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Approximation of almost quadratic mappings via modular functional

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Abstract

In this paper, we present generalized stability results of refined quadratic functional equation

$$f(ax-by)=abf(x-y)+a(a-b)f(x)+b(b-a)f(y),$$

for any fixed nonzero integer numbers $a,b\in\mathbb{Z}$ with $a\neq b$ in modular spaces. As results, we generalize stability results of a quadratic functional equation in [K.-W. Jun, H.-M. Kim, J. Son, Functional Equations in Mathematical Analysis, **2012** (2012), 153–164].

Keywords: Generalized Hyers-Ulam stability, modular functional, modular spaces.

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1. Introduction and preliminaries

In 1940, Ulam [20] brought up the question of the stability of group homomorphisms before the Mathematics Club of the University of Wisconsin:

Let G_1 *be a group and let* (G_2,d) *be a metric group. Given* $\varepsilon>0$*, does there exist a* $\delta>0$ *such that if a function* $h:G_1\to G_2$ *satisfies the inequality* $d(h(xy),h(x)h(y))\leqslant\delta$ *for all* $x,y\in G_1$*, then there exists a homomorphism* $H:G_1\to G_2$ *with* $d(h(x),H(x))<\varepsilon$ *for all* $x\in G_1$.

Ulam's question was partially solved by Hyers [10] in the case of approximately additive mappings, and by Rassias [17] in the case of approximately linear mappings of unbounded control functions. Many mathematicians have been concerned with the stability problems of several functional equations over the last three decades ([1–3, 5, 18]). In particular, the general solution of quadratic functional equation

$$f(ax+y)+af(x-y)=(a+1)f(y)+a(a+1)f(x) \tag{1.1}$$

for any fixed nonzero $a\in\mathbb{Z}$ with $a\neq-1$ can be found in the reference [11] and the generalized Hyers-Ulam stability theorem of the equation (1.1) is as follows. Let X and Y be a vector space and a Banach space, respectively, and let $\varphi:X\times X\to[0,\infty)$ and $f:X\to Y$ be mappings satisfying

$$\|f(ax+y)+af(x-y)-(a+1)f(y)-a(a+1)f(x)\|\leqslant\varphi(x,y)$$

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for all $x, y \in X$ with for any fixed integers $a \neq 0, -1, -2$. Suppose that the series

$$\Phi_1(x,y) := \sum_{i=0}^{\infty} \frac{1}{(a+1)^{2(i+1)}} \varphi((a+1)^i x, (a+1)^i y) < \infty,$$

$$\left(\Phi_2(x,y) := \sum_{i=0}^{\infty} (a+1)^{2i} \varphi\left(\frac{x}{(a+1)^{i+1}}, \frac{y}{(a+1)^{i+1}}\right) < \infty, \ \text{respectively}\right)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying

$$\left\|f(x) - \frac{1}{a+2} f(0) - Q(x)\right\| \leqslant \Phi_1(x,x), \quad (\|f(x) - Q(x)\| \leqslant \Phi_2(x,x), \ \text{respectively})$$

for all $x \in X$, where $\|f(0)\| \leqslant \frac{\varphi(0,0)}{|a(a+1)|}$. The mapping Q is given by

$$Q(x) = \lim_{n\to\infty} \frac{f((a+1)^n x)}{(a+1)^{2n}}, \quad \left(Q(x) = \lim_{n\to\infty} (a+1)^{2n} f\left(\frac{x}{(a+1)^n}\right), \ \text{respectively}\right)$$

for all $x \in X$.

In this paper, we refine the quadratic functional equation (1.1) to the following quadratic functional equation

$$f(ax - by) = abf(x-y) + a(a-b)f(x) + b(b-a)f(y) \tag{1.2}$$

for any fixed nonzero integers $a, b \in \mathbb{Z}$ with $a \neq b$, which is more general than the equation (1.1) by view of $b := -1$ in the equation (1.2) in abstract linear spaces. Then, we present generalized Hyers-Ulam stability of the equation (1.2) with $|a - b| \neq 1$, and, in addition, we investigate generalized Hyers-Ulam stability of the equation (1.2) with $|a - b| = 1$, respectively, in modular spaces with modular functionals.

Now, we recall some basic definitions and remarks of modular spaces with modular functionals, which are primitive notions corresponding to norms or metrics, as in the followings [13, 19].

Definition 1.1 ([19]). Let χ be a linear space.

(a) A functional $\rho : \chi \to [0, \infty]$ is called a modular if
  (1) $\rho(x) = 0$ if and only if $x = 0$;
  (2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$;
  (3) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$;

(b) alternatively, if (3) is replaced by
  (4) $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ together with $\alpha, \beta \geqslant 0$, and for any vectors $x, y \in \chi$,

  then we say that ρ is a convex modular.

