



Zalcman coefficient functional for tilted starlike functions with respect to conjugate points



Daud Mohamad, Nur Hazwani Aqilah Abdul Wahid*

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia.

Abstract

In this paper, we consider a subclass of tilted starlike functions with respect to conjugate points in an open unit disk. For functions in this subclass, we obtain the upper bounds for the initial coefficients and the Zalcman coefficient functional. Furthermore, we present several (known or new) consequences of our results based on the special choices of the involved parameters.

Keywords: Univalent functions, starlike functions with respect to conjugate points, coefficient estimates, Zalcman conjecture, subordination.

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1. Introduction

Let \mathcal{A} denote the class of analytic functions $f(z)$ with the normalized conditions $f(0) = f'(0) - 1 = 0$ in an open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in E. \quad (1.1)$$

We also denote by S the subclass of \mathcal{A} consisting of univalent functions in E .

Let \mathcal{P} denote the class of analytic functions $p(z)$ defined in E which satisfy the condition $\operatorname{Re} p(z) > 0$ and of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in E. \quad (1.2)$$

This class is also known as the class of Carathéodory functions.

*Corresponding author

Email addresses: daud@tmsk.uitm.edu.my (Daud Mohamad), hazwaniaqilah@tmsk.uitm.edu.my (Nur Hazwani Aqilah Abdul Wahid)

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Let H denote the class of Schwarz functions $\omega(z)$ which are analytic in E given by

$$\omega(z) = \sum_{k=2}^{\infty} b_k z^k, \quad z \in E$$

and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$.

Next, we recall the definition of subordination. The analytic function $F(z)$ is subordinate to another analytic function $G(z)$ and is symbolically written as $F(z) \prec G(z)$, if there exists a Schwarz function $\omega(z) \in H$ such that $F(z) = G(\omega(z))$ for all $z \in E$. Further, if $G(z)$ is univalent in E , then $F(z) \prec G(z) \Leftrightarrow F(0) = G(0)$ and $F(E) \subset G(E)$.

In 1987, El-Ashwah and Thomas [10] defined the class S_C^* which satisfies the condition

$$\operatorname{Re} \left\{ \frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \right\} > 0, \quad z \in E.$$

The functions in the class S_C^* are called starlike functions with respect to conjugate points and of the form (1.1). In 1991, Halim [11] defined the class $S_C^*(\delta)$ consisting of functions of the form (1.1) and satisfying the condition

$$\operatorname{Re} \left\{ \frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \right\} > \delta, \quad 0 \leq \delta < 1, \quad z \in E.$$

In 2009, in terms of subordination, Dahhar and Janteng [5] defined the class $S_C^*(A, B)$ consisting of functions of the form (1.1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E. \quad (1.3)$$

From (1.3), it follows that $f(z) \in S_C^*(A, B)$ if and only if

$$\frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad \omega(z) \in H.$$

In 2015, Wahid et al. [33] introduced the class $S_C^*(\alpha, \delta)$ which satisfies the condition

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{zf'(z)}{g(z)} \right\} > \delta, \quad 0 \leq \delta < 1, \quad |\alpha| < \frac{\pi}{2}, \quad z \in E,$$

where $g(z) = \frac{f(z) + \overline{f(\bar{z})}}{2}$. Further, in terms of subordination, they defined the class $S_C^*(\alpha, \delta, A, B)$ consisting of functions of the form (1.1) and satisfying the condition

$$\left\{ e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i \sin \alpha \right\} \frac{1}{t_{\alpha\delta}} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \quad z \in E, \quad (1.4)$$

where $t_{\alpha\delta} = \cos \alpha - \delta > 0$. The functions in the class $S_C^*(\alpha, \delta, A, B)$ are called tilted starlike functions with respect to conjugate points of order δ and of the form (1.1). From (1.4), it follows that $f(z) \in S_C^*(\alpha, \delta, A, B)$ if and only if

$$\left\{ e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i \sin \alpha \right\} \frac{1}{t_{\alpha\delta}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad \omega(z) \in H.$$

The functions in these classes $S_C^*(\delta)$, $S_C^*(A, B)$, $S_C^*(\alpha, \delta)$ and $S_C^*(\alpha, \delta, A, B)$ are the subclass of the class S_C^* , i.e., $S_C^*(0) \equiv S_C^*$, $S_C^*(1, -1) \equiv S_C^*$, $S_C^*(0, 0) \equiv S_C^*$ and $S_C^*(0, 0, 1, -1) \equiv S_C^*$.

Throughout this paper, we shall consider the class $S_C^*(\alpha, \delta, A, B)$ introduced in (1.4). Now we will present some special cases of this class by suitably specializing the parameters α, δ, A , and B .

Remark 1.1.

- (a) If we let $\alpha = \delta = 0, A = 1$ and $B = -1$, then the class $S_C^*(\alpha, \delta, A, B)$ reduces to the class S_C^* .
- (b) If we let $\alpha = 0, A = 1$ and $B = -1$, then the class $S_C^*(\alpha, \delta, A, B)$ reduces to the class $S_C^*(\delta)$.
- (c) If we replace $\alpha = \delta = 0$, then the class $S_C^*(\alpha, \delta, A, B)$ reduces to the class $S_C^*(A, B)$.
- (d) If we replace $A = 1$ and $B = -1$, then the class $S_C^*(\alpha, \delta, A, B)$ reduces to the class $S_C^*(\alpha, \delta)$.

Some problems in geometric function theory are to study the coefficients and the functionals made up of combinations of the coefficients of $f(z)$ in (1.1) for various defined subclasses of S . The upper bounds for the coefficients give several properties of functions, for example, the bound for the second coefficient gives growth and distortion theorems for functions in the class S . This problem has gained much attention after the Bieberbach conjecture, i.e., $|a_n| \leq n, n \geq 2$ for $f(z) \in S$ came into the picture in 1916. Several attempts were made to prove the Bieberbach conjecture and in 1985, De Branges [6] finally solved it using the famous Hayman Regularity theorem.

