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External and weak direct products of UP (BCC)-algebras

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Abstract

The concept of the direct product of finite family of B-algebras is introduced by Lingcong and Endam [J. A. V. Lingcong, J. C. Endam, Int. J. Algebra, **10** (2016), 33–40]. In this paper, we introduce the concept of the direct product of infinite family of UP (BCC)-algebras, we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam, and find the result of the external direct product of special subsets of UP (BCC)-algebras. Also, we introduce the concept of the weak direct product of UP (BCC)-algebras. Finally, we provide several fundamental theorems of (anti-)UP (BCC)-homomorphisms in view of the external direct product UP (BCC)-algebras.

Keywords: UP-algebra, BCC-algebra, external direct product, weak direct product, UP (BCC)-homomorphism, anti-UP (BCC)-homomorphism.

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1. Introduction and preliminaries

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [15, 16]. In 2002, Neggers and Kim [29] constructed a new algebraic structure. They took some properties from BCI and BCK-algebras be called a B-algebra. Furthermore, Kim and Kim [22] introduced a new notion, called a BG-algebra which is a generalization of B-algebra. They obtained several isomorphism theorems of BG-algebras and related properties.

The notion of UP-algebras was introduced by Iampan [11] in 2017, and it is known that the class of KU-algebras [30] is a proper subclass of the class of UP-algebras. It have been examined by several

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researchers, for example, graphs associated with commutative UP-algebras and a graph of equivalence classes of commutative UP-algebras by Ansari et al. in 2018 [2]. In the same year Senapati et al. [36] applied the cubic set structure in UP-algebras and proved the results based on them. In 2019, Satirad et al. [32] proved every nonempty set and every nonempty totally ordered set can be a UP-algebra. In 2020, Romano and Jun [31] introduced the concept of weak implicative UP-filters in UP-algebras, etc. In 2022, Jun et al. [17] have shown that the concept of UP-algebras (see [11]) and the concept of BCC-algebras (see [24]) are the same concept. Therefore, in this article and future research, our research team will use the name BCC instead of UP in honor of Komori, who first defined it in 1984.

The concept of the direct product [35] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group, and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [25] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras and we obtained some of their properties [26]. In the same year, Endam and Teves [8] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [40] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [35] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [21] defined the direct product of GK algebra. They derived the finite form of the direct product of GK algebra and function as well. They investigated and applied the concept of the direct product of GK algebra in GK function and GK kernel and obtained interesting results.

In this paper, we introduce the concept of the direct product of infinite family of BCC-algebras, we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [25]. Moreover, we introduce the concept of the weak direct product of BCC-algebras. Finally, we discuss several (anti-)BCC-homomorphism theorems in view of the external direct product BCC-algebras.

The concept of BCC-algebras (see [24]) can be redefined without the condition (1.1) as follows.

Definition 1.1 ([10])**.** An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *BCC-algebra* if it satisfies the following axioms:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0), \tag{BCC-1}$$

$$(\forall x \in X)(0 * x = x), \tag{BCC-2}$$

$$(\forall x \in X)(x * 0 = 0), \tag{BCC-3}$$

$$(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y). \tag{BCC-4}$$

Example 1.2. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	4	0	6
2	0	1	0	3	4	5	6
3	0	1	2	0	4	0	6
4	0	1	2	3	0	5	6
5	0	1	2	3	4	0	6
6	0	1	2	3	4	5	0

Then $X = (X; *, 0)$ is a BCC-algebra.

Example 1.3 ([33]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$, where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$, where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X); \cdot, \Omega)$ is a BCC-algebra and we shall call it the *generalized power BCC-algebra of type 1 with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X); *, \Omega)$ is a BCC-algebra and we shall call it the *generalized power BCC-algebra of type 2 with respect to* Ω. In particular, $(\mathcal{P}(X); \cdot, \emptyset)$ is a BCC-algebra and we shall call it the *power BCC-algebra of type 1*, and $(\mathcal{P}(X); *, X)$ is a BCC-algebra and we shall call it the *power BCC-algebra of type 2*.

