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Applications of the differential transformation to three-point singular boundary value problems for ordinary differential equations

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Abstract

The differential transform method is used to find numerical approximations of the solution to a class of certain nonlinear three-point singular boundary value problems. The method is based on Taylor's theorem. Coefficients of the Taylor series are determined by constructing a recurrence relation. To deal with the nonlinearity of the problems, the Faà di Bruno's formula containing the partial ordinary Bell polynomials is applied within the differential transform. The error estimation results are also presented. Four concrete problems are studied to show efficiency and reliability of the method. The obtained results are compared to other methods, e.g., reproducing kernel Hilbert space method.

Keywords: Differential transform method, singular boundary value problems, numerical approximation, partial ordinary Bell polynomials, error estimates.

2020 MSC: 34B16, 65L10.

1. Introduction

The aim of the paper is to apply the differential transform method (DTM) on certain three-point singular boundary value problems (SBVPs) given in the following form

$$\alpha(v)\, w''(v) + \beta(v)\, w'(v) + \gamma(v)\, w(v) = u(v, w)\, , 0 \leqslant v \leqslant 1, \tag{1.1}$$

with boundary conditions

$$w(0) = 0, w(1) = a w(b) + c, \tag{1.2}$$

where $b \in (0, 1)$, a and c are finite real constants, $\alpha \in C^2[0, 1]$, $\beta \in C^1[0, 1]$, $\gamma \in C[0, 1]$, $\alpha(0) = 0$ or $\alpha(1) = 0$, $\alpha(v) \neq 0$ in $(0, 1)$, $\beta(0) \neq 0$, $\gamma(0) \neq 0$, $\beta(1) \neq 0$, $\gamma(1) \neq 0$ and $\beta(1) - \alpha'(1) \neq 0$. Singularity may occur when $v = 0$ or $v = 1$.

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Three-point boundary value problems have applications in modelling of physics and engineering problems such as dynamical systems with three degrees of freedom where three states are observed at three different times, in models of elasticity of three-layered equally loaded beam, in optimal control, beam deflection, heat flow, draining and coating flows [4, 9, 24].

Researchers have applied different analytical and numerical techniques to study singular two point linear and nonlinear boundary value problems, see for example [1, 7]. The approaches include different numerical techniques such as spline techniques [6, 14, 21] and finite differences methods [22, 23] to handle such problems. These techniques are popular among researchers, but large computational work associated with root-finding makes them less preferred to solve nonlinear SBVPs. Semi-analytical methods are also convenient for finding solutions to differential equations. However, the calculation and results sometimes look complicated.

Papers [5, 8, 10, 11, 27, 28, 30, 31, 34] are devoted to the study of existence and uniqueness of solutions for singular three-point boundary value problems. The motivation for the present work are the research work of Thomson and Tisdell [28], Geng [5], and Dehghan and Shakeri [4] who studied three-point boundary value problems. Further, we did not find any suitable routine in Mathematica and Matlab software to handle problems discussed in this paper.

We propose a simple approach involving the differential transform in this paper. The differential transform has been introduced by Pukhov as the "Taylor transform" in 1976 and applied to the study of electrical circuits [13]. The differential transform has close relation with the Taylor expansion of real analytic functions. It has applications in solving different types of problems for all classes of differential equations (ordinary, partial, delayed, fractional, fuzzy etc). The recent developments and applications of DTM are discussed in [15, 17–19] and references therein.

In the present paper, the differential transform is used to solve singular boundary value problems. The nonlinearity in the problems is addressed by using the partial ordinary Bell polynomials in the Faà di Bruno's formula. The results obtained by this technique are compared to other methods. Detailed error analysis is provided. However, to the best of our knowledge, no researcher has applied the DTM using Bell polynomials on the practical problems discussed in Section 5.

The paper is organized as follows. In Section 2, we introduce the main idea and basic formulae of the differential transform and provide necessary results for the nonlinearities involving partial ordinary Bell polynomials. In Section 3 we introduce the convergence result and in section 4 the error estimate result is discussed. Numerical results and discussion are presented in Section 5. A conclusion is discussed in Section 6.

2. Preliminaries

In this section we discuss the main idea and basic formulae of the differential transform as well as notation and results related to the transformation of general nonlinear terms.

2.1. The differential transform

Let $w(\nu)$ be a real analytical function in a domain D and $\nu = \nu_0$ be an arbitrary point in D. Then, $w(\nu)$ can be expanded in a Taylor series in a neighbourhood of the point $\nu = \nu_0$.

Definition 2.1 ([19])**.** The differential transform of a real function $w(\nu)$ at a point $\nu_0 \in \mathbb{R}$ is defined as

$$W(k)[\nu_0] = \frac{1}{k!}\left[\frac{d^k w(\nu)}{d\nu^k}\right]_{\nu=\nu_0},$$

where $W(k)[\nu_0]$, the differential transform of the k-th derivative of the function $w(\nu)$ at ν_0.

Definition 2.2 ([19])**.** The inverse differential transform is given by

$$w(\nu) = \sum_{k=0}^{\infty} W(k)[\nu_0](\nu - \nu_0)^k. \tag{2.1}$$

Combining Definitions 2.1 and 2.2 gives an expression of the function w in the form of the Taylor series:

$$w(\nu) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k w(\nu)}{d\nu^k} \right]_{\nu=\nu_0} (\nu - \nu_0)^k = \sum_{k=0}^{\infty} W(k)[\nu_0](\nu - \nu_0)^k.$$

In the real-life applications, we often express the function $w(\nu)$ as a finite sum

$$w(\nu) = \sum_{k=0}^{N} W(k)[\nu_0](\nu - \nu_0)^k. \tag{2.2}$$

The formulae that will be used in the applications presented in Section 5 are collected in the following theorem. Although it is not difficult to prove the formulae, we did not find the proofs of most of the formulae in related literature. As we do not want to rely on assertions without proofs, we prefer to include the proofs in the paper.

Theorem 2.3. *Assume that $W(k)[\nu_0]$ is the differential transform of the function $w(\nu)$ at $\nu = \nu_0$.*

(a) *If $w(\nu) = w_1'(\nu)$, then for any ν_0*

$$W(k)[\nu_0] = (k+1)W_1(k+1)[\nu_0] \text{ for } k = 0, 1, 2, \ldots. \tag{2.3}$$

(b) *If $w(\nu) = w_1^{(n)}(\nu)$, then for any ν_0*

$$W(k)[\nu_0] = (k+1)(k+2)\ldots(k+n)W_1(k+n))[\nu_0] \text{ for } k = 0, 1, 2, \ldots.$$

(c) *If $w(\nu) = w_1(\nu) \cdot w_2(\nu)$, then for any ν_0*

$$W(k)[\nu_0] = \sum_{l=0}^{k} W_1(l)[\nu_0]W_2(k-l)[\nu_0] \text{ for } k = 0, 1, 2, \ldots. \tag{2.4}$$

(d) *If $w(\nu) = w_1(\nu) \cdot w_2(\nu) \ldots \cdot w_n(\nu)$, then for any ν_0*

$$\begin{aligned} W(k)[\nu_0] = \sum_{l_1=0}^{k} \sum_{l_2=0}^{k-l_1} \cdots \sum_{l_{n-1}=0}^{k-l_1-l_2-\cdots-l_{n-2}} & W_1(l_1)[\nu_0]W_2(l_2)[\nu_0] \cdots W_{n-1}(l_{n-1})[\nu_0] \\ & \times W_n(k - l_1 - l_2 - \cdots - l_{n-1})[\nu_0] \end{aligned}$$

for $k = 0, 1, 2, \ldots$.

