



New Ostrowski type inequalities pertaining to conformable fractional operators



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Abstract

The advancements of integral inequalities with the help of fractional operators have recently been the focus of attention in the theory of inequalities. In this study, we first review some fundamental concepts, and then using k -conformable fractional integrals, we establish a new integral identity for differentiable functions. Then, considering this identity as an auxiliary result, several Ostrowski-type inequalities are presented for functions whose modulus of the first derivatives are quasi-convex. The obtained results represent generalizations as well as refinements for some published results.

Keywords: Ostrowski inequality, quasi-convexity, k -fractional conformable integral, E-beta functions, E-gamma functions.

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1. Introduction

The term "convexity" was utilized and discussed generally in the tremendous book by Hardy et al. (see [9]). As of late, the theory of convexity assumes an exceptionally entrancing and astonishing part in the domain of science, hence anyone working, especially in the field of inequalities cannot ignore its importance and significance. The theory of convex analysis provides powerful principles and techniques to study a wide class of problems in both pure and applied mathematics. Numerous paragons of mathematics and applied sciences consistently attempt to utilize and avail the novel thoughts for the pleasure and beautification of convexity theory. This theory plays a consequential and crucial role in applied mathematics, especially in control theory, nonlinear programming, financial mathematics, mathematical statistics, optimization theory, and functional analysis. For the attention of the readers, we encourage them to see the references [11, 13, 17, 19].

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The theory of convexity additionally assumed a significant and meaningful part in the development of the concept of inequalities. In the literature of inequalities, the Ostrowski type inequality [18] appears in different forms for various convex functions. In [23] for the first time, Ostrowski type inequality was studied for Riemann–Liouville fractional integrals and after it, researchers started to get many versions of this for different kinds of fractional integral operators and functions. In the recent past, many generalizations for Ostrowski type inequality have been performed via different directions like on coordinates, on quantum calculus, on different fractional integral operators like Riemann–Liouville, Katugampola, Caputo, Caputo Fabrizio, ψ -generalized fractional operator, etc. For more information, we refer interested readers to go through [2, 3, 15, 25, 28].

Fractional analysis, as a rapidly developing area, has been a tool to bring new derivatives and integrals into the literature with the effort put forward by many researchers in recent years. Fractional calculus has gained considerable popularity and importance in the past few decades in diverse fields of science and engineering. The name fractional calculus does not mean the study about the calculus of fractions and its name also does not suggest that it is the study of any fraction of any calculus-differential and integral. In this way “The Fractional Calculus” is a name for the theory of derivatives and integrals of arbitrary order. Fractional calculus provides a ray of hope in predicting some natural phenomena like the severity of earthquakes, floods, tsunami, landslides, etc. Its prediction helps us to prepare well against such calamities and save many lives. It also helps in finding solutions to some of the most complicated problems related to the industry, health sector, weather forecasting, etc. Consequently, it makes the job easier in many senses. Many mathematicians used fractional mathematical models to predict the rate of infection of Covid-19. This helps the govt of many countries to implement systematic lockdown and control the rate of infection of Covid-19 further. For the attention of the readers see the following references, [1, 4–6, 12, 21, 22, 27–29]. Mubeen and Iqbal [16] investigated the improved version of integral representation for Appel k-series.

Motivated by the ongoing research activities in the direction of inequalities and fractional calculus, first we discuss in Section 2, some necessary known definitions and concepts. In Section 3, we investigate some new Ostrowski type inequalities employing k-fractional conformable integral for quasi-convex functions. Finally, in section Section 4, we give a conclusion.

2. Preliminaries

Here we discuss some necessary known definitions and concept.

Definition 2.1 ([17]). A mapping $\mathfrak{B} : \mathbb{J} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be a convex function if

$$\mathfrak{B}(\gamma v_1 + (1 - \gamma)v_2) \leq \gamma \mathfrak{B}(v_1) + (1 - \gamma)\mathfrak{B}(v_2), \quad (2.1)$$

holds true for all $v_1, v_2 \in \mathbb{J}$ and $\gamma \in [0, 1]$.

Note: If (2.1) is reversed, then \mathfrak{B} is said to be a concave function.

Definition 2.2 ([24]). A mapping $\mathfrak{B} : \mathbb{J} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called a quasi-convex on \mathbb{J} if

$$\mathfrak{B}(\gamma v_2 + (1 - \gamma)v_1) \leq \max\{\mathfrak{B}(v_1), \mathfrak{B}(v_2)\},$$

holds true for all $v_1, v_2 \in \mathbb{J}$ and $\gamma \in [0, 1]$.