Example 1.4 ([7]). Let $\mathbb{N}_0$ be the set of all natural numbers with zero. Define two binary operations $*$ and $\bullet$ on $\mathbb{N}_0$ by

$$(\forall x, y \in \mathbb{N}_0)\left(x * y = \begin{cases} y, & \text{if } x < y, \\ 0, & \text{otherwise,} \end{cases}\right)$$

and

$$(\forall x, y \in \mathbb{N}_0)\left(x \bullet y = \begin{cases} y, & \text{if } x > y \text{ or } x = 0, \\ 0, & \text{otherwise.} \end{cases}\right)$$

Then $(\mathbb{N}_0; *, 0)$ and $(\mathbb{N}_0; \bullet, 0)$ are BCC-algebras.

For more examples of BCC-algebras, see [2, 3, 6, 12, 14, 23, 32, 33, 36, 37].

Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be BCC-algebras. A map $\varphi : A \to B$ is called a *BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker\varphi$, is defined to be the $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker\varphi$ is a BCC-ideal of A, and $\ker\varphi = \{0_A\}$ if and only if φ is injective. A (anti-)BCC-homomorphism φ is called a (anti-)BCC-monomorphism, (anti-)BCC-epimorphism, or (anti-)BCC-isomorphism if φ is injective, surjective, or bijective, respectively.

In a BCC-algebra $X = (X; *, 0)$, the following assertions are valid (see [11, 12]).

$$\begin{aligned}
&(\forall x \in X)(x * x = 0), \qquad (1.1)\\
&(\forall x, y, z \in X)(x * y = 0, y * z = 0 \Rightarrow x * z = 0),\\
&(\forall x, y, z \in X)(x * y = 0 \Rightarrow (z * x) * (z * y) = 0),\\
&(\forall x, y, z \in X)(x * y = 0 \Rightarrow (y * z) * (x * z) = 0),\\
&(\forall x, y \in X)(x * (y * x) = 0),\\
&(\forall x, y \in X)((y * x) * x = 0 \Leftrightarrow x = y * x),\\
&(\forall x, y \in X)(x * (y * y) = 0),\\
&(\forall u, x, y, z \in X)((x * (y * z)) * (x * ((u * y) * (u * z))) = 0),\\
&(\forall u, x, y, z \in X)((((u * x) * (u * y)) * z) * ((x * y) * z) = 0),\\
&(\forall x, y, z \in X)(((x * y) * z) * (y * z) = 0),\\
&(\forall x, y, z \in X)(x * y = 0 \Rightarrow x * (z * y) = 0),\\
&(\forall x, y, z \in X)(((x * y) * z) * (x * (y * z)) = 0),\\
&(\forall u, x, y, z \in X)(((x * y) * z) * (y * (u * z)) = 0).
\end{aligned}$$

According to [11], the binary relation $\leqslant$ on a BCC-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leqslant y \Leftrightarrow x * y = 0).$$

Definition 1.5. A BCC-algebra $X = (X; *, 0)$ is said to be

(i) *bounded* if there is an element $1 \in X$ such that $1 \leqslant x$ for all $x \in X$, that is,

$$(\forall x \in X)(1 * x = 0); \tag{Bounded}$$

(ii) *meet-commutative* [34] if it satisfies the identity

$$(\forall x, y \in X)(x \wedge y = y \wedge x), \tag{Meet-commutative}$$

where

$$(\forall x, y \in X)(x \wedge y = (y * x) * x). \tag{Meet}$$

Example 1.6. Let $X = \{0, 1, 2, 3\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	0	0	0

Then $X = (X; *, 0)$ is a bounded BCC-algebra.

Example 1.7. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	1	2	0	4
4	0	1	2	3	0

Then $X = (X; *, 0)$ is a meet-commutative BCC-algebra.

Definition 1.8 ([9, 11, 13, 18–20, 38])**.** A nonempty subset S of a BCC-algebra $X = (X; *, 0)$ is called

(i) a *BCC-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in S)(x * y \in S);$$

(ii) a *near BCC-filter* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x * y \in S); \tag{1.2}$$

(iii) a *BCC-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.3}$$

$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S); \tag{1.4}$$

(iv) an *implicative BCC-filter* of X if it satisfies the condition (1.3) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S); \tag{1.5}$$

(v) a *comparative BCC-filter* of X if it satisfies the condition (1.3) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S); \tag{1.6}$$

(vi) a *shift BCC-filter* of X if it satisfies the condition (1.3) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S); \tag{1.7}$$

(vii) a *BCC-ideal* of X if it satisfies the condition (1.3) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S); \tag{1.8}$$

(viii) a *strong BCC-ideal* of X if it satisfies the condition (1.3) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{1.9}$$

We get the diagram of the special subsets of BCC-algebras, which is shown with Figure 1.