In 1960, as an approach to prove the Bieberbach conjecture, Lawrence Zalcman proposed a Zalcman conjecture that each $f(z) \in S$ satisfies the inequality $|a_n^2 - a_{2n-1}| \leq (n-1)^2, n \geq 2$. This conjecture is also known as the Zalcman coefficient functional and its importance is that it implies the famous Bieberbach conjecture (see [4]). The generalized version of Zalcman coefficient functional $\lambda a_n^2 - a_{2n-1}, \lambda > 0$ has also been proposed for various subclasses of S [4, 9, 20–22, 25]. For $n = 2$, it simplifies to the well-known Fekete-Szegő functional. This observation clearly defines the role of the generalized Zalcman coefficient functional in the class S . The Zalcman coefficient functional and its generalized version also appeared frequently in the coefficient formulas for the inversion transformation in the theory of univalent functions.

Besides that, in 1999, Ma [23] proposed a generalized Zalcman coefficient functional that each $f(z) \in S$ satisfies the inequality $|a_n a_m - a_{n+m-1}| \leq (n-1)(m-1), n \geq 2, m \geq 2$. But, this inequality only holds for the class of starlike functions with real coefficients. For $f(z) \in S$, Krushkal [17, 19] also proved the Zalcman coefficient functional for $n = 3, 4, 5, 6$. In addition, Krushkal [18] proved the Zalcman coefficient functional for $n \geq 2$ using complex geometry and universal Teichmüller spaces. The estimation for the Zalcman coefficient functional and its generalized version have also been studied for several well-known subclasses of S , for example, starlike functions and typically real functions [4], close-to-convex functions [22], but are still an open problem. This also led to several papers related to the Zalcman conjecture for different subclasses of S [1, 3, 12, 15, 16, 24, 26–28, 30, 31]. However, there were few studies on this problem in the existing literature related to the subclass of starlike functions with respect to other points, i.e., symmetric points, conjugate points and symmetric conjugate points (see [10, 28] for details).

Thus, motivated by the previous works, in the present paper, we obtain the upper bounds for the initial coefficients $|a_n|, n = 2, 3, 4, 5, 6, 7$ for functions in the class $S_C^*(\alpha, \delta, A, B)$. Further, we determine the estimates on the Zalcman coefficient functionals $|a_n^2 - a_{2n-1}|, n = 2, 3, 4$.

The paper is structured as follows. In Section 2, we give some lemmas required for the proofs. In Section 3, we provide estimates for the initial coefficients and some particular cases of the Zalcman coefficient functional for functions from the class $S_C^*(\alpha, \delta, A, B)$. In Section 4, we give the conclusion.

2. Preliminaries

In this section, we give some lemmas to prove our main results.

Lemma 2.1 ([7]). *For a function $p(z) \in P$ of the form (1.2), the sharp inequality $|p_n| \leq 2$ holds for each $n \geq 1$. Equality holds for the function $p(z) = \frac{1+z}{1-z}$.*

Lemma 2.2 ([8]). *Let $p(z) \in P$ of the form (1.2) and $\mu \in \mathbb{C}$. Then*

$$|p_n - \mu p_k p_{n-k}| \leq 2 \max\{1, |2\mu - 1|\}, 1 \leq k \leq n - 1.$$

If $|2\mu - 1| \geq 1$, then the inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ or its rotations. If $|2\mu - 1| < 1$, then the inequality is sharp for the function $p(z) = \frac{1+z^n}{1-z^n}$ or its rotations.

3. Main results

This section is devoted to the proof of our main results. We will now determine the bounds on the first six initial coefficients for functions in the class $S_C^*(\alpha, \delta, A, B)$. Further, this result shall be used in order to estimate the Zalcman coefficient functional for the case of $n = 2$, $n = 3$, and $n = 4$.

3.1. Initial coefficient estimates

Theorem 3.1. *If $f(z) \in \mathcal{A}$ and of the form (1.1) belongs to $S_C^*(\alpha, \delta, A, B)$, then*

$$\begin{aligned} |a_2| &\leq T, \\ |a_3| &\leq \frac{T}{2} |-\xi + \Upsilon - 1|, \\ |a_4| &\leq \frac{T}{6} [| -3\xi + 4\Upsilon - 2| + |\xi^2 - 3\Upsilon\xi + 2\Upsilon^2|], \\ |a_5| &\leq \frac{T}{24} [2| -4\xi + 6\Upsilon - 3| + | -6\xi^2 + \xi(-3 + 22\Upsilon) + 6\Upsilon(1 - 3\Upsilon)| + |\xi^3 - 6\Upsilon\xi^2 + 11\Upsilon^2\xi - 6\Upsilon^3|], \\ |a_6| &\leq \frac{T}{120} [6| -5\xi + 8\Upsilon - 4| + 4| -5\xi^2 + 5\xi(-1 + 4\Upsilon) + 6\Upsilon(2 - 3\Upsilon)| \\ &\quad + | -10\xi^3 + 5\xi^2(-3 + 14\Upsilon) + 10\Upsilon\xi(7 - 15\Upsilon) + 4\Upsilon^2(-19 + 24\Upsilon)| \\ &\quad + |\xi^4 - 10\Upsilon\xi^3 + 35\Upsilon^2\xi^2 - 50\Upsilon^3\xi + 24\Upsilon^4|] \end{aligned}$$

and

$$\begin{aligned} |a_7| &\leq \frac{T}{720} [40| -\xi + 3\Upsilon - 3| + 6| -15\xi^2 + 3\xi(-8 + 21\Upsilon) + 20\Upsilon(2 - 3\Upsilon)| \\ &\quad + 2| -60\xi^2 + \xi(-45 + 314\Upsilon) + 120\Upsilon(1 - 3\Upsilon)| + |\xi^5 - 15\Upsilon\xi^4 + 85\Upsilon^2\xi^3 - 225\Upsilon^3\xi^2 + 274\Upsilon^4\xi - 120\Upsilon^5| \\ &\quad + | -45\xi^3 + 15\xi^2(-1 + 25\Upsilon) + 2\Upsilon\xi(45 - 478\Upsilon) + 20\Upsilon^2(-6 + 37\Upsilon)| \\ &\quad + | -15\xi^4 + 10\xi^3(-4 + 17\Upsilon) + 75\Upsilon\xi^2(4 - 9\Upsilon) + 4\Upsilon^2\xi(-173 + 274\Upsilon) + 120\Upsilon^3(4 - 5\Upsilon)|], \end{aligned}$$

where $\xi = Te^{-i\alpha}$, $T = \Psi t_{\alpha\delta}$, $\Psi = A - B$, $t_{\alpha\delta} = \cos \alpha - \delta$, and $\Upsilon = 1 + B$.