A modular ρ defines a corresponding modular space, i.e., the linear space χ_ρ given by

$$\chi_\rho = \{x \in \chi : \rho(\lambda x) \to 0 \quad \text{as} \quad \lambda \to 0\}.$$

A modular functional ρ is said to satisfy Δ_2-condition if there exists $\kappa > 0$ such that $\rho(2x) \leqslant \kappa\rho(x)$ for all $x \in \chi_\rho$.

Here, if the modular ρ satisfies Δ_2-condition, we observe that $\kappa \geqslant 1$ for nontrivial modular ρ, and that $\kappa \geqslant 2$ for nontrivial convex modular ρ.

Remark 1.2 ([13]).

(a) If ρ is a modular on χ, we note that $\rho(tx)$ is an increasing function in $t \geqslant 0$ for each fixed $x \in \chi$, that is, $\rho(ax) \leqslant \rho(bx)$ whenever $0 \leqslant a < b$.

(b) If ρ is a convex modular on χ and $|\alpha| \leqslant 1$, then $\rho(\alpha x) \leqslant |\alpha|\rho(x)$ for all $x \in \chi$. Moreover, we see that $\rho(\alpha x) \leqslant \alpha\rho(x)$ for all $x \in \chi$, provided $0 < \alpha \leqslant 1$, and in addition, one notes $\rho\big(\sum_{i=1}^{n} \alpha_i x_i\big) \leqslant \sum_{i=1}^{n} \alpha_i \rho(x_i)$ for all $x_i \in \chi$ and $\alpha_i \geqslant 0$ $(i = 1, \dots, n)$ whenever $0 < \sum_{i=1}^{n} \alpha_i := \alpha \leqslant 1$.

Definition 1.3 ([19]). Let χ_ρ be a modular space and let $\{x_n\}$ be a sequence in χ_ρ. Then,

(1) $\{x_n\}$ is ρ-convergent to $x \in \chi_\rho$ and write $x_n \xrightarrow{\rho} x$ if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(2) $\{x_n\}$ is called ρ-Cauchy in χ_ρ if $\rho(x_n - x_m) \to 0$ as $n, m \to \infty$.

(3) A subset K of χ_ρ is called ρ-complete if and only if any ρ-Cauchy sequence is ρ-convergent to an element in K.

The modular ρ has the Fatou property if and only if $\rho(x) \leqslant \liminf_{n\to\infty} \rho(x_n)$ whenever the sequence $\{x_n\}$ is ρ-convergent to x in χ_ρ.

Example 1.4 ([19]). Let (Ω, Σ, μ) be a measure space and let $L^0(\mu)$ be the linear space of all measurable functions on Ω. Define the Orlicz modular ρ_φ by $\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\mu$ for every $f \in L^0(\mu)$, where $\varphi : [0, \infty) \to \mathbb{R}$ is assumed to be a continuous, convex and nondecreasing function such that $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$, $\varphi(t) \to \infty$ as $t \to \infty$. The corresponding modular space $L^0(\mu)_{\rho_\varphi}$ with respect to this modular ρ_φ is called an Orlicz space, and will be denoted by $L^\varphi(\Omega, \mu)$ or briefly L^φ. In other words,

$$L^\varphi \equiv L^\varphi(\Omega, \mu) = \Big\{ f \in L^0(\mu) \mid \rho_\varphi(\lambda f) = \int_\Omega \varphi(|\lambda f|)d\mu \to 0 \text{ as } \lambda \to 0 \Big\},$$

and it is known that the Orlicz space L^φ is a ρ_φ-complete modular space.

Concerning stability problems of functional equations in modular spaces, Sadeghi [19] has proved generalized Hyers-Ulam stability via the fixed point method of a generalized Jensen functional equation $f(rx + sy) = rg(x) + sh(y)$ in convex modular spaces with the Fatou property satisfying Δ_2-condition with $0 < \kappa \leqslant 2$. Recently, many mathematicians have established stability theorems of various functional equations in modular spaces (see, e.g., [8, 12, 14–16]).

This paper is organized as follows. In Section 2, we establish general solution of the functional equation (1.2), and then investigate the generalized Hyers-Ulam stability theorem of the functional equation (1.2) for any fixed nonzero integers $a, b \in \mathbb{Z}$ with $a \neq b$, $|a - b| \neq 1$. In Section 3, we investigate the generalized Hyers-Ulam stability problem of the equation (1.2) for any fixed nonzero integers $a, b \in \mathbb{Z}$ with $a \neq b$, $|a - b| = 1$.

2. Generalized Hyers-Ulam stability of Eq. (1.2) with $|a - b| \neq 1$

In this section, we establish general solution of the functional equation (1.2), and then investigate the generalized Hyers-Ulam stability theorem of the equation (1.2) for any fixed nonzero integers $a, b \in \mathbb{Z}$ with $a \neq b$, $|a - b| \neq 1$. First of all, we present the general solution of the equation (1.2) between linear spaces.