(e) *If $w(\nu) = e^{\alpha\nu}$ then for all $\nu_0 \in \mathbb{R}$*

$$W(k)[\nu_0] = \frac{e^{\alpha\nu_0}\alpha^k}{k!} \text{ for } k = 0, 1, 2, \ldots.$$

(f) *If $w(\nu) = \nu^r$, $r \in \mathbb{R}$ then for any ν such that $|\nu - \nu_0| < |\nu_0|$,*

$$W(k)[\nu_0] = \binom{r}{k}\nu_0^{r-k} \text{ for } k = 0, 1, 2, \ldots,$$

where $\binom{r}{k} = \frac{r(r-1)\ldots(r-k+1)}{k!} = \frac{(r)_k}{k!}$, and $(r)_k$ represents the Pochhammer symbol.

(g) *If $w(\nu) = \nu^n$, $n \in \mathbb{N}_0$, then for $\nu_0 = 0$,*

$$W(k)[\nu_0] = \delta(k - n) \text{ for } k = 0, 1, 2, \ldots,$$

where $\delta(k - n) = \delta_{kn}$ is the Kronecker delta.

Proof. Proofs of the formulae will be derived directly from the Definitions 2.1 and 2.2, respectively.

(a) Proof of this formula can be found in [12].

(b) We proceed by induction. The first step is to prove the formula for $n = 1$, that is, the formula (2.3) that was proved above. Now suppose that the formula is valid for n, and we will prove it for $n+1$. We suppose that if $w(v) = w_1^{(n)}(v)$, then for any v_0 we have $W(k)[v_0] = (k+1)(k+2)\cdots(k+n)W_1(k+n))[v_0]$ for $k = 0, 1, 2, \ldots$. It means that

$$w_1^{(n)}(v) = w(v) = \sum_{k=0}^{\infty} W(k)[v_0](v-v_0)^k = \sum_{k=0}^{\infty}(k+1)(k+2)\cdots(k+n)W_1(k+n))[v_0](v-v_0)^k.$$

Then for $n+1$, we have

$$\begin{aligned} w_1^{(n+1)}(v) = w'(v) = \frac{d}{dv}\sum_{k=0}^{\infty} W(k)[v_0](v-v_0)^k &= \sum_{k=1}^{\infty} kW(k)[v_0](v-v_0)^{k-1} \\ &= \sum_{k=1}^{\infty} k(k+1)(k+2)\cdots(k+n)W_1(k+n))[v_0](v-v_0)^{k-1}. \end{aligned}$$

By changing the limits of the last sum, we obtain

$$w_1^{(n+1)}(v) = \sum_{k=0}^{\infty}(k+1)(k+2)\cdots(k+n)(k+n+1)W_1(k+n+1))[v_0](v-v_0)^k.$$

This implies that if $w(v) = w_1^{(n+1)}(v)$, then for any v_0 we have $W(k)[v_0] = (k+1)(k+2)\cdots(k+n)(k+n+1)W_1(k+n+1))[v_0]$ for $k = 0, 1, 2, \ldots$, which proves the formula.

(c) Suppose that $w_1(v) = \sum_{l=0}^{\infty} W_1(l)[v_0](v-v_0)^l$ and $w_2(v) = \sum_{m=0}^{\infty} W_2(m)[v_0](v-v_0)^m$. Then

$$\begin{aligned} w(v) = w_1(v)\cdot w_2(v) &= \sum_{l=0}^{\infty} W_1(l)[v_0](v-v_0)^l \cdot \sum_{m=0}^{\infty} W_2(m)[v_0](v-v_0)^m \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} W_1(l)[v_0]W_2(m)[v_0](v-v_0)^{l+m}. \end{aligned}$$

If we substitute $k = l+m$, we obtain

$$w(v) = \sum_{l=0}^{\infty}\sum_{k-l=0}^{\infty} W_1(l)[v_0]W_2(k-l)[v_0](v-v_0)^k = \sum_{l=0}^{\infty}\sum_{k=l}^{\infty} W_1(l)[v_0]W_2(k-l)[v_0](v-v_0)^k.$$

By changing the order of summation in the last sum, we get

$$w(v) = \sum_{k=0}^{\infty}\sum_{l=0}^{k} W_1(l)[v_0]W_2(k-l)[v_0](v-v_0)^k,$$

which implies that $W(k)[v_0] = \sum_{l=0}^{k} W_1(l)[v_0]W_2(k-l)[v_0]$ for $k = 0, 1, 2, \ldots$.

(d) We proceed by induction. The first step is to prove the formula for $n = 2$, that is, the formula (2.4) that was proved above. Next we prove validity of the formula for $n+1$ under the assumption that the formula holds for a product of n functions. In other words, we suppose that if $u(v) = w_2(v) \cdot w_3(v) \cdot \dots \cdot w_{n+1}(v)$, then for any v_0,

$$W(k)[v_0] = \sum_{l_2=0}^{k} \sum_{l_3=0}^{k-l_2} \cdots \sum_{l_n=0}^{k-l_2-l_3-\cdots-l_{n-1}} W_2(l_2)[v_0]W_3(l_3)[v_0] \cdots W_n(l_n)[v_0]W_{n+1}(k-l_2-l_3-\cdots-l_n)[v_0]$$

for $k = 0, 1, 2, \dots$. Put $w(v) = w_1(v) \cdot u(v)$. Then, according to (2.4), we have $W(k)[v_0] = \sum_{l_1=0}^{k} W_1(l_1)[v_0]U(k-l_1)[v_0]$ for $k = 0, 1, 2, \dots$, that is,

$$W(k)[v_0] = \sum_{l_1=0}^{k} W_1(l_1)[v_0] \sum_{l_2=0}^{k-l_1} \sum_{l_3=0}^{k-l_1-l_2} \cdots \sum_{l_n=0}^{k-l_1-l_2-l_3-\cdots-l_{n-1}} W_2(l_2)[v_0]W_3(l_3)[v_0] \cdots W_n(l_n)[v_0]W_{n+1}(k-l_1-l_2-l_3-\cdots-l_n)[v_0]$$

for $k = 0, 1, 2, \dots$, which proves the formula.

(e) For any $v_0 \in \mathbb{R}$, we can expand the function $w(v) = e^{\alpha v}$ into Taylor series as

$$w(v) = e^{\alpha v} = \sum_{k=0}^{\infty} \frac{\alpha^k e^{\alpha v_0}}{k!}(v - v_0)^k.$$

Comparison with the Definition 2.2 yields

$$W(k)[v_0] = \frac{e^{\alpha v_0}\alpha^k}{k!}, \text{ for } k = 0, 1, 2, \dots.$$

(f) A proof of the formula can be found in [18].

(g) For $v_0 = 0$, we can expand the function $w(v) = v^n$, where $n \in \mathbb{N}_0$, into Taylor series as

$$w(v) = v^n = \sum_{k=0}^{\infty} \delta(k-n)v^k,$$

where $\delta(k-n) = \delta_{kn}$ is the Kronecker delta. We compare it with the Definition 2.2 and conclude that

$$W(k)[0] = \delta(k-n)$$

for $k = 0, 1, 2, \dots$, which proves the formula. □

For the reader's convenience, the formulae that will be used in the applications are listed in Table 1.

2.2. Faà di Bruno's formula and Bell polynomials

One of the principal disadvantages of most papers based on applications of differential transform is that the differential transform is not directly applied to nonlinear terms like $w^n, n \in \mathbb{N}$ or e^w. The authors of [20] used Adomian polynomials to compute the differential transform of nonlinear terms. However, we can determine the differential transform of nonlinear terms without the necessity to calculate and evaluate symbolic derivatives. We can do it by applying Faà di Bruno's formula to nonlinear terms.