Note: It is clear that any convex function is a quasi-convex function.

Definition 2.3 ([14]). Suppose $\mathfrak{B} \in L_1[v_1, v_2]$, then left and right Riemann-Liouville fractional integrals $\mathfrak{J}_{v_1+}^\alpha \mathfrak{B}$ and $\mathfrak{J}_{v_2-}^\alpha \mathfrak{B}$ of order $\alpha > 0$ with $v_1, v_2 \geq 0$ are defined by

$$\mathfrak{J}_{v_1+}^\alpha \mathfrak{B}(x) = \frac{1}{\Gamma(\alpha)} \int_{v_1}^x (x - \gamma)^{\alpha-1} \mathfrak{B}(\gamma) d\gamma, \quad x > v_1, \quad (2.2)$$

and

$$\mathfrak{J}_{v_2^-}^\alpha \mathfrak{B}(x) = \frac{1}{\Gamma(\alpha)} \int_x^{v_2} (\gamma - x)^{\alpha-1} \mathfrak{B}(\gamma) d\gamma, \quad x < v_2, \tag{2.3}$$

respectively.

$$\Gamma(\alpha) = \int_0^\infty e^{-\gamma} \gamma^{\alpha-1} d\gamma, \quad [\text{Here } \alpha\Gamma(\alpha) = \Gamma(\alpha + 1)].$$

If $\alpha = 0$, then

$$\mathfrak{J}_{v_1^+}^0 \mathfrak{B}(x) = \mathfrak{J}_{v_2^-}^0 \mathfrak{B}(x) = \mathfrak{B}(x),$$

$$B(v_1, v_2) = \frac{\Gamma(v_1) \Gamma(v_2)}{\Gamma(v_1 + v_2)} = \int_0^1 \gamma^{v_1-1} (1 - \gamma)^{v_2-1} d\gamma, \quad \mathbb{R}(v_1), \mathbb{R}(v_2) > 0.$$

Definition 2.4 ([8]). If $k > 0$, then k -Gamma function Γ_k is defined as

$$\Gamma_k(\alpha) = \lim_{m \rightarrow \infty} \frac{m! k^m (mk)^{\frac{\alpha}{k}-1}}{(\alpha)_{m,k}}.$$

If $\text{Re}(\alpha) > 0$, the k -Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{\gamma}{k}} \gamma^{\alpha-1} d\gamma,$$

$\alpha\Gamma_k(\alpha) = \Gamma_k(\alpha + k)$, where $\Gamma_k(\cdot)$ stands for the k -gamma function.

In [23], Set proved some Ostrowski type inequalities involving Riemann–Liouville fractional integral operator as follows:

Lemma 2.5. Let $\mathfrak{B} : \mathbb{J} = [v_1, v_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of an interval \mathbb{J} in \mathbb{R} , where $v_1, v_2 \in \mathbb{J}$, $v_1 < v_2$. If $\mathfrak{B}' \in L_1[v_1, v_2]$, then for $\gamma \in [0, 1]$ and $\alpha \in \mathbb{R}^+$, the following identity holds true

$$\begin{aligned} & \frac{(x - v_1)^\alpha + (v_2 - x)^\alpha}{v_2 - v_1} \mathfrak{B}(x) - \frac{\Gamma(\alpha + 1)}{v_2 - v_1} \{ \mathfrak{J}_{x^-}^\alpha \mathfrak{B}(v_1) + \mathfrak{J}_{x^+}^\alpha \mathfrak{B}(v_2) \} \\ &= \frac{(x - v_1)^{\alpha+1}}{v_2 - v_1} \int_0^1 \gamma^\alpha \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) d\gamma - \frac{(v_2 - x)^{\alpha+1}}{v_2 - v_1} \int_0^1 \gamma^\alpha \mathfrak{B}'(\gamma x + (1 - \gamma)v_2) d\gamma, \end{aligned}$$

where $\Gamma(\cdot)$ is defined as earlier.

In [16], the Riemann–Liouville k -fractional integrals are defined as bellow

$$\mathfrak{J}_{k,v_1^+}^\alpha \mathfrak{B}(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{v_1}^x (x - \gamma)^{\frac{\alpha}{k}-1} \mathfrak{B}(\gamma) d\gamma, \quad x > v_1,$$

and

$$\mathfrak{J}_{k,v_2^-}^\alpha \mathfrak{B}(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^{v_2} (\gamma - x)^{\frac{\alpha}{k}-1} \mathfrak{B}(\gamma) d\gamma, \quad x < v_2.$$

Setting $k = 1$, in the above definition the Riemann–Liouville fractional integral operator is recovered. And the lassical integral is recaptured, if we set $\alpha = k = 1$.