Proof. Let $f(z) \in S_C^*(\alpha, \delta, A, B)$ and of the form (1.1). By definition of subordination, there exists a Schwarz function $\omega(z)$ such that

$$\left[e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i \sin \alpha \right] \frac{1}{t_{\alpha\delta}} = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{3.1}$$

where $g(z) = \frac{f(z) + \overline{f(\bar{z})}}{2}$ and $t_{\alpha\delta} = \cos \alpha - \delta$. Let the function

$$h(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + \sum_{n=1}^{\infty} k_n z^n.$$

Obviously, the function $h(z) \in P$ and

$$\omega(z) = \frac{h(z) - 1}{h(z) + 1}. \tag{3.2}$$

Substituting (3.2) into (3.1), we get

$$e^{i\alpha} \frac{zf'(z)}{g(z)} = \frac{(\Omega^- - T) + h(z)(\Omega^+ + T)}{1 - B + h(z)(1 + B)}, \tag{3.3}$$

where $T = \Psi t_{\alpha\delta}$, $\Psi = A - B$, $t_{\alpha\delta} = \cos \alpha - \delta$, $\Omega^- = e^{i\alpha} (1 - B)$ and $\Omega^+ = e^{i\alpha} (1 + B)$. Substituting the expansions of $f(z)$, $g(z)$, and $h(z)$ in (3.3), and after simplifying, we obtain

$$\begin{aligned} & \Omega^- (z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + 6a_6z^6 + 7a_7z^7 + \dots) \\ & + \Omega^+ (z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 + 5a_5z^5 + 6a_6z^6 + 7a_7z^7 + \dots) \\ & \cdot (1 + k_1z + k_2z^2 + k_3z^3 + k_4z^4 + k_5z^5 + k_6z^6 + k_7z^7 + \dots) \\ & = (\Omega^- - T) (z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 + \dots) \\ & + (\Omega^+ + T) (z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7 + \dots) \\ & \cdot (1 + k_1z + k_2z^2 + k_3z^3 + k_4z^4 + k_5z^5 + k_6z^6 + \dots). \end{aligned} \tag{3.4}$$

By equating the coefficients of z^2 , z^3 , z^4 , z^5 , z^6 , and z^7 in (3.4) and for simplicity, we denote $\xi = Te^{-i\alpha}$ and $\Upsilon = 1 + B$, we get

$$a_2 = \frac{k_1\xi}{2}, \tag{3.5}$$

$$a_3 = \frac{\xi}{8} [2k_2 + k_1^2 (\xi - \Upsilon)], \tag{3.6}$$

$$a_4 = \frac{\xi}{48} [8k_3 + k_1k_2 (6\xi - 8\Upsilon) + k_1^3 (\xi^2 - 3\Upsilon\xi + 2\Upsilon^2)], \tag{3.7}$$

$$\begin{aligned} a_5 = \frac{\xi}{384} [48k_4 + k_1k_3 (32\xi - 48\Upsilon) + k_2^2 (12\xi - 24\Upsilon) + k_1^4 (\xi^3 - 6\Upsilon\xi^2 + 11\Upsilon^2\xi - 6\Upsilon^3) \\ + k_1^2k_2 (12\xi^2 - 44\Upsilon\xi + 36\Upsilon^2)], \end{aligned} \tag{3.8}$$

$$\begin{aligned} a_6 = \frac{\xi}{3840} [384k_5 + k_1k_4 (240\xi - 384\Upsilon) + k_2k_3 (160\xi - 384\Upsilon) + k_1k_2^2 (60\xi^2 - 280\Upsilon\xi + 304\Upsilon^2) \\ + k_1^3k_2 (20\xi^3 - 140\Upsilon\xi^2 + 300\Upsilon^2\xi - 192\Upsilon^3) + k_1^5 (\xi^4 - 10\Upsilon\xi^3 + 35\Upsilon^2\xi^2 - 50\Upsilon^3\xi + 24\Upsilon^4) \\ + k_1^2k_3 (80\xi^2 - 320\Upsilon\xi + 288\Upsilon^2)] \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} a_7 = \frac{\xi}{46080} [3840k_6 + k_2^3 (120\xi^2 - 720\Upsilon\xi + 960\Upsilon^2) + k_3^2 (640\xi - 1920\Upsilon) \\ + k_1^2k_4 (720\xi^2 - 3024\Upsilon\xi + 2880\Upsilon^2) + k_1k_5 (2304\xi - 3840\Upsilon) + k_2k_4 (1440\xi - 3840\Upsilon) \\ + k_1k_2k_3 (960\xi^2 - 5024\Upsilon\xi + 5760\Upsilon^2) + k_1^4k_2 (30\xi^4 - 340\Upsilon\xi^3 + 1350\Upsilon^2\xi^2 - 2192\Upsilon^3\xi + 1200\Upsilon^4) \\ + k_1^6 (\xi^5 - 15\Upsilon\xi^4 + 85\Upsilon^2\xi^3 - 225\Upsilon^3\xi^2 + 274\Upsilon^4\xi - 120\Upsilon^5) \\ + k_1^2k_2^2 (180\xi^3 - 1500\Upsilon\xi^2 + 3824\Upsilon^2\xi - 2960\Upsilon^3) + k_1^3k_3 (160\xi^3 - 1200\Upsilon\xi^2 + 2768\Upsilon^2\xi - 1920\Upsilon^3)]. \end{aligned} \tag{3.10}$$