Here we present some necessary notations and results obtained in [16]. The proofs are not included since they can be found in the cited paper.

Definition 2.4 ([2])**.** The partial ordinary Bell polynomials are the polynomials $\check{P}_{k,l}(\check{v}_1, \dots, \check{v}_{k-l+1})$ in an infinite number of variables $\check{v}_1, \check{v}_2, \dots$ defined by the series expansion

$$\sum_{k \geqslant l} \check{P}_{k,l}(\check{v}_1, \dots, \check{v}_{k-l+1})t^k = \left(\sum_{m \geqslant 1} \check{v}_m t^m\right)^l, l = 0, 1, 2, \dots$$

Table 1: Formulae of the differential transform method.

	Original function	Transformed function
1	$\frac{d^n w(v)}{dv^n}$	$(k+1)(k+2)(k+3)\ldots(k+n)W(k+n)$
2	$w_1(v)w_2(v)$	$\sum_{l=0}^{k} W_1(l)W_2(k-l)$
3	$e^{\alpha v}, v_0 = 0$	$\frac{\alpha^k}{k!}$
4	$w(v) = v^n, v_0 = 0$	$\delta(k-n)$, where $\delta(k-n) = \begin{cases} 1, & k = n, \\ 0, & k \neq n, \end{cases}$
5	$\sin(\alpha v + \beta)$	$\frac{\alpha^k}{k!} \sin\left(\frac{k\pi}{2} + \beta\right)$
6	$\cos(\alpha v + \beta)$	$\frac{\alpha^k}{k!} \cos\left(\frac{k\pi}{2} + \beta\right)$

Lemma 2.5 ([16]). *The partial ordinary Bell polynomials* $\check{P}_{k,l}(\check{v}_1, \ldots, \check{v}_{k-l+1})$, $l = 0, 1, 2, \ldots, k \geqslant l$ *satisfy the recurrence relation*

$$\check{P}_{k,l}(\check{v}_1, \ldots, \check{v}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i.l}{k} \check{v}_i \check{P}_{k-i,l-1}(\hat{v}_1, \ldots, \check{v}_{k-i-l+2}),$$

where $\check{P}_{0,0} = 1$ *and* $\check{P}_{k,0} = 0$ *for* $k \geqslant 1$.

Theorem 2.6 ([16]). *Let* g *and* f *be real functions analytic near* v_0 *and* $g(v_0)$, *respectively, and let* h *be the composition* $h(v) = f(g(v))$. *Differential transforms of functions* g, f, *and* h *are represented by* $G(k)$, $F(k)$, *and* $H(k)$, *respectively. Then* $H(k)$ *satisfies the relations*

$$H(0) = F(0), \qquad H(k) = \sum_{l=1}^{k} F(l) . \check{P}_{k,l}(G(1), \ldots, G(k-l+1)) \text{ for } k \geqslant 1.$$

2.3. *Implementation of method for solving* (1.1)-(1.2)

Equations (1.1) and (1.2) can be rewritten as

$$v(v-1)w''(v) + pw'(v) + qw(v) + f(v) = N(w(v)), 0 \leqslant< v \leqslant 1, \tag{2.5}$$

with boundary conditions

$$w(0) = 0, w(1) = aw(b) + c, \tag{2.6}$$

where $p, q \in \mathbb{R}$, $f(v)$ is a known function and $N(w(v))$ represents nonlinear terms.

Let $F(k)$ be the differential transform of $f(v)$, now applying differential transform on equation (2.5) and (2.6), we obtain

$$\begin{aligned} k(k+1)W(k+1) - k(k-1)W(k) + p(k+1)W(k+1) + qW(k) + F(k) = H(k), \\ W(0) = 0, \end{aligned} \tag{2.7}$$

where $H(k)$ is obtained by Theorem 2.6. Now, we have

$$W(k+1) = \frac{1}{(k+1)(k+p)} \left[H(k) - F(k) + \left(k^2 - k - q\right) W(k)\right]. \tag{2.8}$$

Replacing k with $k-1$ in equation (2.8) and using equation (2.7), we obtain recurrence relation given by

$$W(0) = 0, \qquad W(k) = \frac{1}{k(k+p-1)} \left[H(k-1) - F(k-1) + \left(k^2 - 3k - q + 2\right) W(k-1)\right], k \geqslant 1.$$

The implementation of the differential transform algorithm is shown through flowchart in Figure 1.

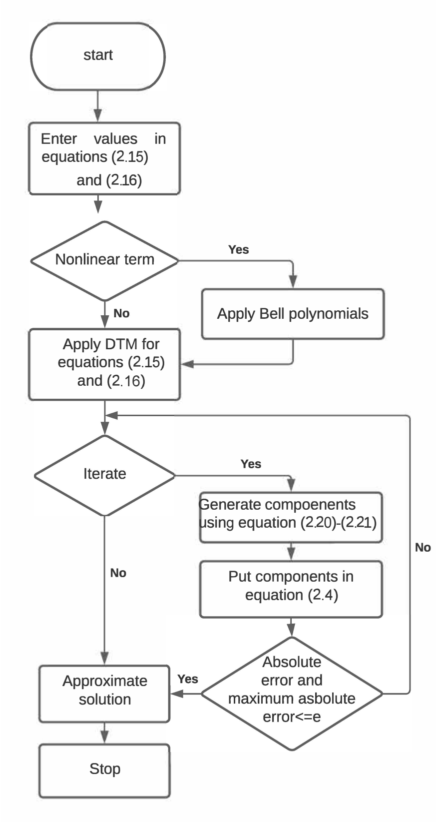

Figure 1: Flow chart of the differential transform algorithm.

3. Convergence analysis

Let $C[0,1]$ be a Banach space with the norm

$$\|w\| = \max_{0\leqslant v\leqslant 1} |w(v)|, \quad w \in C[0,1],$$

and write equation (2.1) in operator form as

$$w = v_0 + T(w),$$

where

$$T(w) = T\left(\sum_{k=0}^{\infty} W(k)\, v^k\right) = \frac{1}{k(k+p-1)}\left[H(k-1) - F(k-1) + \left(k^2 - 3k - q + 2\right) W(k-1)\right] v^k, \quad (3.1)$$

where T is a non-linear operator from a Banach space $C[0,1]$ to $C[0,1]$.

To obtain n term approximate solution of problem (1.1) and (1.2), equation (2.2) can be represented as the n^{th} partial sum

$$\varphi_n(v) = \sum_{k=0}^{n} W(k)\, v^k. \quad (3.2)$$

Equation (3.2) can be written as

$$\varphi_n(v) = v_0 + \sum_{k=1}^{n-1} W(k)\, v^k. \quad (3.3)$$

Equation (3.3) can be written in operator form as

$$\varphi_n(v) = v_0 + T(\varphi_{n-1}), n \geqslant 1. \tag{3.4}$$

Now we show that the sequence $\{\varphi_n\}$ of the n^{th} partial sum represented by (3.3) converges to the exact solution w using the following therorem. The idea of the proof is taken from [25, 26, 29].