Lemma 2.1. *Let X and Y be linear spaces. Then a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the equation*

$$f(rx + y) + rf(-x + y) = r(1 + r)f(x) + (1 + r)f(y), \quad x, y \in X, \tag{2.1}$$

where $r \neq 0, -1$ is any fixed rational number, if and only if, it is quadratic.

Proof. Suppose a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the equation (2.1). Setting $y := 0$ and $y := x$, respectively, in the equation (2.1), we get

$$f(rx) + rf(-x) = r(1 + r)f(x), \quad f((r + 1)x) = (r + 1)^2 f(x) \tag{2.2}$$

for all $x \in X$. Replacing y with $-y$ in (2.1), one obtains

$$f(rx-y)+rf(-x-y)=r(1+r)f(x)+(1+r)f(-y) \tag{2.3}$$

for all $x, y \in X$. Interchanging y with $x-y$ in (2.1), we arrive at

$$f((r+1)x-y)+rf(-y)=r(1+r)f(x)+(1+r)f(x-y) \tag{2.4}$$

for all $x, y \in X$. In addition, if we change y with $rx-y$ in (2.3), one has

$$f(y)+rf(-(r+1)x+y)=r(1+r)f(x)+(1+r)f(-rx+y),$$

which is equivalent to the relation

$$f(-y)+rf((r+1)x-y)=r(1+r)f(-x)+(1+r)f(rx-y) \tag{2.5}$$

for all $x, y \in X$. Thus, associating (2.3) and (2.4) with (2.5), we lead to

$$f(x-y)+f(-x-y)=f(x)+f(-x)+2f(-y)$$

for all $x, y \in X$. Setting $x := y$ and $y := -x$ in the last equality, one obtains

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y), \quad x, y \in X,$$

which is well known as Drygas functional equation, and thus there exist an additive mapping A and a quadratic mapping Q on X such that $f(x) = Q(x) + A(x)$ for all $x \in X$. Therefore, it follows from the relations (2.2) that A is identically zero on X, that is, f is quadratic.

The reverse assertion is trivial. □

Theorem 2.2. *Let X and Y be linear spaces. Then a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the equation (1.2) if and only if f is quadratic.*

Proof. Suppose a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the equation (1.2). Setting $y := 0$ and $y := x$, respectively, in the equation (1.2), we get

$$f(ax)=a^2f(x), \quad f((a-b)x)=(a-b)^2f(x),$$

for all $x \in X$. Thus, the equation (1.2) is equivalent to the following identity

$$\begin{aligned} &f\Big(x-\frac{b}{a}y\Big)=\frac{b}{a}f(x-y)+\Big(1-\frac{b}{a}\Big)f(x)+\frac{b}{a}\Big(\frac{b}{a}-1\Big)f(y),\\ &\Leftrightarrow f(x+ry)+rf(x-y)=(1+r)f(x)+r(1+r)f(y), \quad r:=-\frac{b}{a}\neq 0,-1, \end{aligned}$$

which yields the equation (2.1) for all $x, y \in X$. Hence, the mapping f is quadratic.

The converse is trivial. □

From now on, let X be a vector space and let χ_ρ a ρ-complete convex modular space unless we give any specific reference. For notational convenience, we set the following operator $D_{a,b}f$ as

$$D_{a,b}f(x,y):=f(ax-by)-abf(x-y)-a(a-b)f(x)-b(b-a)f(y)$$

for all $x, y \in X$, which is a perturbing term of the equation (1.2) for an approximate quadratic mapping $f : X \to \chi_\rho$. Then, we investigate some conditions that there exists a true quadratic mapping near an approximate quadratic mapping for the functional equation (1.2), and thus present the generalized Hyers-Ulam stability theorem of the equation (1.2) for any fixed nonzero integers $a, b \in \mathbb{Z}$ with $a \neq b$, $|a-b| \neq 1$.

Theorem 2.3. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho\Big(D_{a,b}f(x,y)\Big) \leqslant \varphi(x,y) \tag{2.6}$$

for which a control function $\varphi : X \times X \to [0,\infty)$ *satisfies the condition*

$$\Phi_1(x,y) := \sum_{i=0}^{\infty} \frac{1}{(a-b)^{2(i+1)}} \varphi\left((a-b)^i x, (a-b)^i y\right) < \infty, \tag{I}$$

for all $x, y \in X$, *where* $|a-b| \geqslant 2$. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$, *defined as* $Q(x) = \lim_{n\to\infty} \frac{f((a-b)^n x)}{(a-b)^{2n}}$, *such that the approximation*

$$\rho\Big(f(x) - Q(x)\Big) \leqslant \Phi_1(x,x) \tag{2.7}$$

holds for all $x \in X$.