Jarad et al. [10] investigated and presented a new generalized fractional operator and also studied some nice correlations between different fractional operators such as Hadamard fractional integral, Riemann–Liouville fractional integral and generalized fractional integral operator and many more.

The left and right FCIO (Fractional Conformable Integral Operators) are defined as

$${}^\beta \mathfrak{J}_{v_1^+}^\alpha \mathfrak{B}(x) = \frac{1}{\Gamma(\beta)} \int_{v_1}^x \left(\frac{(x - v_1)^\alpha - (\gamma - v_1)^\alpha}{\alpha} \right)^{\beta-1} \frac{\mathfrak{B}(\gamma)}{(\gamma - v_1)^{1-\alpha}} d\gamma, \quad x > v_1, \tag{2.4}$$

and

$${}^{\beta} \mathfrak{J}_{v_2^-}^{\alpha} \mathfrak{B}(x) = \frac{1}{\Gamma(\beta)} \int_x^{v_2} \left(\frac{(v_2 - x)^{\alpha} - (v_2 - \gamma)^{\alpha}}{\alpha} \right)^{\beta-1} \frac{\mathfrak{B}(\gamma)}{(v_2 - \gamma)^{1-\alpha}} d\gamma, \quad x < v_2, \quad (2.5)$$

respectively, for $\alpha > 0$ and $\Re(\beta) > 0$. Again, if we set $v_1 = v_2 = 0$ and $\alpha = 1$, then (2.4) and (2.5) reduces to (2.2) and (2.3), respectively.

Now, the k -fractional conformable integral of generalized are defined in [20] as

$${}^{\beta} \mathfrak{J}_{v_1^+}^{\alpha} \mathfrak{B}(x) = \frac{1}{k\Gamma_k(\beta)} \int_{v_1}^x \left(\frac{(x - v_1)^{\alpha} - (\gamma - v_1)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathfrak{B}(\gamma)}{(\gamma - v_1)^{1-\alpha}} d\gamma, \quad x > v_1,$$

and

$${}^{\beta} \mathfrak{J}_{v_2^-}^{\alpha} \mathfrak{B}(x) = \frac{1}{k\Gamma_k(\beta)} \int_x^{v_2} \left(\frac{(v_2 - x)^{\alpha} - (v_2 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} \frac{\mathfrak{B}(\gamma)}{(v_2 - \gamma)^{1-\alpha}} d\gamma, \quad x < v_2,$$

where $\alpha > 0$ and $\Re(\beta) > 0$.

Motivated by the ongoing research activities in the direction of inequalities and fractional calculus, the main objective of this paper is to investigate some new Ostrowski type inequalities employing k -fractional conformable integral for quasi-convex functions.

3. Main results

In this section, first we derive a new fractional equality, that will act as an auxiliary result for the main results of this paper. Next, using this equality and some well-known integral inequalities such as Hölder inequality, Hölder–İşcan inequality, power-mean inequality and improved power-mean inequality, some new bounds of Ostrowski type inequality for differentiable quasi convex functions are presented.

Lemma 3.1. Let $\mathfrak{B} : \mathbb{J} = [v_1, v_2] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of an interval \mathbb{J} in \mathbb{R} , with $v_1, v_2 \in \mathbb{J}$ and $v_1 < v_2$. If $\mathfrak{B}' \in L_1[v_1, v_2]$, $\gamma \in [0, 1]$ and $\alpha, \beta \in \mathbb{R}^+$, then the following fractional identity holds true

$$\begin{aligned} & \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[{}^{\beta} \mathfrak{J}_{x^-}^{\alpha} \mathfrak{B}(v_1) + {}^{\beta} \mathfrak{J}_{x^+}^{\alpha} \mathfrak{B}(v_2) \right] \\ &= \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \right] \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) d\gamma \\ &+ \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \right] \mathfrak{B}'(\gamma x + (1 - \gamma)v_2) d\gamma. \end{aligned} \quad (3.1)$$

Proof. Let,

$$I_1 = \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \right] \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) d\gamma.$$