Theorem 3.1. *Consider $T(w)$ be the nonlinear operator expressed by equation (3.1) and satisfying the Lipschitz condition $\|T(\varphi) - T(\chi)\| \leqslant \gamma\|\varphi - \chi\|$, for all $\varphi, \chi \in C[0,1]$ with Lipschitz constant γ, $0 \leqslant \gamma < 1$. If $\|v_0\| < \infty$, then the sequence $\varphi_n(v) = v_0 + T(\varphi_{n-1})$ converges to the exact solution w.*

Proof. Assuming φ_n be the sequence of n-terms of the series $\sum_{k=0}^{\infty} W(k)v^k$ represented by $\varphi_n = v_0 + T(\varphi_{n-1})$, we prove that

$$\|\varphi_{n+1} - \varphi_n\| \leqslant \gamma^n\|v_0\|. \tag{3.5}$$

For the proof of this, we use principle of mathematical induction. Indeed, suppose that (3.5) is true for $n = 1$, using equation (3.4) and Lipschitz condition, we obtain

$$\|\varphi_2 - \varphi_1\| = \|T(\varphi_1) - T(\varphi_0)\| \leqslant \gamma\|\varphi_1 - \varphi_0\| = \gamma\|v_0\|.$$

Now, we suppose that (3.5) is true for $n = k$,

$$\|\varphi_{k+1} - \varphi_k\| = \|T(\varphi_k) - T(\varphi_{k-1})\| \leqslant \gamma^k\|\varphi_k - \varphi_{k-1}\| = \gamma^k\|v_0\|.$$

Finally, we have to show that the result is appropriate for $n = k + 1$,

$$\|\varphi_{k+2} - \varphi_{k+1}\| = \|T(\varphi_{k+1}) - T(\varphi_k)\| \leqslant \gamma^{k+1}\|\varphi_{k+1} - \varphi_k\| = \gamma^{k+1}\|v_0\|.$$

Then the result is appropriate for all vaules of n. For this, we prove that the sequence $\{\varphi_n\}$ is a Cauchy sequence in the Banach space $C[0,1]$. Indeed, for every $n, m \in \mathbb{N}$, $n \geqslant m$, we have

$$\begin{aligned}
\|\varphi_n - \varphi_m\| &= \|(\varphi_n - \varphi_{n-1}) + (\varphi_{n-1} - \varphi_{n-2}) + \cdots + (\varphi_{m+1} - \varphi_m)\| \\
&\leqslant \|(\varphi_n - \varphi_{n-1})\| + \|(\varphi_{n-1} - \varphi_{n-2})\| + \cdots + \|(\varphi_{m+1} - \varphi_m)\| \\
&\leqslant \gamma^{n-1}\|v_0\| + \gamma^{n-2}\|v_0\| + \cdots + \gamma^{m+1}\|v_0\| + \gamma^m\|v_0\| \\
&\leqslant \gamma^m\left(1 + \gamma + \gamma^2 + \cdots + \gamma^{n-m-1}\right)\|v_0\| \leqslant \gamma^m\left(\frac{1-\gamma^{n-m}}{1-\gamma}\right)\|v_0\|.
\end{aligned} \tag{3.6}$$

Since $0 \leqslant \gamma < 1$, it is $1 - \gamma^{n-m} < 1$, and equation (3.6) becomes

$$\|\varphi_n - \varphi_m\| \leqslant \frac{\gamma^m}{1-\gamma}\|v_0\|. \tag{3.7}$$

Taking as $m \to \infty$ in equation (3.7), we obtain $\|\varphi_n - \varphi_m\| \to 0$, since $\|v_0\| < \infty$. The fact that the sequence $\{\varphi_n\}$ is a Cauchy sequence in the Banach space $C[0,1]$, it indicates that there exists a φ such that

$$\lim_{n\to\infty} \varphi_n = \varphi,$$

while we have

$$w = \sum_{k=0}^{\infty} W(k)v^k = \lim_{n\to\infty} \varphi_n.$$

From this, we conclude that $w = \varphi$, which is the solution of equation (1.1). Hence φ_n converges to w. □

4. Error estimation

For comparison, absolute error and maximum absolute error are computed and defined as

$$E_N(v) := |w(v) - w_N(v)|, \quad E_{N,\infty} := \max_{0\leqslant v\leqslant 1} E_N(v),$$

where $w(v)$ is the analytical solution of the problem (1.1)-(1.2) and $w_N(v)$ is the truncated series solution with degree N. Furthermore, the relative error between exact and approximate solution is defined by

$$R_N(v) := \frac{E_N(v)}{|w(v)|}.$$

In the presented Tables, the following notations have also been used.

1. $w_N(v)$:= approximate solution obtained by present technique;
2. $w_N(v)$:= approximate solution obtained by RKHSM, [5];
3. $R_N(v)$:= relative error between exact and present solution;
4. $R_N(v)$:= relative error between exact and RKHSM, [5].

In Lemma 4.1 we discuss an upper bound for the calculation of absolute errors of the present method. The idea of the proof is taken from [3, 32, 33].

Lemma 4.1. *Let $w(v) \in C^{N+1}[0,1]$ be the analytical solution of the problem (1.1)-(1.2) and $w_N(v) = \sum_{k=0}^{N} W(k)v^k$ the corresponding approximate solution of degree N. Then $\|w(v) - w_N(v)\|_\infty \leqslant \frac{M}{(N+1)!} + \max_{0\leqslant k\leqslant N} |c_k|$, where $M = \max_{0\leqslant v\leqslant 1} |w^{N+1}(v)|$, $c_k = \sum_{k=0}^{N} \left(\frac{w^{(k)}(0)}{k!} - W(k)\right)$.*

Proof. Clearly, we have

$$\|w(v) - w_N(v)\|_\infty \leqslant \|w(v) - \overline{w}_N(v)\|_\infty + \|\overline{w}_N - w_N(v)\|_\infty, \tag{4.1}$$

where $\overline{w}_N = \sum_{k=0}^{N} \frac{w^{(k)}(0)}{k!} v^k$ is the Taylor polynomial of $w(v)$ at $v = 0$. Since $w(v) \in C^{(N+1)}[0,1]$, we have

$$\begin{aligned} w(v) &= \overline{w}_N(v) + \frac{w^{(N+1)}(v_0)}{(N+1)!} v^{N+1}, v_0 \in (0,1), \\ |w(v) - \overline{w}_N(v)| &= \left|\frac{w^{(N+1)}(v_0)}{(N+1)!} v^{N+1}\right| \leqslant \frac{1}{(N+1)!} \max_{0<v_0<1} |w^{(N+1)}(v_0)|. \end{aligned} \tag{4.2}$$

For the calculation of the value of $\|\overline{w}_N(v) - w_N(v)\|_\infty$, let $C = (c_0, c_1, \ldots, c_n)$ and $\Lambda = \left(v^0, v^1, ..., v^N\right)^T$, where $c_k = \frac{w^{(k)}(0)}{k!} - W(k)$, $k = 0, 1, \ldots, N$. Then

$$\begin{aligned} |\overline{w}_N(v) - w_N(v)| &= \left|\sum_{k=0}^{N} \frac{w^{(k)}(0)}{k!} v^k - \sum_{k=0}^{N} W(k)v^k\right| = \left|\sum_{k=0}^{N} \left(\frac{w^{(k)}(0)}{k!} - W(k)\right) v^k\right| \leqslant |C|.|\Lambda|, \\ |\overline{w}_N(v) - w_N(v)| &\leqslant \|C\|_\infty.\|\Lambda\|_\infty. \end{aligned} \tag{4.3}$$

From equations (4.1), (4.2), (4.3), we have

$$\begin{aligned} \|w(v) - w_N(v)\|_\infty &\leqslant \frac{1}{(N+1)!} \max_{0<v_0<1} \left|w^{(N+1)}(v_0)\right| + \|C\|_\infty.\|\Lambda\|_\infty, \\ \|w(v) - w_N(v)\|_\infty &\leqslant \frac{M}{(N+1)!} + \max_{0\leqslant k\leqslant N} |c_k|, \end{aligned}$$

which proves the theorem. □

5. Applications

Four examples are discussed to show the reliability and effectiveness of the present method. The numerical results are compared with other existing results. The MATHEMATICA software version 11.1 has been used for plotting the graphs and for numerical computations.