Proof. Interchanging y with x in (2.6), we get

$$\rho\Big(f((a-b)x) - (a-b)^2 f(x)\Big) \leqslant \varphi(x,x), \tag{2.8}$$

which yields

$$\rho\left(\frac{f((a-b)x)}{(a-b)^2} - f(x)\right) \leqslant \frac{1}{(a-b)^2}\rho\big(f((a-b)x) - (a-b)^2 f(x)\big) \leqslant \frac{1}{(a-b)^2}\varphi(x,x)$$

for all $x \in X$. Then, it follows from (2.8) and the triangle inequality that

$$\begin{aligned}
&\rho\left(\frac{1}{(a-b)^{2m}} f((a-b)^m x) - \frac{1}{(a-b)^{2n}} f((a-b)^n x)\right) \\
&= \rho\left(\sum_{i=0}^{m-n-1} \frac{1}{(a-b)^{2(n+i+1)}} \left(f((a-b)^{n+i+1}x) - (a-b)^2 f((a-b)^{n+i}x)\right)\right) \\
&\leqslant \sum_{i=0}^{m-n-1} \frac{1}{(a-b)^{2(n+i+1)}} \varphi((a-b)^{n+i}x, (a-b)^{n+i}x) \\
&= \sum_{j=n}^{m-1} \frac{1}{(a-b)^{2(j+1)}} \varphi((a-b)^j x, (a-b)^j x)
\end{aligned} \tag{2.9}$$

for all $x \in X$ and for any integers m, n with $m > n \geqslant 0$ because of $\sum_{j=n}^{m-1} \frac{1}{(a-b)^{2(j+1)}} \leqslant 1$. Since the right hand side of (2.9) tends to zero as $n \to \infty$, the sequence $\left\{\frac{f((a-b)^n x)}{(a-b)^{2n}}\right\}$ is Cauchy in the ρ-complete convex modular space χ_ρ, and thus it converges for all $x \in X$. Therefore, one may define a mapping $Q : X \to \chi_\rho$ as

$$Q(x) = \lim_{n\to\infty} \frac{f((a-b)^n x)}{(a-b)^{2n}}, \quad x \in X,$$

which leads to a unique quadratic mapping satisfying the approximation (2.7), as desired, using the direct method ([6, 7, 9]). $\square$

Theorem 2.4. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies the functional inequality* (2.6) *and* φ *satisfies the following conditions*

$$\Phi_2(x, y) := \sum_{i=1}^{\infty} \frac{\kappa^{4i}}{(a-b)^{2i}} \varphi\left(\frac{x}{(a-b)^i}, \frac{y}{(a-b)^i}\right) < \infty, \ \rho((a-b)x) \leqslant \kappa\rho(x), \ 2 \leqslant |a-b| \leqslant \kappa \tag{II}$$

for all $x, y \in X$ *and for some* κ. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$, *defined as* $Q(x) = \lim_{n\to\infty} (a-b)^{2n} f\left(\frac{x}{(a-b)^n}\right)$, $(x \in X)$, *which satisfies the approximation*

$$\rho\left(f(x) - Q(x)\right) \leqslant \frac{1}{(a-b)^2} \Phi_2(x, x)$$

for all $x \in X$.

Proof. Replacing x with $\frac{x}{a-b}$ in (2.8), one arrives at

$$\rho\left(f(x) - (a-b)^2 f(\frac{x}{a-b})\right) \leqslant \varphi\left(\frac{x}{a-b}, \frac{x}{a-b}\right)$$

for all $x \in X$. Thus, one can easily show by induction that

$$\begin{aligned}\rho\left(f(x) - (a-b)^{2n} f\left(\frac{x}{(a-b)^n}\right)\right) \leqslant &\sum_{i=1}^{n-1} \frac{\kappa^{4i-2}}{(a-b)^{2i}} \varphi\left(\frac{x}{(a-b)^i}, \frac{x}{(a-b)^i}\right) \\ &+ \frac{\kappa^{4(n-1)}}{(a-b)^{2(n-1)}} \varphi\left(\frac{x}{(a-b)^n}, \frac{x}{(a-b)^n}\right)\end{aligned} \tag{2.10}$$

for all $x \in X$ and for all positive integer $n > 1$. Therefore, it follows from (2.10) that for any integers m, n with $m, n \geqslant 0$,

