,y)}{2}\right)} \right]^{2\alpha} = \phi\left[\theta\left(\frac{d(Tx,x)+d(Ty,y)}{2}\right)\right].$$

As in the proof of Theorem 3.6, T has a unique fixed point $x \in X$. □

Corollary 3.9. *Let (X, d) be a complete b-rectangular metric space and $T : X \to X$ be a Reich type mapping, i.e., there exists $\lambda \in \left]0, \frac{1}{3}\right[$ such that for all $x, y \in X$,*

$$d(x,y) > 0 \Rightarrow s^3 d(Tx,Ty) \leqslant \lambda\left[(d(x,y) + d(Tx,x) + d(Ty,y))\right].$$

Then T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in]0, +\infty[$, and $\phi(t) = t^{3\lambda}$ for all $t \in [1, +\infty[$.

We prove that T is a θ-ϕ-Reich type contraction. Indeed,

$$\begin{aligned}\theta\left(s^2 d(Tx,Ty)\right) = e^{s^2 d(Tx,Ty)} &\leqslant e^{\lambda(d(x,y)+d(Tx,x)+d(Ty,y))}\\ &= e^{3\lambda\left(\frac{d(x,y)+d(Tx,x)+d(Ty,y)}{3}\right)}\\ &= \phi\left[\theta\left(\frac{d(x,y)+d(Tx,x)+d(Ty,y)}{3}\right)\right].\end{aligned}$$

As in the proof of Theorem 3.6, T has a unique fixed point $x \in X$. □

Corollary 3.10. *Let (X, d) be a complete b-metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $r \in]0, 1[$ such that for all $x, y \in X$,*

$$d(Tx,Ty) > 0 \Rightarrow \theta\left[s^2 d(Tx,Ty)\right] \leqslant \left[\theta(M(x,y))\right]^r,$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2s^2}(d(y,Tx) + d(x,Ty))\right\}.$$

Then T has a unique fixed point.

Proof. Taking $\phi(t) = t^r \in \Phi$ with $r \in]0, 1[$, we conclude that T is a θ-ϕ-contraction. As in the proof of Theorem 3.4, T has a unique fixed point. □

Very recently, Kari et al. [14, Theorem 1] proved the result on (α, η)-complete rectangular b-metric spaces. In this paper, we prove this result in complete b-metric spaces.

Corollary 3.11. *Let $d(X, d)$ be a complete b-rectangular metric space with parameter $s > 1$ and let T be a self mapping on X. If for all $x, y \in X$ with $d(Tx,Ty) > 0$ we have*

$$\theta\left(s^3.d(Tx,Ty)\right) \leqslant \phi\left[\theta\left(\beta_1 d(x,y) + \beta_2 d(Tx,x) + \beta_3 d(Ty,y) + \beta_4 d(y,Tx)\right)\right],$$

where $\theta \in \Theta$, $\phi \in \Phi$, $\beta_i \geqslant 0$ for $i \in \{1, 2, 3, 4\}$, $\sum\limits_{i=0}^{i=4} \beta_i \leqslant 1$, then T has a unique fixed point.

Proof. We prove that T is a θ-ϕ-contraction. Indeed,

$$\begin{aligned}\theta\left(s^2 \cdot d(Tx,Ty)\right) &\leqslant \phi\left[\theta\left(\beta_1 d(x,y) + \beta_2 d(Tx,x) + \beta_3 d(Ty,y) + \frac{\beta_4}{2s^2}(d(y,Tx) + d(x,Ty))\right)\right]\\ &\leqslant \phi\left[\theta(\beta_1+\beta_2+\beta_3+\beta_4)\left(\max\{d(x,y), d(Tx,x), d(Ty,y), \frac{1}{2s^2}(d(y,Tx)+d(x,Ty))\}\right)\right]\\ &\leqslant \phi\left[\theta\left(\max\{d(x,y), d(Tx,x), d(Ty,y), \frac{1}{2s^2}(d(y,Tx)+d(x,Ty))\}\right)\right].\end{aligned}$$

As in the proof of Theorem 3.4, T has a unique fixed point. □

Example 3.12. Let $X = A \cup B$, where $A = \{\frac{1}{6^{n-1}}; n \in \mathbb{N}\}$ and $B = \{0\}$. Define $d : X \times X \to [0, +\infty[$ by

$$d(x, y) = (|x - y|)^2 .$$

Then (X, d) is a b-metric space with coefficient $s = 2$.