Example 5.1. Consider the following linear singular three point boundary value problem [5],

$$\nu w''(\nu) + 2w'(\nu) = \pi\left(2\cos(\pi\nu) - \pi\nu\sin(\pi\nu)\right), 0 < \nu \leqslant 1, \qquad w(0) = 0, w(1) = \frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{2}.$$

The exact solution is given by

$$w(\nu) = \sin(\pi\nu).$$

Table 2: Numerical results for Example 5.1 (N = 10).

ν	$w(\nu)$	$w_N(\nu)$	$w_N(\nu)$ [5]	$R_N(\nu)$	$R_N(\nu)$ [5]
0.08	0.248690	0.24869	0.247643	2.5E-14	4.2E-03
0.16	0.481754	0.481754	0.480385	2.6E-11	2.8E-03
0.24	0.684547	0.684547	0.682673	1.6E-09	2.7E-03
0.32	0.844328	0.844328	0.841789	3.1E-08	3.0E-03
0.40	0.951057	0.951057	0.947664	3.2E-07	3.5E-03
0.48	0.998027	0.998029	0.994087	2.2E-06	3.9E-03
0.56	0.982287	0.982300	0.978457	1.2E-05	3.8E-03
0.64	0.904827	0.904880	0.901389	5.8E-05	3.7E-03
0.72	0.770513	0.770706	0.767705	2.4E-04	3.6E-03
0.80	0.587785	0.588393	0.586084	1.0E-03	2.8E-03
0.88	0.368125	0.369846	0.367952	4.6E-03	4.6E-04
0.96	0.125333	0.129775	0.126626	3.5E-02	1.0E-02

Table 3: Numerical results for Example 5.1 (N = 20).

ν	$w(\nu)$	$w_N(\nu)$	$w_N(\nu)$ [5]	$R_N(\nu)$	$R_N(\nu)$ [5]
0.08	0.248690	0.248690	0.248680	3.2E-16	4.1E-05
0.16	0.481754	0.481754	0.481718	1.1E-16	7.3E-05
0.24	0.684547	0.684547	0.684479	1.6E-16	9.8E-05
0.32	0.844328	0.844328	0.844227	1.3E-16	1.1E-04
0.40	0.951057	0.951057	0.950927	2.1E-16	1.3E-04
0.48	0.998027	0.998027	0.997878	3.1E-16	1.4E-04
0.56	0.982287	0.982287	0.982136	2.7E-15	1.5E-04
0.64	0.904827	0.904827	0.904690	5.0E-14	1.5E-04
0.72	0.770513	0.770513	0.770404	6.9E-13	1.4E-04
0.80	0.587785	0.587785	0.587718	8.3E-12	1.4E-04
0.88	0.368125	0.368125	0.368110	9.8E-11	4.0E-05
0.96	0.125333	0.125333	0.125378	1.7E-09	3.5E-04

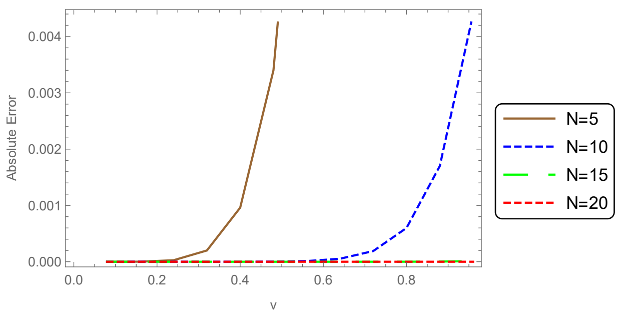

Figure 2: Absolute error of DTM for N=5, 10, 15, and 20 of Example 5.1.

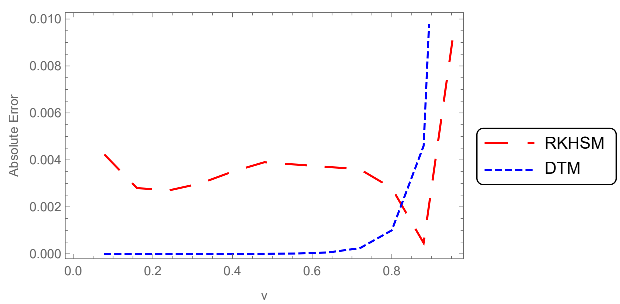

Figure 3: Comparison of Relative error for N=10 of Example 5.1.

Table 4: Maximum absolute errors for w in Example 5.1.

N	$E_{N,\infty}$
5	3.9E-01
10	4.4E-03
15	1.1E-05
20	2.2E-10

Tables 2 and 3 compare the analytical solution, approximate solutions, and relative errors for the present technique to the method discussed in [5] for $N = 10$ and 20, where N denotes the number of series components. It should be emphasized that when considering ten series terms with the present method, the attained accuracy is of order 10^{-2} up to 10^{-14}, whereas the accuracy achieved by the RKHSM [5] when eleven terms were considered is of order 10^{-3}. Moreover, when twenty series terms are considered with the present method, the accuracy is increased and becomes of order 10^{-9} up to 10^{-16}. It should be noted that the achieved accuracy by the [5] even when fifty-one terms were considered was of order 10^{-4} or 10^{-5}. Figure 3 shows a comparison of the relative error obtained by the present approach with RKHSM [5]. It is obvious, that in comparison to [5], the numerical results show that the present method achieves a better approximate solution.

Figure 2 and Table 4 show the absolute errors and maximum absolute errors for the present approach for various values of N. The tables and graph show that as the number of components increases, the absolute error, relative error, and maximum absolute error all decrease. As a result, adding more terms improves the method's accuracy.

Example 5.2. Consider the following linear singular three point boundary value problem [5],

$$v\,(1-v)\,w''\,(v) + (1-v)\,w'\,(v) + w\,(v) = (1-v)\cosh v + \sinh v + (1-v)\,v \sinh v,\ 0 < v \leqslant 1,$$
$$w\,(0) = 0, w\,(1) + \frac{1}{2} w \left(\frac{4}{5}\right) = \frac{\sinh \frac{4}{5}}{2} + \sinh 1.$$

The exact solution is given by

$$w\,(v) = \sinh v.$$

Table 5: Numerical results for Example 5.2 (N = 6).

v	$w(v)$	$w_N(v)$	$w_N(v)$ [5]	$R_N(v)$	$R_N(v)$ [5]
0.08	0.080085	0.080085	0.080056	5.9E-11	3.6E-04
0.16	0.160684	0.160684	0.160650	3.3E-09	2.0E-04
0.24	0.242311	0.242311	0.244227	3.7E-08	1.4E-04
0.32	0.325489	0.325489	0.325451	2.0E-07	1.1E-04
0.40	0.410752	0.410752	0.410713	7.9E-07	9.5E-05
0.48	0.498646	0.498644	0.498608	2.3E-06	7.5E-05
0.56	0.589732	0.589728	0.589699	5.8E-06	5.5E-05
0.64	0.684594	0.684585	0.684569	1.2E-05	3.6E-05
0.72	0.783840	0.783820	0.783825	2.5E-05	1.9E-05
0.80	0.888106	0.888064	0.888106	4.7E-05	1.8E-05
0.88	0.998058	0.997976	0.998484	2.1E-05	4.2E-04
0.96	1.114400	1.11425	1.114990	1.3E-04	5.2E-04

Table 6: Numerical results for Example 5.2 (N = 8).