$$\begin{aligned}&\rho\Big((a-b)^{2m} f\Big(\frac{x}{(a-b)^m}\Big) - (a-b)^{2(n+m)} f\Big(\frac{x}{(a-b)^{n+m}}\Big)\Big) \\ &\leqslant \kappa^{2m} \rho\Big(f\Big(\frac{x}{(a-b)^m}\Big) - (a-b)^{2n} f\Big(\frac{x}{(a-b)^{m+n}}\Big)\Big) \\ &\leqslant \kappa^{2m} \sum_{j=1}^{n-1} \frac{\kappa^{4j-2}}{(a-b)^{2j}} \varphi\Big(\frac{x}{(a-b)^{j+m}}, \frac{x}{(a-b)^{j+m}}\Big) + \kappa^{2m} \frac{\kappa^{4(n-1)}}{(a-b)^{2(n-1)}} \varphi\Big(\frac{x}{(a-b)^{n+m}}, \frac{x}{(a-b)^{n+m}}\Big) \\ &\leqslant \frac{(a-b)^{2m}}{\kappa^{2(m+1)}} \sum_{j=m+1}^{n+m-1} \frac{\kappa^{4j}}{(a-b)^{2j}} \varphi\Big(\frac{x}{(a-b)^j}, \frac{x}{(a-b)^j}\Big) + \frac{(a-b)^{2m}}{\kappa^{2m}} \frac{\kappa^{4(n+m-1)}}{(a-b)^{2(n+m-1)}} \varphi\Big(\frac{x}{(a-b)^{n+m}}, \frac{x}{(a-b)^{n+m}}\Big)\end{aligned} \tag{2.11}$$

for all $x \in X$. Since the right hand side of (2.11) tends to zero as $m \to \infty$, the sequence $\{(a-b)^{2n} f(\frac{x}{(a-b)^n})\}$ is Cauchy for all $x \in X$, and it thus converges by the completeness of χ_ρ. Thus, one can define a mapping $Q : X \to \chi_\rho$ as

$$Q(x) = \lim_{n\to\infty} (a-b)^{2n} f\left(\frac{x}{(a-b)^n}\right)$$

for all $x \in X$, which yields the approximation

$$\begin{aligned}&\rho\left(f(x) - Q(x)\right) \\ &\leqslant \frac{1}{(a-b)^2} \rho\Big((a-b)^2 f(x) - (a-b)^{2(n+1)} f\left(\frac{x}{(a-b)^n}\right) + (a-b)^{2(n+1)} f\left(\frac{x}{(a-b)^n}\right) - (a-b)^2 Q(x)\Big) \\ &\leqslant \frac{\kappa^2}{(a-b)^2} \rho\Big(f(x) - (a-b)^{2n} f\left(\frac{x}{(a-b)^n}\right)\Big) + \frac{\kappa^2}{(a-b)^2} \rho\Big((a-b)^{2n} f\left(\frac{x}{(a-b)^n}\right) - Q(x)\Big)\end{aligned}$$

$$\leqslant \frac{1}{(a-b)^2}\sum_{i=1}^{n-1}\frac{\kappa^{4i}}{(a-b)^{2i}}\varphi\left(\frac{x}{(a-b)^i},\frac{x}{(a-b)^i}\right)+\frac{\kappa^2}{(a-b)^2}\frac{\kappa^{4(n-1)}}{(a-b)^{2(n-1)}}\varphi\left(\frac{x}{(a-b)^n},\frac{x}{(a-b)^n}\right)$$
$$+\frac{\kappa^2}{(a-b)^2}\,\rho\left((a-b)^{2n}f\left(\frac{x}{(a-b)^n}\right)-Q(x)\right)$$

for all $x \in X$. Letting $n \to \infty$, we see that there exists a unique quadratic mapping satisfying the approximation $\rho\Big(f(x)-Q(x)\Big) \leqslant \frac{1}{(a-b)^2}\Phi_2(x,x)$ for all $x \in X$. □

Remark 2.5. In particular, if we consider $\chi := \chi_\rho$ as a Banach space with norm $\rho := \|\cdot\|$ and thus $\rho\Big((a-b)x\Big) = |a-b|\rho(x)$, $\kappa := |a-b|$, then Theorem 2.3 together with Theorem 2.4 reduces to stability results of the equation (1.2) in normed linear spaces.

Example 2.6. We remark that the quadratic functional equation (1.2) is not stable for a special case as we see the following example, which is a modification of the example contained in [4].

Let $\varepsilon > 0$ and a, b be any given nonzero integers with $a \neq b$, $|a-b| > 1$. Let us define a function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} \frac{\varepsilon}{\alpha^2 M}, & \text{if } |x| > 1,\\ \frac{\varepsilon}{\alpha^2 M}x^2, & \text{if } |x| \leqslant 1,\end{cases}$$

where $M := 1+3|a||b|+a^2+b^2$, and then define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty}\frac{\varphi(\alpha^n x)}{\alpha^{2n}},\ \ \alpha := |a|+|b|.$$