Define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{6^n}, & \text{if } x \in \{\frac{1}{6^{n-1}}\}, \\ 1, & \text{if } x = 0. \end{cases}$$

Then $T(x) \in X$. Let $\theta(t) = \sqrt{t} + 1$, $\phi(t) = \frac{t+1}{2}$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. Consider the following possibilities.

Case 1: $x = \frac{1}{6^{n-1}}, y = \frac{1}{6^{n-1}}$ for $m > n \geqslant 0$. Then

$$d(Tx, Ty) = \left(\frac{1}{6^n} - \frac{1}{6^m}\right)^2 = \left(\frac{6^m - 6^n}{6^{m+n}}\right)^2 .$$

So

$$\theta\left(s^3 d(Tx, Ty)\right) = \sqrt{8}\left(\frac{6^m - 6^n}{6^{m+n}}\right) + 1$$

and

$$\phi\left[\theta(d(x, y))\right] = \phi\left[\theta\left(\frac{6^{m-1} - 6^{n-1}}{6^{m+n-2}}\right)^2\right] = 3\left(\frac{6^m - 6^n}{6^{m+n-2}}\right) + 1.$$

On the other hand,

$$\theta(s^3 d(Tx, Ty) - \phi\left[\theta(d(x, y))\right] = \sqrt{8}\left(\frac{6^m - 6^n}{6^{m+n}}\right) + 1 - 3\left(\frac{6^m - 6^n}{6^{m+n}}\right) + 12 = \sqrt{8} - 3\left[\left(\frac{6^m - 6^n}{6^{m+n}}\right)\right] \leqslant 0.$$

This implies that

$$\theta(s^3 d(Tx, Ty) \leqslant \phi\left[\theta(d(x, y))\right] \leqslant \phi\left[\theta(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2s^2}\right\})\right].$$

Case 2: $x = \frac{1}{6^{n-1}}, y = 0$.

Then $T(x) = \frac{1}{6^n}$, $T(y)0$, then $d(Tx, Ty) = (\frac{1}{6^n})^2$. So we have

$$\theta(s^3 d(Tx, Ty) = \frac{\sqrt{8}}{6^n} + 1.$$

Thus

$$M(x, y) = \phi\left[\theta(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(y, Tx) + d(x, Ty)}{2s^2}\right\})\right] \geqslant d(x, y) = \left(\frac{1}{6^{n-1}}\right)^2$$

and

$$\phi\left[\theta\left(d(x, y)\right)\right] = \frac{3}{6^n} + 1.$$

On the other hand,

$$\theta(s^3 d(Tx, Ty) - \phi\left[\theta\left(d(x, y)\right)\right] = \frac{\sqrt{8}}{6^n} + 1 - \frac{3}{6^n} + 1 = \frac{\sqrt{8} - 3}{6^n} \leqslant 0.$$

This implies that

$$\begin{aligned} \theta(s^3 d(Tx, Ty) \leqslant \phi\left[\theta(d(y, Ty))\right] \leqslant \phi\left[\theta(d(1, \frac{1}{3}))\right] &\leqslant \phi\left[\theta(d(y, Ty)\right] \\ &\leqslant \phi\left[\theta(\max\{d(x, y), d(x, Tx), d(y, Ty)\}, d(y, Tx))\right]. \end{aligned}$$

Hence the condition (3.8) is satisfied. Therefore, T has a unique fixed point $z = 1$.

4. Application to nonlinear integral equations

In this section, we endeavor to apply Theorems 3.2 and 3.4 to prove the existence and uniqueness of the integral equation of Fredholm type:

$$x(t) = \lambda \int_a^b K(t, r, x(r))ds, \tag{4.1}$$

where $a, b \in \mathbb{R}$, $x \in C([a, b], \mathbb{R})$ and $K : [a, b]^2 \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Theorem 4.1. *Consider the nonlinear integral equation problem* (4.1) *and assume that the kernel function* K *satisfies the condition* $|K(t, r, x(r)) - K(t, r, y(r))| \leqslant \frac{1}{s^2}(|x(t) - y(t)|)$ *for all* $t, r \in [a, b]$ *and* $x, y \in \mathbb{R}$. *Then the equation* (4.1) *has a unique solution* $x \in C([a, b]$ *for some constant* λ *depending on the constant* s.