Further, taking modulus on both sides of (3.5)-(3.10) and suitably arranging the terms in (3.6)-(3.10), respectively, we have

$$|a_2| = \frac{|\xi| |k_1|}{2}, \tag{3.11}$$

$$|a_3| = \frac{|\xi|}{4} |k_2 - \mu_1 k_1^2|, \tag{3.12}$$

$$|a_4| = \frac{|\xi|}{48} [8|k_3 - \nu_1 k_1 k_2| + |k_1|^3 |\xi^2 - 3\Upsilon\xi + 2\Upsilon^2|], \tag{3.13}$$

$$|a_5| = \frac{|\xi|}{384} [48|k_4 - \eta_1 k_1 k_3| + |k_2| |12\xi - 24\Upsilon| |k_2 - \eta_2 k_1^2| + |k_1|^4 |\xi^3 - 6\Upsilon\xi^2 + 11\Upsilon^2\xi - 6\Upsilon^3|], \tag{3.14}$$

$$|a_6| = \frac{|\xi|}{3840} [384|k_5 - \lambda_1 k_1 k_4| + |k_3| |160\xi - 384\Upsilon| |k_2 - \lambda_2 k_1^2|] \tag{3.15}$$

$$+ |k_1| |k_2| |60\xi^2 - 280\Upsilon\xi + 304\Upsilon^2| |k_2 - \lambda_3 k_1^2| + |k_1|^5 |\xi^4 - 10\Upsilon\xi^3 + 35\Upsilon^2\xi^2 - 50\Upsilon^3\xi + 24\Upsilon^4| \Big],$$

and

$$|a_7| = \frac{|\xi|}{46080} \left[3840 |k_6 - \gamma_1 k_3^2| + |k_2|^2 |120\xi^2 - 720\Upsilon\xi + 960\Upsilon^2| |k_2 - \gamma_2 k_1^2| \right. \\ + |k_1| |2304\xi - 3840\Upsilon| |k_5 - \gamma_3 k_1 k_4| + |k_2| |1440\xi - 3840\Upsilon| |k_4 - \gamma_4 k_1 k_3| \quad (3.16) \\ + |k_1|^6 |\xi^5 - 15\Upsilon\xi^4 + 85\Upsilon^2\xi^3 - 225\Upsilon^3\xi^2 + 274\Upsilon^4\xi - 120\Upsilon^5| \\ \left. + |k_1|^3 |160\xi^3 - 1200\Upsilon\xi^2 + 2768\Upsilon^2\xi - 1920\Upsilon^3| |k_3 - \gamma_5 k_1 k_2| \right],$$

where

$$\begin{aligned} \mu_1 &= \frac{-\xi + \Upsilon}{2}, & \nu_1 &= \frac{-3\xi + 4\Upsilon}{4}, \\ \eta_1 &= \frac{-2\xi + 3\Upsilon}{3}, & \eta_2 &= \frac{-3\xi^2 + 11\Upsilon\xi - 9\Upsilon^2}{3\xi - 6\Upsilon}, \\ \lambda_1 &= \frac{-5\xi + 8\Upsilon}{8}, & \lambda_2 &= \frac{-5\xi^2 + 20\Upsilon\xi - 18\Upsilon^2}{10\xi - 24\Upsilon}, \\ \lambda_3 &= \frac{-5\xi^3 + 35\Upsilon\xi^2 - 75\Upsilon^2\xi + 48\Upsilon^3}{15\xi^2 - 70\Upsilon\xi + 76\Upsilon^2}, & \gamma_1 &= \frac{-\xi + 3\Upsilon}{6}, \\ \gamma_2 &= \frac{-45\xi^3 + 375\Upsilon\xi^2 - 956\Upsilon^2\xi + 740\Upsilon^3}{30\xi^2 - 180\Upsilon\xi + 240\Upsilon^2}, & \gamma_3 &= \frac{-15\xi^2 + 63\Upsilon\xi - 60\Upsilon^2}{48\xi - 80\Upsilon}, \\ \gamma_4 &= \frac{-30\xi^2 + 157\Upsilon\xi - 180\Upsilon^2}{45\xi - 120\Upsilon}, & \gamma_5 &= \frac{-15\xi^4 + 170\Upsilon\xi^3 - 675\Upsilon^2\xi^2 + 1096\Upsilon^3\xi - 600\Upsilon^4}{80\xi^3 - 600\Upsilon\xi^2 + 1384\Upsilon^2\xi - 960\Upsilon^3}. \end{aligned}$$

Hence, by applying Lemmas 2.1 and 2.2, and by triangle inequality, from (3.11)-(3.16), respectively, we get the desired results. □

Corollary 3.2. For special values of the parameters α, δ, A and B , we have following.

(a) If $f(z) \in \mathcal{A}$ given by (1.1) belongs to $S_C^*(0, 0, 1, -1) \equiv S_C^*$, then

$$|a_2| \leq 2, \quad |a_3| \leq 3, \quad |a_4| \leq 4, \quad |a_5| \leq 5, \quad |a_6| \leq 6 \text{ and } |a_7| \leq 7.$$

(b) If $f(z) \in \mathcal{A}$ given by (1.1) belongs to $S_C^*(0, \delta, 1, -1) \equiv S_C^*(\delta)$, then

$$\begin{aligned} |a_2| &\leq 2\Phi, \\ |a_3| &\leq \Phi(2\Phi + 1), \\ |a_4| &\leq \frac{\Phi}{3} [2(1 + 3\Phi + 2\Phi^2)], \\ |a_5| &\leq \frac{\Phi}{6} [8\Phi + 3 + 3\Phi|4\Phi + 1| + 4\Phi^3], \\ |a_6| &\leq \frac{\Phi}{15} [3|5\Phi + 2| + 10\Phi|2\Phi + 1| + 5\Phi^2|4\Phi + 3| + 4\Phi^4], \\ |a_7| &\leq \frac{\Phi}{90} [10|2\Phi + 3| + 15\Phi^2|6\Phi + 1| + 18\Phi|5\Phi + 4| + 15\Phi|8\Phi + 3| + 8\Phi^5 + 20\Phi^3|3\Phi + 4|], \end{aligned}$$

where $\Phi = 1 - \delta$.