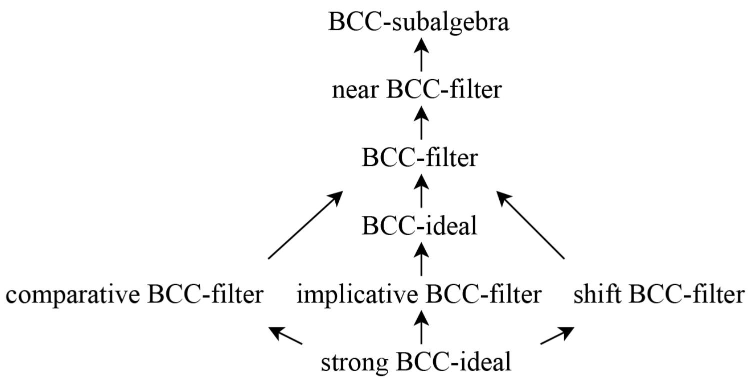

Figure 1: Special subsets of BCC-algebras.

2. External direct product of BCC-algebras

Lingcong and Endam [25] discussed the notion of the direct product of *B*-algebras, 0-commutative *B*-algebras, and *B*-homomorphisms and obtained related properties, one of which is a direct product of two *B*-algebras, which is also a *B*-algebra. Then, they extended the concept of the direct product of *B*-algebra to finite family *B*-algebra, and some of the related properties were investigated as follows.

Definition 2.1 ([25])**.** Let $(X_i; *_i)$ be an algebra for each $i \in \{1, 2, \dots, k\}$. Define the *direct product* of algebras $X_1, X_2, \dots, X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times \cdots \times X_k = \{(x_1, x_2, \dots, x_k) \mid x_i \in X_i \ \forall i = 1, 2, \dots, k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, \dots, x_k) \otimes (y_1, y_2, \dots, y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, \dots, x_k *_k y_k)$$

for all $(x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k) \in \prod_{i=1}^{k} X_i$.

Now, we extend the concept of the direct product to infinite family of BCC-algebras and provide some of its properties.

Definition 2.2. Let X_i be a nonempty set for each $i \in I$. Define the *external direct product* of sets X_i for all $i \in I$ to be the set $\prod_{i\in I} X_i$, where

$$\prod_{i\in I} X_i = \{f : I \to \bigcup_{i\in I} X_i \mid f(i) \in X_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i\in I} X_i$ with a function $(x_i)_{i\in I} : I \to \bigcup_{i\in I} X_i$, where $i \mapsto x_i \in X_i$ for all $i \in I$.

Remark 2.3. Let X_i be a nonempty set and S_i a subset of X_i for all $i \in I$. Then $\prod_{i\in I} S_i$ is a nonempty subset of the external direct product $\prod_{i\in I} X_i$ if and only if S_i is a nonempty subset of X_i for all $i \in I$.

Definition 2.4. Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\otimes$ on the external direct product $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes)$ as follows:

$$(\forall (x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i)((x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I}).$$

We shall show that $\otimes$ is a binary operation on $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i y_i \in X_i$ for all $i \in I$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x'_i)_{i\in I}, (y'_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x'_i)_{i\in I} = (y'_i)_{i\in I}$. We shall show that $(x_i)_{i\in I} \otimes (x'_i)_{i\in I} = (y_i)_{i\in I} \otimes (y'_i)_{i\in I}$. Then

$$x_i = y_i \text{ for all } i \in I \text{ and } x'_i = y'_i \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i x'_i = y_i *_i y'_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \otimes (x'_i)_{i\in I} = (x_i *_i x'_i)_{i\in I} = (y_i *_i y'_i)_{i\in I} = (y_i)_{i\in I} \otimes (y'_i)_{i\in I}.$$

Hence, $\otimes$ is a binary operation on $\prod_{i\in I} X_i$.

Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(f_{x_i}(j) = \begin{cases} x_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right). \tag{2.1}$$

Then $f_{x_i} \in \prod_{i\in I} X_i$.