Proof. Let $X = C([a, b]$ and $T : X \to X$ be defined by

$$T(x)(t) = \lambda \int_a^b K(t, r, x(r))ds,$$

for all $x \in X$. Let $d : X \times X \to [0, +\infty[$ be given by

$$d(x, y) = \left(\max_{t\in[a,b]} |x(t) - y(t)|\right)^2$$

for all $x, y \in X$. It is clear that (X, d) is a complete b-metric space.

We will find the condition on λ under which the operator has a unique fixed point which will the solution of the integral equation (4.1). Assume that $x, y \in X$ and $t, r \in [a, b]$. Then we get

$$\begin{aligned}
|Tx(t) - Ty(t)|^2 &= |\lambda|^s \left(|\int_a^b K(t, r, x(r))dr - \int_a^b K(t, r, y(r))dr|\right)^2 \\
&= |\lambda^2|| \int_a^b K(t, r, x(r)) - K(t, r, y(r))dr|^2 \\
&\leqslant |\lambda|^2 \int_a^b |K(t, r, x(r)) - K(t, r, y(r))dr|^2 \\
&\leqslant |\lambda|^2 \int_a^b \left(\frac{1}{s^2}(|x(r) - y(r)|)\,dr\right)^2 \\
&= |\lambda|^2 \frac{1}{s^4}\left[\int_a^b ((|x(r)| - |y(r)|))\,dr\right]^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\max_{t\in[a,b]} (|Tx(t) - Ty(t)|) &= \max_{t\in[a,b]} |\lambda|^2 \int_a^b |K(t, r, x(r)) - K(t, r, y(r))dr|^s \\
&\leqslant \max_{t\in[a,b]} \frac{1}{s^2}|\lambda|^2 \int_a^b ((|x(r) - y(r)|)\,dr)^2 \leqslant |\lambda|^2 \frac{1}{s^4} \int_a^b \left(\left(\max_{r\in[a,b]} |x(r) - y(r)|\right) dr\right)^2.
\end{aligned}$$

Since by the definition of the b-rectangular metric space, we have $d(Tx, Ty) > 0$ and $d(x, y) > 0$ for all $x \neq y$, we can take natural exponential sides and get

$$e^{[s^3 d(Tx,Ty)]} = e^{\left[s^3|\lambda|^2 \max_{t\in[a,b]} \int_a^b |K(t,r,x(r))-K(t,r,y(r))dr|^2\right]}$$

$$\leqslant e^{\left[(\frac{|\lambda|}{s})^2 \int_a^b \left(\left(\max_{r\in[a,b]} |x(r)-y(r)|\right)dr\right)^2\right]} = \left[e^{\left[\int_a^b \left(\left(\max_{r\in[a,b]} |x(r)-y(r)|\right)dr\right)^2\right]}\right]^{(\frac{|\lambda|}{s})^2},$$

provided that $|\lambda| < s$, which implies that

$$e^{[s^3 d(Tx,Ty)]} \leqslant \left[e^{\left[\int_a^b \left(\left(\max_{r\in[a,b]} |x(r)-y(r)|\right)dr\right)^2\right]}\right]^{k}.$$

Hence

$$F\left(s^3 d(Tx,Ty)\right) + \phi(d(x,y)) \leqslant F(d(x,y))$$

for all $x, y \in X$ with $\theta(t) = e^t$, $\phi(t) = t^k$ and $k = (\frac{|\lambda|}{s})^2$. It follows that T satisfies the conditions (3.1) and (3.8). Therefore there exists a unique solution of the nonlinear Fredholm inequality (4.1). □

5. Conclusion

We defined θ-ϕ-contraction on a b-metric space into itself by extending θ-ϕ-contraction introduced Zheng et al. in metric space and also we proved θ-type theorem in the setting of b-metric spaces as well as θ-ϕ-type theorem in the framework of b-rectangular metric spaces. Moreover, we gave some applications to nonlinear integral equations. We also gave illustrative examples to exhibit the utility of our results.

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