ν	$w(\nu)$	$w_N(\nu)$	$w_N(\nu)$ [5]	$R_N(\nu)$	$R_N(\nu)$ [5]
0.08	0.080085	0.080085	0.080084	4.5E-15	6.0E-06
0.16	0.160684	0.160684	0.160683	1.1E-12	4.6E-06
0.24	0.242311	0.242311	0.242310	3.0E-11	4.0E-06
0.32	0.325489	0.325489	0.325488	2.9E-10	3.4E-06
0.40	0.410752	0.410752	0.410751	1.7E-09	3.0E-06
0.48	0.498646	0.498646	0.498644	7.4E-09	2.4E-06
0.56	0.589732	0.589732	0.589731	2.5E-08	2.0E-06
0.64	0.684594	0.684594	0.684593	7.2E-08	1.3E-06
0.72	0.783840	0.783840	0.783840	1.8E-07	6.8E-07
0.80	0.888106	0.888106	0.888106	4.1E-07	1.7E-13
0.88	0.998058	0.998058	0.998074	8.8E-07	1.5E-05
0.96	1.114400	1.114400	1.114430	1.7E-06	2.7E-05

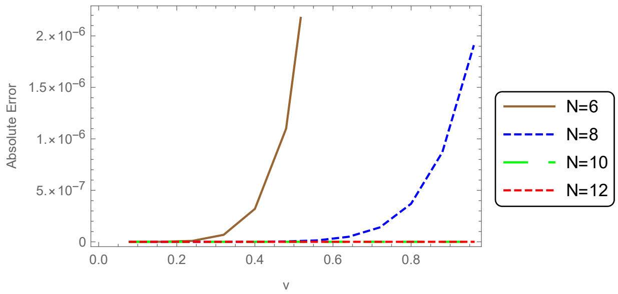

Figure 4: Absolute error of DTM for N=6, 8, 10, and 12 of Example 5.2.

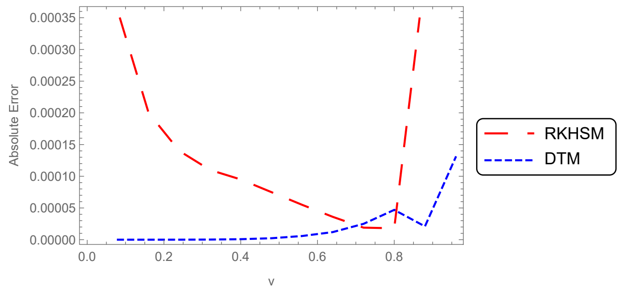

Figure 5: Comparison of Relative error for N=6 of Example 5.2.

Table 7: Maximum absolute errors for w in Example 5.2.

N	$E_{N,\infty}$
6	1.5E-04
8	1.9E-06
10	4.1E-13
12	9.4E-11

Tables 5 and 6 compare the analytical solution, approximate solutions, and relative errors for the present technique to the method discussed in [5] for $N = 6$ and 8, where N denotes the number of series components. It should be emphasized that when considering six series terms with the present method, the attained accuracy is of order 10^{-4} up to 10^{-11}, whereas the accuracy achieved by the RKHSM [5] when eleven terms were considered is of order 10^{-4}. Moreover, when eight series terms are considered with the present method, the accuracy is increased and becomes of order 10^{-6} up to 10^{-15}. It should be noted that the achieved accuracy by the [5] even when fifty-one terms were considered was of order 10^{-5} or 10^{-6}. Figure 5 shows a comparison of the relative error obtained by the present approach with RKHSM [5]. It is obvious, that in comparison to [5], the numerical results show that the present method achieves a better approximate solution.

Figure 4 and Table 7 show the absolute errors and maximum absolute errors for the present approach for various values of N. The tables and graph show that as the number of components increases, the absolute error, relative error, and maximum absolute error all decrease. As a result, adding more terms improves the method's accuracy.

Example 5.3. Consider the following nonlinear singular boundary value problem [5],

$$v(1-v)w''(v)+6w'(v)+2w(v)+w^2(v)=6\cosh v+\sinh v\left(2+v-v^2+\sinh v\right),\ 0\leqslant v\leqslant 1,$$
$$w(0)=0, w(1)+\frac{1}{2}w\left(\frac{5}{6}\right)=\frac{\sinh\frac{5}{6}}{2}+\sinh 1.$$

The exact solution is given by

$$w(v)=\sinh v.$$

Table 8: Numerical results for Example 5.3 (N = 6).

v	$w(v)$	$w_N(v)$	$w_N(v)$ [5]	$R_N(v)$	$R_N(v)$ [5]
0.08	0.080085	0.080085	0.080085	5.9E-11	3.4E-07
0.16	0.160684	0.160684	0.160683	3.3E-09	1.0E-06
0.24	0.242311	0.242311	0.242310	3.7E-08	1.1E-06
0.32	0.325489	0.325489	0.325489	2.0E-07	9.5E-07
0.40	0.410752	0.410752	0.410752	7.9E-07	1.3E-06
0.48	0.498646	0.498644	0.498645	2.3E-06	1.8E-06
0.56	0.589732	0.589728	0.589731	5.8E-06	1.7E-06
0.64	0.684594	0.684585	0.684593	1.2E-05	2.0E-06
0.72	0.783840	0.783820	0.783843	2.5E-05	3.0E-06
0.80	0.888106	0.888064	0.888119	4.7E-05	1.5E-05
0.88	0.998058	0.997976	0.997949	2.1E-05	1.5E-05
0.96	1.114400	1.11425	1.114860	1.3E-04	4.1E-04

Table 9: Numerical results for Example 5.3 (N = 8).

v	$w(v)$	$w_N(v)$	$w_N(v)$ [5]	$R_N(v)$	$R_N(v)$ [5]
0.08	0.080085	0.080085	0.080085	4.5E-15	3.0E-08
0.16	0.160684	0.160684	0.160684	1.1E-12	6.7E-08
0.24	0.242311	0.242311	0.242311	3.0E-11	2.6E-07
0.32	0.325489	0.325489	0.325489	2.9E-10	2.8E-07
0.40	0.410752	0.410752	0.410752	1.7E-09	3.1E-07
0.48	0.498646	0.498646	0.498645	7.4E-09	4.0E-07
0.56	0.589732	0.589732	0.589732	2.5E-08	3.0E-07
0.64	0.684594	0.684594	0.684594	7.2E-08	3.0E-07
0.72	0.783840	0.783840	0.783840	1.8E-07	5.2E-09
0.80	0.888106	0.888106	0.888108	4.1E-07	1.8E-06
0.88	0.998058	0.998058	0.998061	8.8E-07	2.7E-06
0.96	1.114400	1.114400	1.114430	1.7E-06	2.2E-05

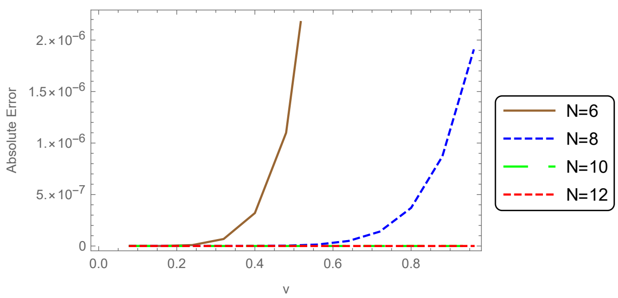

Figure 6: Absolute error of DTM for N=6, 8, 10, and 12 of Example 5.3.