Then the function f satisfies the functional inequality

$$|f(ax-by)-abf(x-y)-a(a-b)f(x)-b(b-a)f(y)| \leqslant \varepsilon(x^2+y^2) \tag{2.12}$$

for all $x, y \in \mathbb{R}$, but there exist no a quadratic function $Q : \mathbb{R} \to \mathbb{R}$ and a constant $\beta > 0$ such that

$$|f(x)-Q(x)| \leqslant \beta x^2$$

for all $x \in \mathbb{R}$. In fact, we see immediately that $|\varphi(x)| \leqslant \frac{\varepsilon}{\alpha^2 M}$, $|f(x)| \leqslant \frac{\varepsilon}{M(\alpha^2-1)}$ for all $x \in \mathbb{R}$, and thus,

$$|f(ax-by)-abf(x-y)-a(a-b)f(x)-b(b-a)f(y)| \leqslant M\frac{\varepsilon}{M(\alpha^2-1)} = \frac{\varepsilon}{(\alpha^2-1)}$$

for all $x, y \in \mathbb{R}$. Now, the inequality (2.12) is trivial for case $x = y = 0$. Let $x^2+y^2 \geqslant \frac{1}{\alpha^2}$. Then, we figure out that

$$\begin{aligned}&|f(ax-by)-abf(x-y)-a(a-b)f(x)-b(b-a)f(y)|\\ &\leqslant \frac{\varepsilon}{M(\alpha^2-1)}M \leqslant \frac{\varepsilon}{(\alpha^2-1)}\alpha^2(x^2+y^2) < \varepsilon(x^2+y^2)\end{aligned}$$

for all $x, y \in \mathbb{R}$ with $x^2+y^2 \geqslant \frac{1}{\alpha^2}$. Next, we consider the case $0 < x^2+y^2 < \frac{1}{\alpha^2}$. Then we can choose the smallest positive integer $N \in \mathbb{N}$ such that

$$\frac{1}{\alpha^{2(N+1)}} \leqslant x^2+y^2 < \frac{1}{\alpha^{2N}}. \tag{2.13}$$

Thus one has $|a-b|^{2(N-1)}x^2 \leqslant \alpha^{2(N-1)}x^2 < \alpha^{-2}$, $|a-b|^{2(N-1)}y^2 \leqslant \alpha^{2(N-1)}y^2 < \alpha^{-2}$, and so,

$$|a-b|^{(N-1)}|x-y| \leqslant \alpha^{(N-1)}(|x|+|y|) < \frac{2}{\alpha} \leqslant 1,$$

and
$$|a-b|^{(N-1)}|ax-by| \leqslant \alpha^{(N-1)}(|ax|+|by|) < |a|\frac{1}{\alpha}+|b|\frac{1}{\alpha}=1.$$
Thus, we have
$$\varphi(ax-by)-ab\varphi(x-y)-a(a-b)\varphi(x)-b(b-a)\varphi(y)=0 \tag{2.14}$$
for each $n=0,1,\ldots,N-1$. Therefore, it follows from relations (2.13) and (2.14) that
$$\begin{aligned}
&|f(ax-by)-abf(x-y)-a(a-b)f(x)-b(b-a)f(y)||\\
&\leqslant \sum_{n=0}^{\infty}\frac{1}{\alpha^{2n}}|\varphi(\alpha^n(ax-by))-ab\varphi(\alpha^n(x-y))-a(a-b)\varphi(\alpha^n x)-b(b-a)\varphi(\alpha^n y)|\\
&\leqslant \sum_{n=N}^{\infty}\frac{1}{\alpha^{2n}}|\varphi(\alpha^n(ax-by))-ab\varphi(\alpha^n(x-y))-a(a-b)\varphi(\alpha^n x)-b(b-a)\varphi(\alpha^n y)|\\
&\leqslant \frac{\varepsilon M}{\alpha^2 M}\sum_{n=N}^{\infty}\frac{1}{\alpha^{2n}}=\frac{\varepsilon\alpha^2}{\alpha^{2(N+1)}(\alpha^2-1)}\\
&\leqslant \frac{\varepsilon\alpha^2}{(\alpha^2-1)}(x^2+y^2)<\varepsilon(x^2+y^2),
\end{aligned}$$
which yields the inequality (2.12).

However, we assume for a contradiction that there exist a quadratic function $Q:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ such that
$$|f(x)-Q(x)|\leqslant\beta x^2$$
for all $x\in\mathbb{R}$. Since $|Q(x)|\leqslant|f(x)|+\beta x^2\leqslant\frac{\varepsilon}{M(\alpha^2-1)}+\beta x^2$ is bounded at zero and f is a continuous function, the function Q is of the form $Q(x)=cx^2, x\in\mathbb{R}$ for some constant c. Hence one obtains
$$|f(x)|\leqslant(\beta+|c|)x^2$$
for all $x\in\mathbb{R}$. On the other hand, there exists a positive integer $m\in\mathbb{N}$ such that $m\frac{\varepsilon}{\alpha^2 M}>\beta+|c|$, and thus if $x\in(0,\frac{1}{\alpha^{m-1}})$, then we have $\alpha^n x\in(0,1)$ for all $n\leqslant m-1$, and so
$$f(x)=\sum_{n=0}^{\infty}\frac{\varphi(\alpha^n x)}{\alpha^{2n}}\geqslant\sum_{n=0}^{m-1}\frac{\varepsilon}{\alpha^2 M}\frac{(\alpha^n x)^2}{\alpha^{2n}}=\frac{\varepsilon}{\alpha^2 M}mx^2>(\beta+|c|)x^2,$$
which yields a contradiction. Hence, there exist no a quadratic function $Q:\mathbb{R}\to\mathbb{R}$ and a constant $\beta>0$ such that
$$|f(x)-Q(x)|\leqslant\beta x^2$$
for all $x\in\mathbb{R}$.