(c) If $f(z) \in \mathcal{A}$ given by (1.1) belongs to $S_C^*(0, 0, A, B) \equiv S_C^*(A, B)$, then

$$|a_2| \leq \Psi,$$

$$\begin{aligned}
 |a_3| &\leq \frac{\Psi}{2} |-\Psi + \Upsilon - 1|, \\
 |a_4| &\leq \frac{\Psi}{6} [|-3\Psi + 4\Upsilon - 2| + |\Psi^2 - 3\Upsilon\Psi + 2\Upsilon^2|], \\
 |a_5| &\leq \frac{\Psi}{24} [2|-4\Psi + 6\Upsilon - 3| + |-6\Psi^2 + \Psi(-3 + 22\Upsilon) + 6\Upsilon(1 - 3\Upsilon)| + |\Psi^3 - 6\Upsilon\Psi^2 + 11\Upsilon^2\Psi - 6\Upsilon^3|], \\
 |a_6| &\leq \frac{\Psi}{120} [6|-5\Psi + 8\Upsilon - 4| + 4|-5\Psi^2 + 5\Psi(-1 + 4\Upsilon) + 6\Upsilon(2 - 3\Upsilon)| \\
 &\quad + |-10\Psi^3 + 5\Psi^2(-3 + 14\Upsilon) + 10\Upsilon\Psi(7 - 15\Upsilon) + 4\Upsilon^2(-19 + 24\Upsilon)| \\
 &\quad + |\Psi^4 - 10\Upsilon\Psi^3 + 35\Upsilon^2\Psi^2 - 50\Upsilon^3\Psi + 24\Upsilon^4|], \\
 |a_7| &\leq \frac{\Psi}{720} [|-90\Psi^2 + 18\Psi(-8 + 21\Upsilon) + 120\Upsilon(2 - 3\Upsilon)| \\
 &\quad + |-45\Psi^3 + 15\Psi^2(-1 + 25\Upsilon) + 2\Upsilon\Psi(45 - 478\Upsilon) + 20\Upsilon^2(-6 + 37\Upsilon)| \\
 &\quad + |\Psi^5 - 15\Upsilon\Psi^4 + 85\Upsilon^2\Psi^3 - 225\Upsilon^3\Psi^2 + 274\Upsilon^4\Psi - 120\Upsilon^5| \\
 &\quad + |-120\Psi^2 + 2\Psi(-45 + 314\Upsilon) + 240\Upsilon(1 - 3\Upsilon)| + 40|-\Psi + 3\Upsilon - 3| \\
 &\quad + |-15\Psi^4 + 10\Psi^3(-4 + 17\Upsilon) + 75\Upsilon\Psi^2(4 - 9\Upsilon) + 4\Upsilon^2\Psi(-173 + 274\Upsilon) + 120\Upsilon^3(4 - 5\Upsilon)|].
 \end{aligned}$$

(d) If $f(z) \in \mathcal{A}$ given by (1.1) belongs to $S_C^*(\alpha, \delta, 1, -1) \equiv S_C^*(\alpha, \delta)$, then

$$\begin{aligned}
 |a_2| &\leq 2t_{\alpha\delta}, \\
 |a_3| &\leq t_{\alpha\delta} |2t_{\alpha\delta}e^{-i\alpha} + 1|, \\
 |a_4| &\leq \frac{t_{\alpha\delta}}{3} [2|3t_{\alpha\delta}e^{-i\alpha} + 1| + 4t_{\alpha\delta}^2], \\
 |a_5| &\leq \frac{t_{\alpha\delta}}{6} [|8t_{\alpha\delta}e^{-i\alpha} + 3| + 3t_{\alpha\delta}|4t_{\alpha\delta}e^{-i\alpha} + 1| + 4t_{\alpha\delta}^3], \\
 |a_6| &\leq \frac{t_{\alpha\delta}}{15} [3|5t_{\alpha\delta}e^{-i\alpha} + 2| + 10t_{\alpha\delta}|2t_{\alpha\delta}e^{-i\alpha} + 1| + 5t_{\alpha\delta}^2|4t_{\alpha\delta}e^{-i\alpha} + 3| + 4t_{\alpha\delta}^4], \\
 |a_7| &\leq \frac{t_{\alpha\delta}}{90} [10|2t_{\alpha\delta}e^{-i\alpha} + 3| + 15t_{\alpha\delta}^2|6t_{\alpha\delta}e^{-i\alpha} + 1| + 18t_{\alpha\delta}|5t_{\alpha\delta}e^{-i\alpha} + 4| + 15t_{\alpha\delta}|8t_{\alpha\delta}e^{-i\alpha} + 3| \\
 &\quad + 8t_{\alpha\delta}^5 + 20t_{\alpha\delta}^3|3t_{\alpha\delta}e^{-i\alpha} + 4|].
 \end{aligned}$$

Corollaries 3.2 (a)-(d) show the upper bounds for the initial coefficients for functions from the subclasses studied by El-Ashwah and Thomas [10], Halim [11], Dahhar and Janteng [5], and Wahid et al. [33], respectively. It is observed that $|a_n| \leq n$, $n = 2, 3, 4, 5, 6, 7$ in Corollary 3.2 (a) coincides with the result of the well-known class of starlike functions S^* (see [7] for details).

3.2. Zalcman coefficient functional

Theorem 3.3. If $f(z) \in \mathcal{A}$ and of the form (1.1) belongs to $S_C^*(\alpha, \delta, A, B)$, then

$$|a_2^2 - a_3| \leq \frac{\Gamma|\xi + \Upsilon - 1|}{2},$$

where $\xi = \Gamma e^{-i\alpha}$, $\Gamma = \Psi t_{\alpha\delta}$, $\Psi = A - B$, $t_{\alpha\delta} = \cos \alpha - \delta$ and $\Upsilon = 1 + B$.