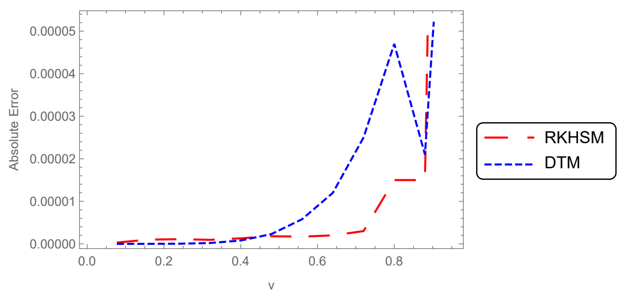

Figure 7: Comparison of Relative error for N=6 of Example 5.3.

Table 10: Maximum absolute errors for w in Example 5.3.

N	$E_{N,\infty}$
6	1.5E-04
8	1.9E-06
10	4.1E-13
12	9.4E-11

Tables 8 and 9 compare the analytical solution, approximate solutions, and relative errors for the present technique to the method discussed in [5] for $N = 6$ and 8, where N denotes the number of series components. It should be emphasized that when considering six series terms with the present method, the attained accuracy is of order 10^{-4} up to 10^{-11}, whereas the accuracy achieved by the RKHSM [5] when eleven terms were considered is of order 10^{-4} up to 10^{-7}. Moreover, when eight series terms are considered with the present method, the accuracy is increased and becomes of order 10^{-6} up to 10^{-15}. It should be noted that the achieved accuracy by the [5] even when twenty-one terms were considered was of order 10^{-5} up to 10^{-8}. Figure 7 shows a comparison of the relative error obtained by the present approach with RKHSM [5]. It is obvious, that in comparison to [5], the numerical results show that the present method achieves a better approximate solution.

Figure 6 and Table 10 show the absolute errors and maximum absolute errors for the present approach for various values of N. The tables and graph show that as the number of components increases, the absolute error, relative error, and maximum absolute error all decrease. As a result, adding more terms improves the method's accuracy.

Example 5.4. Consider the following nonlinear singular three point boundary value problem, [5],

$$v(1-v)w''(v) + 10w'(v) + 2w(v) + w^5(v) = \sin^5 v - (1-v)v\sin v + 2\sin v + 10\cos v,\ 0 < v \leqslant 1,$$

$$w(0) = 0, w(1) + \frac{1}{2}w\left(\frac{5}{6}\right) = \frac{\sin\frac{5}{6}}{2} + \sin 1.$$

The exact solution is given by

$$w(v) = \sin v.$$

Table 11: Numerical results for Example 5.4 (N = 8).

v	$w(v)$	$w_N(v)$	$w_N(v)$ [5]	$R_N(v)$	$R_N(v)$ [5]
0.08	0.079914	0.079914	0.079914	4.8E-15	1.3E-07
0.16	0.159318	0.159318	0.159318	1.1E-12	5.6E-07
0.24	0.237703	0.237703	0.237702	3.0E-11	5.6E-07
0.32	0.314567	0.314567	0.314567	3.0E-10	3.5E-07
0.40	0.389418	0.389418	0.389418	1.8E-09	4.6E-07
0.48	0.461779	0.461779	0.461779	8.0E-09	8.0E-07
0.56	0.531186	0.531186	0.531186	2.8E-08	9.2E-07
0.64	0.597195	0.597195	0.597195	8.2E-08	2.2E-07
0.72	0.659385	0.659385	0.659384	2.1E-07	7.5E-07
0.80	0.717356	0.717356	0.717355	5.1E-07	1.9E-06
0.88	0.770739	0.770738	0.770787	1.1E-06	6.2E-05
0.96	0.819192	0.819190	0.818854	2.3E-06	4.1E-04

Table 12: Numerical results for Example 5.4 (N = 10).

ν	$w(\nu)$	$w_N(\nu)$	$w_N(\nu)$ [5]	$R_N(\nu)$	$R_N(\nu)$ [5]
0.08	0.079914	0.079914	0.079914	1.7E-16	4.0E-08
0.16	0.159318	0.159318	0.159318	1.7E-16	3.4E-08
0.24	0.237703	0.237703	0.237703	1.5E-14	1.0E-07
0.32	0.314567	0.314567	0.314567	2.8E-13	1.0E-07
0.40	0.389418	0.389418	0.389418	2.6E-12	1.2E-07
0.48	0.461779	0.461779	0.461779	1.6E-11	1.6E-07
0.56	0.531186	0.531186	0.531186	7.9E-11	1.1E-07
0.64	0.597195	0.597195	0.597195	3.0E-10	1.3E-07
0.72	0.659385	0.659385	0.659385	1.0E-09	8.4E-08
0.80	0.717356	0.717356	0.717356	2.9E-09	1.7E-08
0.88	0.770739	0.770739	0.770735	7.2E-09	5.2E-06
0.96	0.819192	0.819192	0.819168	1.9E-08	2.8E-05

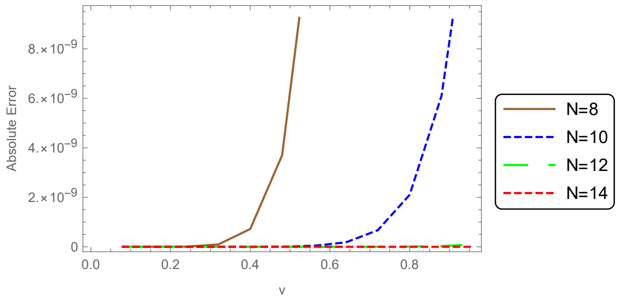

Figure 8: Absolute error of DTM for N=8, 10, 12, and 14 of Example 5.4.

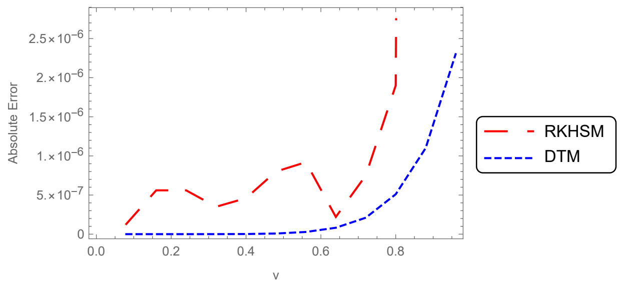

Figure 9: Comparison of Relative error for N=8 of Example 5.4.

Table 13: Maximum absolute errors for w in Example 5.4.

N	$E_{N,\infty}$
8	1.8E-06
10	1.5E-08
12	9.4E-11
14	4.1E-13

Tables 11 and 12 compare the analytical solution, approximate solutions, and relative errors for the present technique to the method discussed in [5] for $N = 8$ and 10, where N denotes the number of series components. It should be emphasized that when considering eight series terms with the present method, the attained accuracy is of order 10^{-6} up to 10^{-15}, whereas the accuracy achieved by the RKHSM [5] when eleven terms were considered is of order 10^{-4} up to 10^{-7}. Moreover, when ten series terms are considered with the present method, the accuracy is increased and becomes of order 10^{-8} up to 10^{-16}. It should be noted that the achieved accuracy by the [5] even when twenty-one terms were considered was of order 10^{-5} up to 10^{-8}. Figure 9 shows a comparison of the relative error obtained by the present approach with RKHSM [5]. It is obvious, that in comparison to [5], the numerical results show that the present method achieves a better approximate solution.

Figure 8 and Table 13 show the absolute errors and maximum absolute errors for the present approach for various values of N. The tables and graph show that as the number of components increases, the absolute error, relative error, and maximum absolute error all decrease. As a result, adding more terms improves the method's accuracy.