Lemma 2.5. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i, \\ 0_j *_j 0_j, & \text{otherwise,} \end{cases}\right).$$

By (1.1), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right).$$

By (2.1), we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$. □

The following theorem shows that the direct product of BCC-algebras in term of infinite family of BCC-algebras is also a BCC-algebra.

Theorem 2.6. $X_i = (X_i; *_i, 0_i)$ *is a BCC-algebra for all* $i \in I$ *if and only if* $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ *is a BCC-algebra, where the binary operation* $\otimes$ *is defined in Definition 2.4.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$.

(BCC-1) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-1), we have $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$ for all $i \in I$. Thus

$$\begin{aligned}((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((x_i)_{i\in I} \otimes (z_i)_{i\in I}))\\ &= (y_i *_i z_i)_{i\in I} \otimes ((x_i *_i y_i)_{i\in I} \otimes (x_i *_i z_i)_{i\in I})\\ &= (y_i *_i z_i)_{i\in I} \otimes ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I}\\ &= ((y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)))_{i\in I} = (0_i)_{i\in I}.\end{aligned}$$

(BCC-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(0_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(BCC-3) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (BCC-3), we have $x_i *_i 0_i = 0_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \otimes (0_i)_{i\in I} = (x_i *_i 0_i)_{i\in I} = (0_i)_{i\in I}.$$

(BCC-4) Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i)_{i\in I}$. Then $(x_i *_i y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$, so $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Since X_i satisfies (BCC-4), we have $x_i = y_i$ for all $i \in I$. Therefore, $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. Let $i \in I$.

(BCC-1) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-1), we have $(f_{y_i} \otimes f_{z_i}) \otimes ((f_{x_i} \otimes f_{y_i}) \otimes (f_{x_i} \otimes f_{z_i})) = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left(((f_{y_i} \otimes f_{z_i}) \otimes ((f_{x_i} \otimes f_{y_i}) \otimes (f_{x_i} \otimes f_{z_i})))(j) = \begin{cases} (y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)), & \text{if } j = i,\\ (0_j *_j 0_j) *_j ((0_j *_j 0_j) *_j (0_j *_j 0_j)), & \text{otherwise,}\end{cases}\right),$$

this implies that $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$.

(BCC-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-2), we have $(0_i)_{i\in I} \otimes f_{x_i} = f_{x_i}$. Now,

$$(\forall j \in I)\left(((0_i)_{i\in I} \otimes f_{x_i})(j) = \begin{cases} 0_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

this implies that $0_i *_i x_i = x_i$.

(BCC-3) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies (BCC-3), we have $f_{x_i} \otimes (0_i)_{i\in I} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left((f_{x_i} \otimes (0_i)_{i\in I})(j) = \begin{cases} x_i *_i 0_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

this implies that $x_i *_i 0_i = 0_i$.

(BCC-4) Let $x_i, y_i \in X_i$ be such that $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} y_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right).$$

By assumption and (1.1), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} 0_i, & \text{if } j = i,\\ 0_j, & \text{otherwise,}\end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} 0_i, & \text{if } j = i,\\ 0_j, & \text{otherwise,}\end{cases}\right).$$

Thus $f_{x_i} \otimes f_{y_i} = (0_i)_{i\in I}$ and $f_{y_i} \otimes f_{x_i} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ satisfies (BCC-4), we have $f_{x_i} = f_{y_i}$. Therefore, $x_i = y_i$. Hence, $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$. □

We call the BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ in Theorem 2.6 the external direct product BCC-algebra induced by a BCC-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Theorem 2.7. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a bounded BCC-algebra, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.6, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is bounded for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is bounded.

Assume that X_i is bounded for all $i \in I$. Then there exists $1_i \in X_i$ be such that $1_i \leqslant x_i$ for all $x_i \in X_i$. That is, $1_i * x_i = 0_i$ for all $i \in I$. Now, $(1_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Thus

$$(1_i)_{i\in I} \otimes (x_i)_{i\in I} = (1_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

That is, $(1_i)_{i\in I} \leqslant (x_i)_{i\in I}$. Hence, $\prod_{i\in I} X_i$ is bounded.