By using integrating by parts, we have

$$\begin{aligned} I_1 &= \left(\left(\frac{1 - (1 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}} \frac{\mathfrak{B}(\gamma x + (1 - \gamma)v_1)}{x - v_1} \right) \Big|_0^1 \\ &- \int_0^1 \frac{\mathfrak{B}(\gamma x + (1 - \gamma)v_1)}{x - v_1} \frac{\beta}{k} \left(\frac{1 - (1 - \gamma)^{\alpha}}{\alpha} \right)^{\frac{\beta}{k}-1} (1 - \gamma)^{\alpha-1} d\gamma. \end{aligned}$$

This gives,

$$I_1 = \frac{1}{\alpha^{\frac{\beta}{k}} x - v_1} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{(x - v_1)^{\frac{\alpha\beta}{k} + 1}} \mathfrak{J}_{x^-}^{\alpha} \mathfrak{B}(v_1). \quad (3.2)$$

Similarly, we have

$$I_2 = -\frac{1}{\alpha^{\frac{\beta}{k}} v_2 - x} \mathfrak{B}(x) + \frac{\Gamma_k(\beta + k)}{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}} \mathfrak{J}_{x^+}^{\alpha} \mathfrak{B}(v_2). \quad (3.3)$$

Now, multiplying (3.2) by $\frac{(x-v_1)^{\frac{\alpha\beta}{k}+1}}{v_2-v_1}$ and (3.3) by $\frac{(v_2-x)^{\frac{\alpha\beta}{k}+1}}{v_2-v_1}$ and subtracting these equations, we get the desired results. \square

Remark 3.2. If we set $k=1$, in Lemma 3.1, then [7, Lemma 2], is recovered.

Theorem 3.3. Let $\mathfrak{B} : [v_1, v_2] \rightarrow \mathbb{R}$ be a differentiable mapping on \mathbb{J} of the interval $\mathbb{J} \in \mathbb{R}$ with $v_1, v_2 \in \mathbb{J}$ and $v_1 < v_2$. If $\mathfrak{B}' \in L_1[v_1, v_2]$ and $|\mathfrak{B}'|$ is quasi-convex function on $[v_1, v_2]$, then the following integral inequality holds true

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[\mathfrak{J}_{x^-}^{\alpha} \mathfrak{B}(v_1) + \mathfrak{J}_{x^+}^{\alpha} \mathfrak{B}(v_2) \right] \right| \\ & \leq \frac{B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right)}{(v_2 - v_1) \alpha^{\frac{\beta}{k} + 1}} \left[(x - v_1)^{\frac{\alpha\beta}{k} + 1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_1)| \right\} \right. \\ & \quad \left. + (v_2 - x)^{\frac{\alpha\beta}{k} + 1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_2)| \right\} \right], \end{aligned} \quad (3.4)$$

where $\alpha, \beta, k > 0$ and $B(\cdot, \cdot), \Gamma(\cdot)$ is the E-beta and E-gamma functions as defined earlier.

Proof. Let \mathfrak{N} be the left side of Lemma 3.1. Now using the Lemma 3.1 and quasi-convexity of $|\mathfrak{B}'|$, we have

$$\begin{aligned} |\mathfrak{N}| & \leq \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)| d\gamma \\ & \quad + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)| d\gamma. \end{aligned} \quad (3.5)$$

By the quasi-convexity of $|\mathfrak{B}'|$, we have

$$\begin{aligned} & \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] |\mathfrak{B}'(\gamma x + (1 - \gamma)u)| d\gamma \\ & \leq \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_1)| \right\} d\gamma, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} & \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)| d\gamma \\ & \leq \int_0^1 \left[\left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \right] \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_2)| \right\} d\gamma. \end{aligned} \quad (3.7)$$

Now, changing of variable with $y = (1 - \gamma)^\alpha$, we can get

$$\int_0^1 \left[\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right]^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 [1 - (1 - \gamma)^\alpha]^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}}} B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right). \tag{3.8}$$

Using (3.6), (3.7) and (3.8) in (3.5) we have the desired proof. □

Remark 3.4. Under the same suppositions of Theorem 3.3, if we choose $|\mathfrak{B}'(x)| \leq M$ for $x \in [v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[\beta \mathfrak{J}_{x^-}^\alpha \mathfrak{B}(v_1) + \beta \mathfrak{J}_{x^+}^\alpha \mathfrak{B}(v_2) \right] \right| \\ & \leq \frac{B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right)}{(v_2 - v_1) \alpha^{\frac{\beta}{k} + 1}} M \left[(x - v_1)^{\frac{\alpha\beta}{k} + 1} + (v_2 - x)^{\frac{\alpha\beta}{k} + 1} \right]. \end{aligned} \tag{3.9}$$

Remark 3.5. Under the same suppositions of Theorem 3.3, if we take $k = 1$, then [7, Theorem 3], is recovered.