Proof. In view of (3.5) and (3.6), we have

$$|a_2^2 - a_3| = \left| \frac{2k_1^2\xi^2}{8} - \frac{2k_2\xi}{8} - \frac{k_1^2(\xi^2 - \Upsilon\xi)}{8} \right| = \left| \frac{\xi}{4} \right| |k_2 - \sigma_1 k_1^2|, \tag{3.17}$$

where $\sigma_1 = \frac{\xi + \Upsilon}{2}$. By applying Lemma 2.2, from (3.17), we see that

$$|k_2 - \sigma_1 k_1^2| \leq 2|\xi + \Upsilon - 1|.$$

Hence, using the triangle inequality, (3.17) yields the desired result. This completes the proof of Theorem 3.3. \square

Corollary 3.4. For special values of the parameters α , δ , A , and B , we obtain following.

(a) Let $f(z) \in S_C^*(0, 0, 1, -1) \equiv S_C^*$. Then

$$|a_2^2 - a_3| \leq 1.$$

(b) Let $f(z) \in S_C^*(0, \delta, 1, -1) \equiv S_C^*(\delta)$. Then

$$|a_2^2 - a_3| \leq \Phi |2\Phi - 1|,$$

where $\Phi = 1 - \delta$.

(c) Let $S_C^*(0, 0, A, B) \equiv S_C^*(A, B)$. Then

$$|a_2^2 - a_3| \leq \frac{A\Psi}{2}.$$

(d) Let $f(z) \in S_C^*(\alpha, \delta, 1, -1) \equiv S_C^*(\alpha, \delta)$. Then

$$|a_2^2 - a_3| \leq t_{\alpha\delta} \sqrt{4t_{\alpha\delta}^2 - 4t_{\alpha\delta} \cos \alpha + 1}.$$

Theorem 3.5. If $f(z) \in \mathcal{A}$ and of the form (1.1) belongs to $S_C^*(\alpha, \delta, A, B)$, then

$$|a_3^2 - a_5| \leq \frac{T}{24} [|-6\xi^2 + \xi(-3 - 10\Upsilon) + 6\Upsilon(-1 + 3\Upsilon)| + 2|-4\xi + 6\Upsilon - 3| + |5\xi^3 - 6\Upsilon\xi^2 - 5\Upsilon^2\xi + 6\Upsilon^3|],$$

where $\xi = Te^{-i\alpha}$, $T = \Psi t_{\alpha\delta}$, $\Psi = A - B$, $t_{\alpha\delta} = \cos \alpha - \delta$, and $\Upsilon = 1 + B$.

Proof. From (3.6) and (3.8), we have

$$|a_3^2 - a_5| = \left| \frac{\xi}{384} [k_2^2(12\xi + 24\Upsilon) + k_1^2k_2(12\xi^2 + 20\Upsilon\xi - 36\Upsilon^2) - 48k_4 - k_1k_3(32\xi - 48\Upsilon) + k_1^4(5\xi^3 - 6\Upsilon\xi^2 - 5\Upsilon^2\xi + 6\Upsilon^3)] \right| \tag{3.18}$$

Further, by suitably arranging the terms, (3.18) yields

$$|a_3^2 - a_5| = \left| \frac{\xi}{384} \{k_2(12\xi + 24\Upsilon)[k_2 - \chi_1k_1^2] - 48[k_4 - \chi_2k_1k_3] + k_1^4(5\xi^3 - 6\Upsilon\xi^2 - 5\Upsilon^2\xi + 6\Upsilon^3)\} \right|,$$

where

$$\chi_1 = \frac{-3\xi^2 - 5\Upsilon\xi + 9\Upsilon^2}{3\xi + 6\Upsilon}$$

and

$$\chi_2 = \frac{-2\xi + 3\Upsilon}{3}.$$

Hence, applying Lemma 2.2, we see that

$$|k_2 - \chi_1k_1^2| \leq 2 \left| \frac{-6\xi^2 + \xi(-3 - 10\Upsilon) + 6\Upsilon(-1 + 3\Upsilon)}{3\xi + 6\Upsilon} \right| \tag{3.19}$$

and

$$|k_4 - \chi_2k_1k_3| \leq 2 \left| \frac{-4\xi + 6\Upsilon - 3}{3} \right|. \tag{3.20}$$

Then, making use of (3.19), (3.20) and Lemma 2.1, and by triangle inequality, we obtain the desired result. Thus, the proof of Theorem 3.5 is completed. \square

Corollary 3.6. For special values of the parameters involved, we obtain following.

(a) Let $f(z) \in S_C^*(0, 0, 1, -1) \equiv S_C^*$. Then

$$|a_3^2 - a_5| \leq \frac{23}{3}.$$

(b) Let $f(z) \in S_C^*(0, \delta, 1, -1) \equiv S_C^*(\delta)$. Then

$$|a_3^2 - a_5| \leq \frac{\Phi}{6} [3\Phi |4\Phi + 1| + |8\Phi + 3| + 20\Phi^3],$$

where $\Phi = 1 - \delta$.

(c) Let $f(z) \in S_C^*(0, 0, A, B) \equiv S_C^*(A, B)$. Then

$$|a_3^2 - a_5| \leq \frac{\Psi}{24} [|-6\Psi^2 + \Psi(-3 - 10\Upsilon) + 6\Upsilon(-1 + 3\Upsilon)| + 2|-4\Psi + 6\Upsilon - 3| + |5\Psi^3 - 6\Upsilon\Psi^2 - 5\Upsilon^2\Psi + 6\Upsilon^3|].$$

(d) Let $f(z) \in S_C^*(\alpha, \delta, 1, -1) \equiv S_C^*(\alpha, \delta)$. Then

$$|a_3^2 - a_5| \leq \frac{t_{\alpha\delta}}{6} [3t_{\alpha\delta} |4t_{\alpha\delta} e^{-i\alpha} + 1| + |8t_{\alpha\delta} e^{-i\alpha} + 3| + 20t_{\alpha\delta}^3].$$