Conversely, assume that $\prod_{i\in I} X_i$ is bounded. Then there exists $(1_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $(1_i)_{i\in I} \leqslant (x_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. That is, $(1_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $i \in I$. Now, $1_i \in X_i$. Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ is bounded, we have $(1_i)_{i\in I} \otimes f_{x_i} = (0_i)_{i\in I}$. By Lemma 2.5, we have $f_{1_i *_i x_i} = (0_i)_{i\in I}$. By (2.1), we have $1_i *_i x_i = 0_i$. That is, $1_i \leqslant x_i$. Hence, X_i is bounded for all $i \in I$. □

Theorem 2.8. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a meet-commutative BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a meet-commutative BCC-algebra, where the binary operation $\otimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.6, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a BCC-algebra, where the binary operation $\otimes$ is defined in Definition 2.4. We are left to prove that X_i is meet-commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is meet-commutative.

Assume that X_i is meet-commutative for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is meet-commutative, we have $x_i \wedge y_i = y_i \wedge x_i$ for all $i \in I$. That is, $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$ for all $i \in I$. Thus

$$\begin{aligned}
(x_i)_{i\in I} \wedge (y_i)_{i\in I} &= ((y_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes (x_i)_{i\in I} \\
&= (y_i *_i x_i)_{i\in I} \otimes (x_i)_{i\in I} \\
&= ((y_i *_i x_i) *_i x_i)_{i\in I} \\
&= ((x_i *_i y_i) *_i y_i)_{i\in I} \\
&= (x_i *_i y_i)_{i\in I} \otimes (y_i)_{i\in I} \\
&= ((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I} \\
&= (y_i)_{i\in I} \wedge (x_i)_{i\in I}.
\end{aligned}$$

Hence, $\prod_{i\in I} X_i$ is meet-commutative.

Conversely, assume that $\prod_{i\in I} X_i$ is meet-commutative. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Since $\prod_{i\in I} X_i$ is meet-commutative, we have $f_{x_i} \wedge f_{y_i} = f_{y_i} \wedge f_{x_i}$. That is, $(f_{y_i} \otimes f_{x_i}) \otimes f_{x_i} = (f_{x_i} \otimes f_{y_i}) \otimes f_{y_i}$. By Lemma 2.5, we have $f_{(y_i *_i x_i) *_i x_i} = f_{(x_i *_i y_i) *_i y_i}$. By (2.1), we have $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$. Hence, X_i is meet-commutative for all $i \in I$. □

Next, we introduce the concept of the weak direct product of infinite family of BCC-algebras and obtain some of its properties as follows.

Definition 2.9. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Define the *weak direct product* of a BCC-algebra X_i for all $i \in I$ to be the structure $\prod^w_{i\in I} X_i = (\prod^w_{i\in I} X_i; \otimes)$, where

$$\prod^w_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^w_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 2.10. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod^w_{i\in I} X_i$ is a BCC-subalgebra of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^w_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^w_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j, & \text{if } j \in I_1 - I_2, \\ x_j *_j y_j, & \text{if } j \in I_1 \cap I_2, \\ 0_j *_j y_j, & \text{if } j \in I_2 - I_1, \\ 0_j *_j 0_j, & \text{otherwise}, \end{cases} \right).$$

By (BCC-2) and (BCC-3), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} 0_j, & \text{if } j \in I_1 - I_2, \\ x_j *_j y_j, & \text{if } j \in I_1 \cap I_2, \\ y_j, & \text{if } j \in I_2 - I_1, \\ 0_j, & \text{otherwise}, \end{cases} \right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j)$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^w_{i\in I} X_i$. Hence, $\prod^w_{i\in I} X_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 2.11. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-subalgebra of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. Thus $x_i *_i y_i \in S_i$ for all $i \in I$, so $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). Since $\prod_{i\in I} S_i$ is a BCC-subalgebra of $\prod_{i\in I} X_i$ and by Lemma 2.5, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a BCC-subalgebra of X_i for all $i \in I$. □

Theorem 2.12. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a near BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a near BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a near BCC-filter of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i \in S_i$ for all $i \in I$, it follows from (1.2) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a near BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a near BCC-filter of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). Since $\prod_{i\in I} S_i$ is a near BCC-filter of $\prod_{i\in I} X_i$ and by Lemma 2.5, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a near BCC-filter of X_i for all $i \in I$. □

Theorem 2.13. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (1.4) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.4), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $y_i \in S_i$. Hence, S_i is a BCC-filter of X_i for all $i \in I$. □

Theorem 2.14. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an implicative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an implicative BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an implicative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (1.5) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an implicative BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an implicative BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$.

Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. Since $\prod_{i\in I} S_i$ is an implicative BCC-filter of $\prod_{i\in I} X_i$ and by Lemma 2.5, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is an implicative BCC-filter of X_i for all $i \in I$. □

Theorem 2.15. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a comparative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a comparative BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a comparative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (y_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i ((y_i *_i z_i) *_i y_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$, it follows from (1.6) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a comparative BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a comparative BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{x_i} \otimes ((f_{y_i} \otimes f_{z_i}) \otimes f_{y_i}) = f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$. Since $\prod_{i\in I} S_i$ is a comparative BCC-filter of $\prod_{i\in I} X_i$, we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $y_i \in S_i$. Hence, S_i is a comparative BCC-filter of X_i for all $i \in I$. □

Theorem 2.16. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a shift BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a shift BCC-filter of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a shift BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$, it follows from (1.7) that $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$ for all $i \in I$. Thus $(((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (y_i)_{i\in I}) \otimes (z_i)_{i\in I} = (((z_i *_i y_i) *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a shift BCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a shift BCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. Since $\prod_{i\in I} S_i$ is a shift BCC-filter of $\prod_{i\in I} X_i$ and by Lemma 2.5, we have $f_{((z_i *_i y_i) *_i y_i) *_i z_i} = ((f_{z_i} \otimes f_{y_i}) \otimes f_{y_i}) \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.1), we have $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$. Hence, S_i is a shift BCC-filter of X_i for all $i \in I$. □

Theorem 2.17. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a BCC-ideal of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$, it follows from (1.8) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a BCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a BCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. Since $\prod_{i\in I} S_i$ is a BCC-ideal of $\prod_{i\in I} X_i$ and by Lemma 2.5, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is a BCC-ideal of X_i for all $i \in I$. □

Theorem 2.18. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong BCC-ideal of the external direct product BCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a strong BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes ((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((z_i *_i y_i) *_i (z_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$, it follows from (1.9) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong BCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong BCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{z_i} \otimes f_{y_i}) \otimes (f_{z_i} \otimes f_{x_i}) = f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$. Since $\prod_{i\in I} S_i$ is a strong BCC-ideal of $\prod_{i\in I} X_i$, we have $f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i \in S_i$. Hence, S_i is a strong BCC-ideal of X_i for all $i \in I$. □

Moreover, we discuss several BCC-homomorphism theorems in view of the external direct product of BCC-algebras.

Definition 2.19 ([5]). Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}).$$

Then $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function (see [5]).

Theorem 2.20 ([5]). *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$.*

(i) *ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 2.19;*

(ii) *ψ_i is surjective for all $i \in I$ if and only if ψ is surjective;*

(iii) *ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.*

Theorem 2.21. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is a BCC-homomorphism for all $i \in I$ if and only if ψ is a BCC-homomorphism which is defined in Definition 2.19;*

(ii) *ψ_i is a BCC-monomorphism for all $i \in I$ if and only if ψ is a BCC-monomorphism;*

(iii) *ψ_i is a BCC-epimorphism for all $i \in I$ if and only if ψ is a BCC-epimorphism;*

(iv) *ψ_i is a BCC-isomorphism for all $i \in I$ if and only if ψ is a BCC-isomorphism;*

(v) *$\ker \psi = \prod_{i\in I} \ker \psi_i$ and $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.*

Proof.

(i) Assume that ψ_i is a BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I}\\ &= (\psi_i(x_i *_i x'_i))_{i\in I}\\ &= (\psi_i(x_i) *_i \psi_i(x'_i))_{i\in I} = (\psi_i(x_i))_{i\in I} \otimes (\psi_i(x'_i))_{i\in I} = \psi(x_i)_{i\in I} \otimes \psi(x'_i)_{i\in I}.\end{aligned}$$

Hence, ψ is a BCC-homomorphism.