Remark 3.6. Under the same suppositions of Remark 3.4, if we set $k = 1$, then [7, Remark 4], is recovered.

Theorem 3.7. Let $\mathfrak{B} : [v_1, v_2] \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of the interval $\mathbb{J} \in \mathbb{R}$ for $v_1, v_2 \in \mathbb{J}$ with $v_1 < v_2$. If $\mathfrak{B}' \in L_1[v_1, v_2]$ and $|\mathfrak{B}'|^q$ is quasi-convex function on $[v_1, v_2]$, where $q > 1$, $q^{-1} + p^{-1} = 1$ and $\alpha, B \in \mathbb{R}^+$, then the following fractional integral inequality holds true

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[\beta \mathfrak{J}_{x^-}^\alpha \mathfrak{B}(v_1) + \beta \mathfrak{J}_{x^+}^\alpha \mathfrak{B}(v_2) \right] \right| \\ & \leq \left(\frac{B\left(\frac{1}{\alpha}, \frac{\beta p}{k} + 1\right)}{\alpha^{\frac{\beta p}{k} + 1}} \right)^{\frac{1}{p}} \left[\frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_1)| \right\} \right. \\ & \quad \left. + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_2)| \right\} \right], \end{aligned} \tag{3.10}$$

where $\alpha, B, k > 0$.

Proof. From Lemma 3.1 and the well-known Hölder’s inequality, we have

$$\begin{aligned} |\mathfrak{K}| & \leq \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \times \left[\int_0^1 |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \right]^{\frac{1}{q}} \\ & \quad + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \times \left[\int_0^1 |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)|^q d\gamma \right]^{\frac{1}{q}}. \end{aligned} \tag{3.11}$$

By the quasi-convexity of $|\mathfrak{B}'|$

$$\int_0^1 |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \leq \max \left\{ |\mathfrak{B}'(x)|^q, |\mathfrak{B}'(v_1)|^q \right\}, \tag{3.12}$$

and

$$\int_0^1 |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)|^q d\gamma \leq \max \left\{ |\mathfrak{B}'(x)|^q, |\mathfrak{B}'(v_1)|^q \right\}. \tag{3.13}$$

Also that

$$\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta p}{k} + 1}} B \left(1 + \frac{\beta p}{k}, \frac{1}{\alpha} \right). \tag{3.14}$$

Using (3.12), (3.13) and (3.14) in (3.11) we have the desired proof. □

Remark 3.8. Under the same suppositions of Theorem 3.7, if we choose $|\mathfrak{B}'(x)| \leq M$ for $x \in [v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[{}^{\beta}_k \mathfrak{J}_x^\alpha \mathfrak{B}(v_1) + {}^{\beta}_k \mathfrak{J}_{x^+}^\alpha \mathfrak{B}(v_2) \right] \right| \\ & \leq \left(\frac{B \left(\frac{1}{\alpha}, \frac{\beta}{k} + 1 \right)}{\alpha^{\frac{\beta}{k} + 1}} \right)^{\frac{1}{p}} M \left[\frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \right]. \end{aligned} \tag{3.15}$$

Remark 3.9. Under the same suppositions of Theorem 3.7, if we set $k = 1$, then [7, Theorem 5], is recovered.

Remark 3.10. Under the same suppositions of Remark 3.8, if we set $k = 1$, then [7, Remark 6], is recovered.

Theorem 3.11. *let $\mathfrak{B} : [v_1, v_2] \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of the interval \mathbb{J} in \mathbb{R} , $v_1, v_2 \in \mathbb{J}$ with $v_1 < v_2$. If $\mathfrak{B}' \in L_1 [v_1, v_2]$ and $|\mathfrak{B}'|^q$ is quasi-convex function on $[v_1, v_2]$, where $q \geq 1$ and $\alpha, \beta \in \mathbb{R}^+$, then the following fractional integral inequality holds true*

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[{}^{\beta}_k \mathfrak{J}_x^\alpha \mathfrak{B}(v_1) + {}^{\beta}_k \mathfrak{J}_{x^+}^\alpha \mathfrak{B}(v_2) \right] \right| \\ & \leq \left[\frac{1}{\alpha^{1 + \frac{\beta}{k}}} B \left(\frac{1}{\alpha}, 1 + \frac{\beta}{k} \right) \right] \left[\frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_1)| \right\} \right. \\ & \quad \left. + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ |\mathfrak{B}'(x)|, |\mathfrak{B}'(v_2)| \right\} \right], \end{aligned} \tag{3.16}$$

where $\alpha, \beta, k > 0$ and $B(\cdot, \cdot), \Gamma(\cdot)$ is the E-beta and E-gamma functions as defined earlier.