3. Generalized Hyers-Ulam stability of Eq. (1.2) for $|a-b|=1$

In this section, we investigate the generalized Hyers-Ulam stability problem of the equation (1.2) for any fixed nonzero integers $a,b\in\mathbb{Z}$ with $a\neq b$, $|a-b|=1$.

First of all, we begin with investigating the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) for $a-b=1$. Thus, we denote the following notation $D_{a,b}f:=D_{a,a-1}f$ by
$$D_{a,a-1}f(x,y):=f(ax-(a-1)y)-a(a-1)f(x-y)-af(x)+(a-1)f(y)$$
for any fixed nonzero integers $a\in\mathbb{Z}$ with $a\neq1$.

Now, we present some conditions such that there exists a true quadratic mapping Q near an approximate quadratic mapping f as follows.

Theorem 3.1. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho\Big(D_{a,a-1}f(x,y)\Big) \leqslant \psi(x,y) \tag{3.1}$$

and ψ *satisfies the condition*

$$\Psi_1(x,y) := \begin{cases} \sum_{i=0}^{\infty} \frac{1}{a^{2i}}\psi(a^i x, a^i y) < \infty, & \text{if } a \neq -1, \\ \sum_{i=0}^{\infty} \frac{1}{2^{2i}}\psi(2^i x, 2^i y) < \infty, & \text{if } a = -1 \end{cases} \tag{III}$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$ *satisfying*

$$\begin{aligned} &\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{a^2}\Psi_1(x,0), \ \ \textit{if} \ \ a \neq -1, \\ &\Big(\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{4}\Psi_1(2x,x), \ \ \textit{if} \ \ a = -1 \ (b = -2), \ \ \textit{respectively}\Big), \end{aligned} \tag{3.2}$$

for all $x \in X$. *The mapping* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}}, \ \ \textit{if} \ \ a \neq -1, \ \left(Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \ \ \textit{if} \ \ a = -1 \ (b = -2), \ \ \textit{respectively}\right),$$

for all $x \in X$.

Proof. First, we prove our theorem for the case $a \neq -1$. Setting $y := 0$ in (3.1), one has

$$\rho\Big(f(ax) - a^2 f(x)\Big) \leqslant \psi(x,0),$$

which yields

$$\rho\left(\frac{1}{a^{2n}}f(a^n x) - f(x)\right) \leqslant \sum_{i=0}^{n-1} \frac{1}{a^{2(i+1)}}\psi(a^i x, 0)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Thus, we see that a mapping $Q : X \to \chi_\rho$, defined as $Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}}$, is a unique quadratic mapping satisfying the estimation (3.2) by using the direct method. Therefore, one can cover the proof for case $a \neq -1$, as desired.

Now, one proves the theorem for case $a = -1$ and $b = -2$ under functional inequality

$$\rho\Big(D_{-1,-2}f(x,y) := f(-x+2y) - 2f(x-y) + f(x) - 2f(y)\Big) \leqslant \psi(x,y)$$

for all $x, y \in X$. Replacing x and y in previous inequality with $2x$ and x, respectively, one has

$$\rho\Big(f(2x) - 4f(x)\Big) \leqslant \psi(2x,x)$$

for all $x \in X$. Thus, it follows from the last inequality that

$$\rho\left(\frac{1}{4^n}f(2^n x) - f(x)\right) \leqslant \sum_{i=0}^{n-1} \frac{1}{4^{i+1}}\psi(2^{i+1}x, 2^i x)$$

for all $x \in X$ and for all $n \in \mathbb{N}$. Thereafter, applying the direct method, one can discover the unique quadratic mapping $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ satisfying the approximation (3.2). This completes the proof. □

The following stability theorem under the condition (IV) is dual of the previous theorem, and its proof goes through by the similar way to that of Theorem 3.1.