Theorem 3.7. If $f(z) \in \mathcal{A}$ and of the form (1.1) belongs to $S_C^*(\alpha, \delta, A, B)$, then

$$|a_4^2 - a_7| \leq \frac{\Gamma}{720} [24|-6\xi + 10\Upsilon - 5| + 6|15\xi^2 + 3\xi(5 - 21\Upsilon) + 20\Upsilon(-2 + 3\Upsilon)| + |-135\xi^3 + 15\xi^2(-8 + 7\Upsilon) + 4\Upsilon\xi(-77 + 159\Upsilon) + 20\Upsilon^2(36 - 37\Upsilon)| + |15\xi^4(-7 - 4\Upsilon) + 10\xi^3(-4 + 29\Upsilon - 36\Upsilon^2) + 5\Upsilon\xi^2(-36 + 47\Upsilon + 32\Upsilon^2) + 36\Upsilon^2\xi(17 - 26\Upsilon) + 120\Upsilon^3(-4 + 5\Upsilon)| + 40|\xi + 3\Upsilon| + |\xi^5(19 - 60\Upsilon) + 5\Upsilon\xi^4(-9 + 44\Upsilon) + 15\Upsilon^2\xi^3(-3 - 16\Upsilon) + 5\Upsilon^3\xi^2(45 + 16\Upsilon) - 274\Upsilon^4\xi + 120\Upsilon^5| + 15|-\xi^2 + 6\Upsilon\xi - 8\Upsilon^2|],$$

where $\xi = \Gamma e^{-i\alpha}$, $\Gamma = \Psi t_{\alpha\delta}$, $\Psi = A - B$, $t_{\alpha\delta} = \cos \alpha - \delta$, and $\Upsilon = 1 + B$.

Proof. Using (3.7) and (3.10), we obtain

$$|a_4^2 - a_7| = \left| \frac{\xi}{46080} [k_3^2(640\xi + 1920\Upsilon) + k_1 k_2 k_3(960\xi^2 + 2464\Upsilon\xi - 5760\Upsilon^2) + k_2 k_4(-1440\xi + 3840\Upsilon) + k_2^3(-120\xi^2 + 720\Upsilon\xi - 960\Upsilon^2) + k_1 k_5(-2304\xi + 3840\Upsilon) + k_1^2 k_4(-720\xi^2 + 3024\Upsilon\xi - 2880\Upsilon^2) + k_1^3 k_3(160\xi^3 - 480\Upsilon\xi^3 + 720\Upsilon\xi^2 + 320\Upsilon^2\xi^2 - 2448\Upsilon^2\xi + 1920\Upsilon^3) + k_1^4 k_2(210\xi^4 + 120\Upsilon\xi^4 - 340\Upsilon\xi^3 + 720\Upsilon^2\xi^3 - 630\Upsilon^2\xi^2 - 320\Upsilon^3\xi^2 + 1872\Upsilon^3\xi - 1200\Upsilon^4) + k_1^6(19\xi^5 - 60\Upsilon\xi^5 - 45\Upsilon\xi^4 + 220\Upsilon^2\xi^4 - 45\Upsilon^2\xi^3 - 240\Upsilon^3\xi^3 + 225\Upsilon^3\xi^2 + 80\Upsilon^4\xi^2 - 274\Upsilon^4\xi + 120\Upsilon^5) + k_1^2 k_2^2(540\xi^3 - 420\Upsilon\xi^2 - 2544\Upsilon^2\xi + 2960\Upsilon^3) - 3840k_6] \right|.$$

Further, by suitably arranging the terms in above equation, we get

$$|a_4^2 - a_7| = \left| \frac{\xi}{46080} \{-3840[k_6 - v_1 k_1 k_5] + k_4(-1440\xi + 3840\Upsilon)[k_2 - v_2 k_1^2] + k_1 k_2(960\xi^2 + 2464\Upsilon\xi - 5760\Upsilon^2)[k_3 - v_3 k_1 k_2] + k_1^3(160\xi^3 - 480\Upsilon\xi^3 + 720\Upsilon\xi^2 + 320\Upsilon^2\xi^2 - 2448\Upsilon^2\xi + 1920\Upsilon^3)[k_3 - v_4 k_1 k_2] + k_3^2(640\xi + 1920\Upsilon) + k_1^6(19\xi^5 - 60\Upsilon\xi^5 - 45\Upsilon\xi^4 + 220\Upsilon^2\xi^4 - 45\Upsilon^2\xi^3 - 240\Upsilon^3\xi^3 + 225\Upsilon^3\xi^2 + 80\Upsilon^4\xi^2 - 274\Upsilon^4\xi + 120\Upsilon^5) + k_2^3(-120\xi^2 + 720\Upsilon\xi - 960\Upsilon^2)\} \right|, \tag{3.21}$$

where

$$\begin{aligned} v_1 &= \frac{-3\xi + 5\Upsilon}{5}, \\ v_2 &= \frac{15\xi^2 - 63\Upsilon\xi + 60\Upsilon^2}{-30\xi + 80\Upsilon}, \\ v_3 &= \frac{-135\xi^3 + 105\Upsilon\xi^2 + 636\Upsilon^2\xi - 740\Upsilon^3}{240\xi^2 + 616\Upsilon\xi - 1440\Upsilon^2}, \\ v_4 &= \frac{15\xi^4(-7 - 4\Upsilon) + 10\Upsilon\xi^3(17 - 36\Upsilon) + 5\Upsilon^2\xi^2(63 + 32\Upsilon) - 936\Upsilon^3\xi + 600\Upsilon^4}{80\xi^3(1 - 3\Upsilon) + 40\Upsilon\xi^2(9 + 4\Upsilon) - 1224\Upsilon^2\xi + 960\Upsilon^3}. \end{aligned}$$