Proof. Assuming $q \geq 1$, and from Lemma 3.1 and power mean inequality, we get

$$\begin{aligned} |\mathfrak{N}| & \leq \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{1 - \frac{1}{q}} \\ & \quad \times \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \right]^{\frac{1}{q}} \\ & \quad + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{1 - \frac{1}{q}} \\ & \quad \times \left[\int_0^1 \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)|^q d\gamma \right]^{\frac{1}{q}}. \end{aligned} \tag{3.17}$$

Changing the variable with $y=(1-\gamma)^\alpha$ and using the quasi-convexity of $|\mathfrak{B}'|^q$

$$\begin{aligned} & \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \mathfrak{B}'(\gamma x + (1-\gamma)v_1) \right|^q d\gamma \\ & \leq \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 [1-(1-\gamma)^\alpha]^{\frac{\beta}{k}} \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\} d\gamma \\ & \leq \frac{1}{\alpha^{\frac{\beta}{k}+1}} B \left(\frac{1}{\alpha'}, \frac{\beta}{k} + 1 \right) \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\}. \end{aligned} \tag{3.18}$$

Similarly

$$\begin{aligned} & \int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \mathfrak{B}'(\gamma x + (1-\gamma)v_2) \right|^q d\gamma \\ & \leq \frac{1}{\alpha^{\frac{\beta}{k}}} \int_0^1 [1-(1-\gamma)^\alpha]^{\frac{\beta}{k}} \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_2) \right|^q \right\} d\gamma \\ & \leq \frac{1}{\alpha^{\frac{\beta}{k}+1}} B \left(\frac{1}{\alpha'}, \frac{\beta}{k} + 1 \right) \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_2) \right|^q \right\}, \end{aligned} \tag{3.19}$$

and

$$\int_0^1 \left(\frac{1-(1-\gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{1}{\alpha^{\frac{\beta}{k}+1}} B \left(\frac{1}{\alpha'}, \frac{\beta}{k} + 1 \right). \tag{3.20}$$

Using equations (3.18), (3.19) and (3.20) in (3.17), we have the desired proof. □

Remark 3.12. Under the same suppositions of Theorem 3.11, if we choose $|\mathfrak{B}'(x)| \leq M$ for $x \in [v_1, v_2]$, then we have

$$\begin{aligned} & \left| \frac{(x-v_1)^{\frac{\alpha\beta}{k}} + (v_2-x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}}(v_2-v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta+k)}{v_2-v_1} \left[{}^{\beta}_k \mathfrak{J}_x^\alpha \mathfrak{B}(v_1) + {}^{\beta}_k \mathfrak{J}_x^\alpha \mathfrak{B}(v_2) \right] \right| \\ & \leq \left[\frac{1}{\alpha^{1+\frac{\beta}{k}}} B \left(\frac{1}{\alpha'}, 1 + \frac{\beta}{k} \right) \right] M \left[\frac{(x-v_1)^{\frac{\alpha\beta}{k}+1}}{v_2-v_1} + \frac{(v_2-x)^{\frac{\alpha\beta}{k}+1}}{v_2-v_1} \right]. \end{aligned} \tag{3.21}$$

Remark 3.13. Under the same suppositions of Theorem 3.11, if we take $k = 1$, then the Theorem 3.11 becomes [7, Theorem 7].

Remark 3.14. Under the same suppositions of Remark 3.12, if we choose $k = 1$, then the Remark 3.12 becomes [7, Remark 8].

In [19, Theorems 1.4 and 1.5] new and different representations (“Hölder’s–İşcan Integral inequality and Improved power-mean integral inequality”), i.e., modified/extended versions of “Hölder’s Integral inequality” and Power-mean integral inequality are given. We use the ability of this technique and obtained the following inequalities, which gives better results than the “classical Hölder’s integral inequality” and Power mean inequality.