Theorem 3.2. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho\Big(D_{a,a-1}f(x,y)\Big) \leqslant \psi(x,y)$$

and ψ *satisfies the condition*

$$\Psi_2(x,y) := \begin{cases} \sum\limits_{i=1}^{\infty} \frac{\kappa^{4i}}{a^{2i}}\psi\left(\frac{x}{a^i}, \frac{y}{a^i}\right) < \infty, & \rho(ax) \leqslant \kappa\rho(x),\ |a| \leqslant \kappa, \quad \textit{if } a \neq -1, \\ \sum\limits_{i=1}^{\infty} \frac{\kappa^{4i}}{2^{2i}}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, & \rho(2x) \leqslant \kappa\rho(x),\ 2 \leqslant \kappa, \quad \textit{if } a = -1, \end{cases} \tag{IV}$$

for all $x, y \in X$ *and for some* κ. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$ *satisfying*

$$\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{a^2}\Psi_2(x,0), \ \ \textit{if} \ \ a \neq -1, \ \left(\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{4}\Psi_2(2x,x), \ \ \textit{if} \ \ a = -1, \ \ \textit{respectively}\right)$$

for all $x \in X$. *The mapping* Q *is given by*

$$Q(x) = \lim_{n\to\infty} a^{2n} f\left(\frac{x}{a^n}\right), \ \ \textit{if} \ \ a \neq -1, \ \left(Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right), \ \ \textit{if} \ \ a = -1, \ \ \textit{respectively}\right)$$

for all $x \in X$.

Proof. The proof can be covered by the similar argument to that of Theorem 2.4. □

In the last part, we study the generalized Hyers-Ulam stability of the quadratic functional equation (1.2) for case $a - b = -1$. Thus, we use the following notation:

$$D_{a,a+1}f(x,y) := f(ax - (a+1)y) - a(a+1)f(x-y) + af(x) - (a+1)f(y)$$

for any fixed nonzero integer $a \in \mathbb{Z}$ with $a \neq -1$.

The following theorem is an alternative stability results of functional equation (1.2) under the condition (V), which is similarly verified as in the proof of Theorem 3.1.

Theorem 3.3. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho\Big(D_{a,a+1}f(x,y)\Big) \leqslant \psi(x,y)$$

and ψ *satisfies the condition*

$$\Psi_3(x,y) := \begin{cases} \sum\limits_{i=0}^{\infty} \frac{1}{a^{2i}}\psi(a^i x, a^i y) < \infty, & \textit{if } a \neq 1, \\ \sum\limits_{i=0}^{\infty} \frac{1}{2^{2i}}\psi(2^i x, 2^i y) < \infty, & \textit{if } a = 1\ (b = 2) \end{cases} \tag{V}$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$ *satisfying*

$$\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{a^2}\Psi_3(x,0), \ \ \textit{if} \ \ a \neq 1, \ \left(\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{4}\Psi_3(2x,x), \ \ \textit{if} \ \ a = 1\ (b=2), \ \ \textit{respectively}\right)$$

for all $x \in X$. *The mapping* Q *is given by*

$$Q(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}}, \ \ \textit{if} \ \ a \neq 1, \ \left(Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}, \ \ \textit{if} \ \ a = 1\ (b=2), \ \ \textit{respectively}\right)$$

for all $x \in X$.

The following theorem is an alternative stability results of functional equation (1.2) under the condition (VI), which is similarly verified as in the proof of Theorem 3.2.

Theorem 3.4. *Assume that a mapping* $f : X \to \chi_\rho$ *with* $f(0) = 0$ *satisfies*

$$\rho\Big(D_{a,a+1}f(x,y)\Big) \leqslant \psi(x,y)$$

and ψ *satisfies the condition*

$$\Psi_4(x,y) := \begin{cases} \sum_{i=1}^{\infty} \frac{\kappa^{4i}}{a^{2i}}\psi\left(\frac{x}{a^i}, \frac{y}{a^i}\right) < \infty, & \rho(ax) \leqslant \kappa\rho(x),\ |a| \leqslant \kappa, \quad \text{if } a \neq 1, \\ \sum_{i=1}^{\infty} \frac{\kappa^{4i}}{2^{2i}}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) < \infty, & \rho(2x) \leqslant \kappa\rho(x),\ 2 \leqslant \kappa, \quad \text{if } a = 1\ (b = 2) \end{cases} \tag{VI}$$

for all $x, y \in X$. *Then there exists a unique quadratic mapping* $Q : X \to \chi_\rho$ *satisfying*

$$\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{a^2}\Psi_4(x,0), \ \textit{if}\ \ a \neq 1, \ \left(\rho\Big(f(x) - Q(x)\Big) \leqslant \frac{1}{4}\Psi_4(2x,x), \ \textit{if}\ \ a = 1\ (b = 2), \ \textit{respectively}\right)$$

for all $x \in X$. *The mapping* Q *is given by*

$$Q(x) = \lim_{n\to\infty} a^{2n} f\left(\frac{x}{a^n}\right), \ \textit{if}\ \ a \neq 1, \ \left(Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right), \ \textit{if}\ \ a = 1\ (b = 2), \ \textit{respectively}\right)$$

for all $x \in X$.

4. Conclusion

In this paper, we have proved stability theorems of a refined quadratic functional equation (1.2) in modular spaces by the direct method. As results, we have generalized stability results of a quadratic functional equation (1.1) in normed spaces.

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