6. Conclusion

In this paper, certain three-point singular boundary value problems are solved by differential transform method. The numerical solutions are in good agreement with other methods. The nonlinear term in the differential equation is transformed by a convenient modification of the Faà di Bruno's formula. Relative errors, absolute errors and maximum absolute errors are illustrated in tables and figures in detail. The method is convenient for implementation and provides results with high accuracy and without complicated calculations. The method discussed in the paper is expected to work for delay differential equations or differential algebraic equations.

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References

[1] R. Agarwal, D. Regan, I. Rachunkova, S. Stanek, *Two-point higher-order bvps with singularities in phase variables*, Computer and Mathematics with Applications, **46** (2003), 1799–1826. 1

[2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions (Translated from the French by J. W. Nienhuys)*, D. Reidel Publishing Company, Boston-Dordrecht, (1974). 2.4

[3] G. J. Cooper, *Error bounds for numerical solutions of ordinary differential equations*, Numer. Math., **18** (2009), 162–170. 4

[4] M. Dehghan, F. Shakeri, *A semi-numerical technique for solving the multi-point boundary value problems and engineering applications*, Int. J. Numer. Methods Heat Fluid Flow, **21** (2011), 794–809. 1

[5] F. Geng, *Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method*, Appl. Math. Comput., **215** (2009), 2095–2112. 1, 2, 4, 5.1, 2, 3, 5.1, 5.2, 5, 6, 5.2, 5.3, 8, 9, 5.3, 5.4, 11, 12, 5.4

[6] M. K. Kadalbajoo, V. K. Aggarwal, *Numerical solution of singular boundary value problems via Chebyshev polynomial and b-spline*, Appl. Math. Comput., **160** (2005), 851–863. 1

[7] M. Kumar, N. Singh, *Modified adomian decomposition method and computer implementation for solving singular boundary value problems arising in various physical problems*, Comput. Chemical Eng., **34** (2010), 1750–1760. 1

[8] A. Y. Lepin, V. D. Ponomarev, *On a positive solution of a three-point boundary value problem*, Differ. Equ., **42** (2006), 291–293. 1

[9] G. R. Liu, T. Y. Wu, *Multipoint boundary value problems by differential quadrature method*, Math. Comput. Modelling, **35** (2002), 215–227. 1

[10] A. P. Palamides, N. M. Stavrakakis, *Existence and uniqueness of a positive solution for a third-order three-point boundary-value problem*, Electron. J. Differential Equations, **2010** (2010), 12 pages. 1

[11] B. Pandit, A. K. Verma, R. P. Agarwal, *Numerical approximations for a class of nonlinear higher order singular boundary value problem by using homotopy perturbation and variational iteration method*, Comput. Math. Methods, **2021** (2021), 18 pages. 1

[12] Z. Pátíková, J. Rebenda, *Applications of the differential transform to second-order half-linear Euler equations*, J. Comput. Sci., **59** (2022), 6 pages. 2.1

[13] G. E. Pukhov, *Differential transforms and circuit theory*, J. Circuit Theor. Appl., **10** (1982), 265–276. 1

[14] A. S. V. Ravi Kanth, *Cubic spline polynomial for non-linear singular two-point boundary value problems*, Appl. Math. Comput., **189** (2007), 2017–2022. 1

[15] A. S. V. Ravi Kanth, K. Aruna, *Solution of Singular Two-Point Boundary Value Problems Using Differential Transformation Method*, Phys. Lett. A, **372** (2008), 4671–4673. 1

[16] J. Rebenda, *An application of Bell polynomials in numerical solving of nonlinear differential equations*, arXiv, **2019** (2019), 10 pages. 2.2, 2.5, 2.6

[17] J. Rebenda, *Application of Differential Transform to Multi-Term Fractional Differential Equations with Non-Commensurate Orders*, Symmetry, **11** (2019), 10 pages. 1

[18] J. Rebenda, Z. Pátíková, *Differential Transform Algorithm for Functional Differential Equations with Time-Dependent Delays*, Complexity, **2020** (2020), 12 pages. 2.1

[19] J. Rebenda, Z. Šmarda, *A differential transformation approach for solving functional differential equations with multiple delays*, Commun. Nonlinear Sci. Numer. Simul., **48** (2017), 311–318. 1, 2.1, 2.2

[20] J. Rebenda, Z. Šmarda, *Numerical algorithm for nonlinear delayed differential systems of* n*th order*, Adv. Difference Equ., **2019** (2019), 1–13. 2.2

[21] P. Roul, V. M. K. Prasad Goura, *B-spline collocation methods and their convergence for a class of nonlinear derivative dependent singular boundary value problems*, Applied Mathematics and Computation, **341** (2019), 428–450. 1

[22] P. Roul, V. M. K. Prasad Goura, R. Agarwal, *A compact finite difference method for a general class of nonlinear singular boundary value problems with Neumann and Robin boundary conditions*, Appl. Math. Comput., **350** (2019), 283–304. 1

[23] P. Roul, K. Thula, *A new high-order numerical method for solving singular two-point boundary value problems*, J. Comput. Appl. Math., **2018** (2018), 556–574. 1

[24] A. Saadatmandi, M. Dehghan, *The use of sinc-collocation method for solving multi-point boundary value problems*,Commun. Nonlinear Sci. Numer. Simul., **17** (2012), 593–601. 1

[25] N. Sharma, L. N. Mishra, V. N. Mishra, S. Pandey, *Solution of Delay Differential equation via* N_1^v *iteration algorithm*, Eur. J. Pure Appl. Math., **13** (2020), 1110–1130. 3

[26] R. Singh, J. Kumar, *An efficient numerical technique for the solution of nonlinear singular boundary value problems*, Comput. Phys. Commun., **185** (2014), 1282–1289. 3

[27] M. Singh, A. K. Verma, R. P. Agarwal, *On an iterative method for a class of* 2 *point &* 3 *point nonlinear SBVPs*, J. Appl. Anal. Comput., **9** (2019), 1242–1260. 1

[28] H. Thompson, C. Tisdell, *Three-point boundary value problems for second-order, ordinary, differential equation*, Math. Comput. Modelling, **34** (2001), 311–318. 1

[29] Umesh, M. Kumar, *Numerical solution of singular boundary value problems using advanced Adomian decomposition method*, Eng. Comput., **2020** (2020), 11 pages. 3

[30] A. K. Verma, B. Pandit, R. P. Agarwal, *Existence and nonexistence results for a class of fourth-order coupled singular boundary value problems arising in the theory of epitaxial growth*, Math. Methods Appl. Sci., **2020** (2020), 34 pages. 1

[31] A. K. Verma, B. Pandit, L. Verma, R. P. Agarwal, *A Review on a Class of Second Order Nonlinear Singular BVPs*, Mathematics, **5** (2020), 50 pages. 1

[32] P. G. Warne, D. A. Polignone Warne, J. S. Sochacki, G. E. Parker, D. C. Carothers, *Explicit a-priori error bounds and adaptive error control for approximation of nonlinear initial value differential systems*, Comput. Math. Appl., **52** (2006), 1695–1710. 4

[33] L. J. Xie, C. I. Zhou, S. Xu, *An effective numerical method to solve a class of nonlinear singular boundary value problems using improved differential transform method*, SpringerPlus, **5** (2016), 1–19. 4

[34] X. U. Zhang, L. S. Liu, C. X. Wu, *Nontrivial solution of third-order nonlinear eigenvalue problems*, Appl. Math. Comput., **176** (2006), 714–721. 1