Theorem 3.15. Let $\mathfrak{B} : [v_1, v_2] \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of the interval \mathbb{J} in \mathbb{R} , $v_1, v_2 \in \mathbb{J}$ with $v_1 < v_2$. If $\mathfrak{B}' \in L_1[v_1, v_2]$ and $|\mathfrak{B}'|^q$ is quasi-convex function on $[v_1, v_2]$, where $q \geq 1$ and $\alpha, \beta \in \mathbb{R}^+$, then the

following fractional integral inequality holds true

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[{}_k^{\beta} \mathfrak{J}_{x^-}^{\alpha} \mathfrak{B}(v_1) + {}_k^{\beta} \mathfrak{J}_{x^+}^{\alpha} \mathfrak{B}(v_2) \right] \right| \\ & \leq \frac{1}{2^q} \left\{ \left(\frac{B\left(\frac{2}{\alpha}, \frac{\beta p}{k} + 1\right)}{\alpha^{\frac{\beta p}{k} + 1}} \right)^{\frac{1}{p}} + \left(\frac{B\left(\frac{1}{\alpha}, \frac{\beta p}{k} + 1\right) - B\left(\frac{2}{\alpha}, \frac{\beta p}{k} + 1\right)}{\alpha^{\frac{\beta p}{k} + 1}} \right)^{\frac{1}{p}} \right\} \\ & \times \left[\frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ \left| \mathfrak{B}'(x) \right|, \left| \mathfrak{B}'(v_1) \right| \right\} + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max \left\{ \left| \mathfrak{B}'(x) \right|, \left| \mathfrak{B}'(v_2) \right| \right\} \right], \end{aligned} \tag{3.22}$$

where $q^{-1} = 1 - p^{-1}$, $\alpha, \beta, k > 0$ and $B(.,.)$, $\Gamma(.)$ is the E-beta and E-gamma functions as defined earlier.

Proof. from Lemma 3.1 and Hölder–İşcan integral inequality, we have

$$\begin{aligned} |\mathfrak{K}| & \leq \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left\{ \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \left[\int_0^1 (1 - \gamma) \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right|^q d\gamma \right]^{\frac{1}{q}} \right. \\ & \left. + \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \left[\int_0^1 \gamma \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right|^q d\gamma \right]^{\frac{1}{q}} \right\} \\ & + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left\{ \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \left[\int_0^1 (1 - \gamma) \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_2) \right|^q d\gamma \right]^{\frac{1}{q}} \right. \\ & \left. + \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma \right]^{\frac{1}{p}} \times \left[\int_0^1 \gamma \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right|^q d\gamma \right]^{\frac{1}{q}} \right\}. \end{aligned} \tag{3.23}$$

It can be noticed that,

$$\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma = \frac{B\left(\frac{2}{\alpha}, \frac{\beta p}{k} + 1\right)}{\alpha^{\frac{\beta p}{k} + 1}}, \tag{3.24}$$

$$\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta p}{k}} d\gamma = \frac{B\left(\frac{1}{\alpha}, \frac{\beta p}{k} + 1\right) - B\left(\frac{2}{\alpha}, \frac{\beta p}{k} + 1\right)}{\alpha^{\frac{\beta p}{k} + 1}}. \tag{3.25}$$

Since, $|\mathfrak{B}'|^q$ is quasi-convexity

$$\begin{aligned} \int_0^1 (1 - \gamma) \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right|^q d\gamma & = \int_0^1 (1 - \gamma) \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\} d\gamma \\ & = \frac{1}{2} \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\}, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \int_0^1 \gamma \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right|^q d\gamma & = \int_0^1 \gamma \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\} d\gamma \\ & = \frac{1}{2} \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\}. \end{aligned} \tag{3.27}$$

Using (3.24), (3.25), (3.26), (3.27) in (3.23), we have the desired proof. □

Theorem 3.16. Let $\mathfrak{B} : [v_1, v_2] \rightarrow \mathbb{R}$ be a differentiable mapping on $\overset{\circ}{\mathbb{J}}$ of the interval \mathbb{J} in \mathbb{R} , where $v_1, v_2 \in \mathbb{J}$ with $v_1 < v_2$. If $\mathfrak{B}' \in L_1 [v_1, v_2]$ and $|\mathfrak{B}'|^q$ is quasi-convex function on $[v_1, v_2]$, where $p > 1$ and $\alpha, \beta \in \mathbb{R}^+$, then the following fractional integral inequality holds true