Conversely, assume that ψ is a BCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since ψ is a BCC-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{x_i}) \otimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i), & \text{if } j = i,\\ \psi_j(0_j *_j 0_j), & \text{otherwise,}\end{cases}\right). \tag{2.2}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \otimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i), & \text{if } j = i,\\ \psi_j(0_j) \circ_j \psi_j(0_j), & \text{otherwise,}\end{cases}\right). \tag{2.3}$$

By (2.2) and (2.3), we have $\psi_i(x_i *_i y_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is a BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.20 (i).

(iii) It is straightforward from (i) and Theorem 2.20 (ii).

(iv) It is straightforward from (i) and Theorem 2.20 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I}\\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I\\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I\\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I}\\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I}\\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I\\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

Finally, we discuss several anti-BCC-homomorphism theorems in view of the external direct product of BCC-algebras.

Theorem 2.22. *Let* $X_i = (X_i; *_i, 0_i)$ *and* $S_i = (S_i; \circ_i, 1_i)$ *be BCC-algebras and* $\psi_i : X_i \to S_i$ *be a function for all* $i \in I$*. Then*

(i) ψ_i *is an anti-BCC-homomorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-homomorphism which is defined in Definition 2.19;*

(ii) ψ_i *is an anti-BCC-monomorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-monomorphism;*

(iii) ψ_i *is an anti-BCC-epimorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-epimorphism;*

(iv) ψ_i *is an anti-BCC-isomorphism for all* $i \in I$ *if and only if* ψ *is an anti-BCC-isomorphism.*

Proof.

(i) Assume that ψ_i is an anti-BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I}\\ &= (\psi_i(x_i *_i x'_i))_{i\in I}\\ &= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I} = (\psi_i(x'_i))_{i\in I} \otimes (\psi_i(x_i))_{i\in I} = \psi(x'_i)_{i\in I} \otimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-BCC-homomorphism.

Conversely, assume that ψ is an anti-BCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Since ψ is an anti-BCC-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{y_i}) \otimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i), & \text{if } j = i,\\ \psi_j(0_j *_j 0_j), & \text{otherwise,}\end{cases}\right). \tag{2.4}$$

Since

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{y_i}) \otimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i), & \text{if } j = i,\\ \psi_j(0_j) \circ_j \psi_j(0_j), & \text{otherwise,}\end{cases}\right). \tag{2.5}$$

By (2.4) and (2.5), we have $\psi_i(x_i *_i y_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an anti-BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.20 (i).

(iii) It is straightforward from (i) and Theorem 2.20 (ii).

(iv) It is straightforward from (i) and Theorem 2.20 (iii). □

3. Conclusions and Future Work

In this paper, we have introduced the concept of the direct product of infinite family of BCC-algebras, we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [25]. We proved that the external direct product of BCC-algebras is also a BCC-algebra, the external direct product of a bounded BCC-algebra is also a bounded BCC-algebra, and the external direct product of meet-commutative BCC-algebra is also a meet-commutative BCC-algebra. Also, we have introduced the concept of the weak direct product of BCC-algebras. We proved that the weak direct product of BCC-algebras is a BCC-subalgebra and the external direct product of BCC-subalgebras (resp., near BCC-filters, BCC-filters, comparative BCC-filters, shift BCC-filters, implicative BCC-filters, BCC-ideals, strong BCC-ideals) is also a BCC-subalgebra (resp., near BCC-filter, BCC-filter, comparative BCC-filter, shift BCC-filter, implicative BCC-filter, BCC-ideal, strong BCC-ideal) of the external direct product BCC-algebras. Finally, we have provided several fundamental theorems of (anti-)BCC-homomorphisms in view of the external direct product BCC-algebras.

Based on the concept of the external direct product of BCC-algebras in this article, we can apply it to the study of the external direct product in other algebraic systems. Researching the external and weak direct products that we will study in the future will be the internal direct products of BCC-algebras.

The research topics of interest by our research team being studied in the external direct product of BCC-algebras are as follows.

(1) To study fuzzy set theory (with respect to a triangular norm) based on the concept of Somjanta et al. [38] and Thongarsa et al. [4, 39].

(2) To study bipolar fuzzy set theory based on the concept of Muhiuddin [27].

(3) To study interval-valued fuzzy set theory based on the concept of Muhiuddin et al. [28].

(4) To study interval-valued intuitionistic fuzzy set theory based on the concept of Senapati et al. [37].

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