By Lemma 2.2, we observe that

$$|k_6 - v_1 k_1 k_5| \leq 2 \left| \frac{-6\xi + 10\Upsilon - 5}{5} \right|, \tag{3.22}$$

$$|k_2 - v_2 k_1^2| \leq 2 \left| \frac{15\xi^2 + 3\xi(5 - 21\Upsilon) + 20\Upsilon(-2 + 3\Upsilon)}{-15\xi + 40\Upsilon} \right|, \tag{3.23}$$

$$|k_3 - v_3 k_1 k_2| \leq \left| \frac{-135\xi^3 + 5\xi^2(-24 + 21\Upsilon) + 4\Upsilon\xi(-77 + 159\Upsilon) + 20\Upsilon^2(36 - 37\Upsilon)}{60\xi^2 + 154\Upsilon\xi - 360\Upsilon^2} \right| \tag{3.24}$$

$$\begin{aligned} |k_3 - v_4 k_1 k_2| &\leq \left| \frac{5\xi^4(-21 - 12\Upsilon) + 10\xi^3(-4 + 29\Upsilon - 36\Upsilon^2) + 5\Upsilon\xi^2(-36 + 47\Upsilon + 32\Upsilon^2)}{20\xi^3(1 - 3\Upsilon) + 10\Upsilon\xi^2(9 + 4\Upsilon) - 306\Upsilon^2\xi + 240\Upsilon^3} \right| \tag{3.25} \\ &\quad \left| \frac{+12\Upsilon^2\xi(51 - 78\Upsilon) + 120\Upsilon^3(-4 + 5\Upsilon)}{20\xi^3(1 - 3\Upsilon) + 10\Upsilon\xi^2(9 + 4\Upsilon) - 306\Upsilon^2\xi + 240\Upsilon^3} \right|. \end{aligned}$$

Thus, using (3.22)-(3.25) and Lemma 2.1, and by triangle inequality, (3.21) yields the desired result. This completes the proof of Theorem 3.7. \square

Corollary 3.8. For special values of the parameters $\alpha, \delta, A,$ and $B,$ we obtain following.

(a) Let $f(z) \in S_C^*(0, 0, 1, -1) \equiv S_C^*.$ Then

$$|a_4^2 - a_7| \leq \frac{73}{5}.$$

(b) Let $f(z) \in S_C^*(0, \delta, 1, -1) \equiv S_C^*(\delta).$ Then

$$|a_4^2 - a_7| \leq \frac{\Phi}{90} [6|12\Phi + 5| + 45\Phi|2\Phi + 1| + 30\Phi^2|9\Phi + 4| + 20\Phi^3|21\Phi + 4| + 20\Phi + 15\Phi^2 + 152\Phi^5],$$

where $\Phi = 1 - \delta.$

(c) Let $S_C^*(0, 0, A, B) \equiv S_C^*(A, B).$ Then

$$\begin{aligned} |a_4^2 - a_7| &\leq \frac{\Psi}{720} [24|-6\Psi + 10\Upsilon - 5| + 6|15\Psi^2 + 3\Psi(5 - 21\Upsilon) + 20\Upsilon(-2 + 3\Upsilon)| \\ &\quad + |-135\Psi^3 + 15\Psi^2(-8 + 7\Upsilon) + 4\Upsilon\Psi(-77 + 159\Upsilon) + 20\Upsilon^2(36 - 37\Upsilon)| \\ &\quad + |15\Psi^4(-7 - 4\Upsilon) + 10\Psi^3(-4 + 29\Upsilon - 36\Upsilon^2) + 5\Upsilon\Psi^2(-36 + 47\Upsilon + 32\Upsilon^2) \\ &\quad + 36\Upsilon^2\Psi(17 - 26\Upsilon) + 120\Upsilon^3(-4 + 5\Upsilon)| + 40|\Psi + 3\Upsilon| + |\Psi^5(19 - 60\Upsilon) + 5\Upsilon\Psi^4(-9 + 44\Upsilon) \\ &\quad + 15\Upsilon^2\Psi^3(-3 - 16\Upsilon) + 5\Upsilon^3\Psi^2(45 + 16\Upsilon) - 274\Upsilon^4\Psi + 120\Upsilon^5| + 15|-\Psi^2 + 6\Upsilon\Psi - 8\Upsilon^2|]. \end{aligned}$$

(d) Let $f(z) \in S_C^*(\alpha, \delta, 1, -1) \equiv S_C^*(\alpha, \delta).$ Then

$$\begin{aligned} |a_4^2 - a_7| &\leq \frac{t_{\alpha\delta}}{90} [6|12t_{\alpha\delta}e^{-i\alpha} + 5| + 45t_{\alpha\delta}|2t_{\alpha\delta}e^{-i\alpha} + 1| + 30t_{\alpha\delta}^2|9t_{\alpha\delta}e^{-i\alpha} + 4| \\ &\quad + 20t_{\alpha\delta}^3|21t_{\alpha\delta}e^{-i\alpha} + 4| + 20t_{\alpha\delta} + 15t_{\alpha\delta}^2 + 152t_{\alpha\delta}^5]. \end{aligned}$$

Corollaries 3.4, 3.6, and 3.8 show the upper bounds for the Zalcman coefficient functional for $n = 2$, $n = 3$, and $n = 4$, respectively, for functions from the subclasses studied by El-Ashwah and Thomas [10], Halim [11], Dahhar and Janteng [5], and Wahid et al. [33]. Estimation of Zalcman coefficient functional in Corollaries 3.4, 3.6, and 3.8 was not given before.

4. Conclusion

Motivated significantly by several recent works, we have obtained the upper bounds for the initial coefficients and some particular cases of the Zalcman coefficient functional for functions in the class $S_C^*(\alpha, \delta, A, B)$. It is shown that for the specific choices of parameters α , δ , A , and B , the results presented in this paper can be reduced to a number of unknown results for the subclasses S_C^* , $S_C^*(\delta)$, $S_C^*(A, B)$, and $S_C^*(\alpha, \delta)$ which are stated in the corollaries. For future work, by using the result in Theorem 3.1, it is natural to devote further investigation to other properties for functions in the class $S_C^*(\alpha, \delta, A, B)$ such as the Fekete-Szegő functional, Hankel and Toeplitz determinants, Krushkal inequality and logarithmic coefficients. One may also refer to, for example, [2, 13, 14, 24, 26, 29] for some ideas on these properties for newly defined subclasses of analytic and univalent functions. Meanwhile, the results obtained in Theorems 3.3 and 3.5 could lead to an alternative method different from [32] in finding the upper bound for the third Hankel determinant for functions in the class $S_C^*(\alpha, \delta, A, B)$ (see for instance [2, 24]). Moreover, the method used in this paper perhaps could be applied to solve problems of bounds on the coefficients and functionals made up of combinations of the coefficients of $f(z)$ for different subclasses of univalent functions.

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