$$\begin{aligned} & \left| \frac{(x - v_1)^{\frac{\alpha\beta}{k}} + (v_2 - x)^{\frac{\alpha\beta}{k}}}{\alpha^{\frac{\beta}{k}} (v_2 - v_1)} \mathfrak{B}(x) - \frac{\Gamma_k(\beta + k)}{v_2 - v_1} \left[{}_{\beta}^{\mathfrak{J}}_{x^-} \mathfrak{B}(v_1) + {}_{\beta}^{\mathfrak{J}}_{x^+} \mathfrak{B}(v_2) \right] \right| \\ & \leq \frac{B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right)}{\alpha^{\frac{\beta}{k} + 1}} \times \left[\frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max\left\{|\mathfrak{B}'(x)|, |\mathfrak{B}'(v_1)|\right\} \right. \\ & \quad \left. + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \max\left\{|\mathfrak{B}'(x)|, |\mathfrak{B}'(v_2)|\right\} \right], \end{aligned} \tag{3.28}$$

where $q^{-1} = 1 - p^{-1}$, $\alpha, \beta, k > 0$ and $B(\cdot, \cdot), \Gamma(\cdot)$ is the E-beta and E-gamma functions as defined earlier.

Proof. From Lemma 3.1 and improved power mean integral inequality for $q > 1$, we have

$$\begin{aligned} |\mathfrak{K}| & \leq \frac{(x - v_1)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left\{ \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{\frac{1}{p}} \right. \\ & \quad \times \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \right]^{\frac{1}{q}} \\ & \quad \left. + \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{\frac{1}{p}} \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \right]^{\frac{1}{q}} \right\} \\ & \quad + \frac{(v_2 - x)^{\frac{\alpha\beta}{k} + 1}}{v_2 - v_1} \left\{ \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{\frac{1}{p}} \right. \\ & \quad \times \left[\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_2)|^q d\gamma \right]^{\frac{1}{q}} \\ & \quad \left. + \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma \right]^{\frac{1}{p}} \left[\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma \right]^{\frac{1}{q}} \right\}. \end{aligned} \tag{3.29}$$

It can be noticed that

$$\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)}{\alpha^{\frac{\beta}{k} + 1}}, \tag{3.30}$$

$$\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} d\gamma = \frac{B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)}{\alpha^{\frac{\beta}{k} + 1}}. \tag{3.31}$$

Since, $|\mathfrak{B}'|^q$ is quasi-convexity

$$\int_0^1 (1 - \gamma) \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} |\mathfrak{B}'(\gamma x + (1 - \gamma)v_1)|^q d\gamma = \frac{B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)}{\alpha^{\frac{\beta}{k} + 1}} \max\left\{|\mathfrak{B}'(x)|^q, |\mathfrak{B}'(v_1)|^q\right\}, \tag{3.32}$$

and

$$\int_0^1 \gamma \left(\frac{1 - (1 - \gamma)^\alpha}{\alpha} \right)^{\frac{\beta}{k}} \left| \mathfrak{B}'(\gamma x + (1 - \gamma)v_1) \right| d\gamma$$

$$= \frac{B\left(\frac{1}{\alpha}, \frac{\beta}{k} + 1\right) - B\left(\frac{2}{\alpha}, \frac{\beta}{k} + 1\right)}{\alpha^{\frac{\beta}{k} + 1}} \max \left\{ \left| \mathfrak{B}'(x) \right|^q, \left| \mathfrak{B}'(v_1) \right|^q \right\}. \quad (3.33)$$

Using (3.30), (3.31), (3.32), (3.33) in (3.29), we have the desired proof. \square

4. Conclusion

The majority of current work on inequality has focused on using fractional calculus to uncover new bounds for several well-known inequalities. In this sense, we have looked into the relationship between inequality theory and the k -fractional conformable integral operator. A new general fractional identity and some new estimations of Ostrowski-type inequalities for differentiable quasi-convex functions are investigated. Some novel special cases of the main results are discussed as well. Choosing $k = 1$, we recover some results for conformable fractional integrals given by Akdemir et al. [7]. In the future, researchers can work on the basic algebraic properties associated with different classes of fractional operators. The new work aims to motivate researchers working in fractional calculus, interval analysis, and other related topics.

We would like to conclude our current investigation by noting that fractional-order analogues of a variety of well-known integral inequalities have been frequently explored in a number of recent scientific articles by implementing some trivial or redundant parametric variations of some widely and extensively studied fractional integral and fractional derivative operators (see, [26] for example for a detailed analysis about the triviality and insignificance of the so-called “post-quantum” calculus incorporating a redundant or superficial forced-in parameter).

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