J. Math. Computer Sci., 29 (2023), 118–130

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Properties of analytic functions associated with Mitttag-Leffler-type Borel distribution series

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Abstract

Keeping in view of the latest trends toward special functions, due its various applications in Physics and applied Mathematics, we introduce a subclass of analytic functions with the help of Borel distribution series. Furthermore, we investigate some useful geometric and algebric properties of these functions. We discuss coefficient estimates, growth and distortion theorems, radii of close-to-convexity, starlikeness, convexity and convolution properties to this subclass. subject classification numbers as needed.

Keywords: Analytic, starlike, convexity, Mittag-Leffler, Borel distribution, coefficient bounds.

2020 MSC: 30C45, 30C50, 30C80.

1. Introduction

Let A indicate the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be a subclass of A that contains univalent functions and satisfies the usual normalization condition $u(0) = u'(0) - 1 = 0$. The subset of A comprising of functions $u(z)$ that are all univalent in E is represented by S. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leqslant \upsilon < 1$, if it fulfills

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, (z \in E).$$

We indicate this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leqslant \upsilon < 1$, if it fulfills

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, (z \in E).$$

We indicate this class with $K(\upsilon)$.

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doi: 10.22436/jmcs.029.02.02

Received: 2022-05-13 *Revised:* 2022-05-27 *Accepted:* 2022-06-29

The regular classes of starlike and convex functions in E are $S^*(0) = S^*$ and $K(0) = K$, respectively. Let T indicate the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geqslant 0,\ z \in E), \tag{1.2}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon)$, $C(\upsilon) = T \cap K(\upsilon)$. Silverman [26] has thoroughly studied the class $T^*(\upsilon)$ and related classes, which have some interesting properties.

For $u \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

their convolution indicate by $(u * g)$, is specified as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z), \quad (z \in E).$$

Note that $u * g \in A$.

The following subclasses were introduced and examined by Goodman [9, 10] and Ronning [20, 21].

(1). If a function $u \in A$ satisfies the condition, it is said to be in the class $UCV(\rho, \gamma)$, a uniformly $\gamma-$convex function

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)} - \rho\right\} > \gamma \left|\frac{zu''(z)}{u'(z)}\right|,$$

where $\gamma \geqslant 0, -1 < \rho \leqslant 1$ and $\rho + \gamma \geqslant 0$.

(2). If a function $u \in A$ satisfies the condition, it is said to be in the class $SP(\rho, \gamma)$, uniformly γ-starlike function

$$\Re\left\{\frac{zu'(z)}{u(z)} - \rho\right\} > \gamma \left|\frac{zu'(z)}{u(z)} - 1\right|,$$

where $\gamma \geqslant 0, -1 < \rho \leqslant 1$ and $\rho + \gamma \geqslant 0$.

Further, Subramanian et al. [29], Santosh et al. [22], Thirupathi Reddy, and Venkateswarlu [30] have also studied interesting properties for the classes $UCV(\rho, \gamma)$ and $SP(\rho, \gamma)$.

Mittag-Leffler function and Borel distribution

The study of operators is essential in geometric function theory, complex analysis, and other related fields. Convolution of certain analytic functions may be used to express a variety of derivative and integral operators. It should be highlighted that this formalism aids future mathematical study and a better understanding of the geometric properties of such operators. Let $E_\tau(z)$ and $E_{\tau,\nu}(z)$ be functions defined by

$$E_\tau(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\tau n + 1)}, \quad (z \in \mathbb{C}, \Re(\tau) > 0),$$

and

$$E_{\tau,\nu}(z) = \frac{1}{\Gamma(\nu)} + \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\tau n + \nu)}, \quad (\tau, \nu \in \mathbb{C}, \Re(\tau) > 0, \Re(\nu) > 0).$$

It can be written in other form

$$E_{\tau,\nu}(z) = \frac{1}{\Gamma(\nu)} + \sum_{n=2}^{\infty} \frac{z^{n-1}}{\Gamma(\tau(n-1) + \nu)}, \quad (\tau, \nu \in \mathbb{C}, \Re(\tau) > 0, \Re(\nu) > 0).$$

The function $E_\tau(z)$ was introduced by Mittag-Leffler [15] and is, therefore, known as the Mittag-Leffler function. A more general function $E_{\tau,\nu}$ generalizing $E_\tau(z)$ was introduced by Wiman [33] and defined by

$$E_{\tau,\nu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+\nu)},\quad (z,\tau,\nu\in\mathbb{C},\Re(\tau)>0,\Re(\nu)>0).$$

Observe that the function $E_{\tau,\nu}$ contains many well-known functions as its special case, for example,

$$\begin{aligned}
&E_{1,1}(z)=e^z,\quad E_{1,2}(z)=\frac{e^z-1}{z},\\
&E_{2,1}\left(z^2\right)=\cosh z,\quad E_{2,1}\left(-z^2\right)=\cos z,\quad E_{2,2}\left(z^2\right)=\frac{\sinh z}{z},\\
&E_{2,2}\left(-z^2\right)=\frac{\sin z}{z}c,\quad E_3(z)=\frac{1}{2}\left[e^{z^{1/3}}+2e^{-\frac{1}{2}z^{1/3}}\cos\left(\frac{\sqrt{3}}{2}z^{1/3}\right)\right],\\
&\text{and } E_4(z)=\frac{1}{2}\left[\cos z^{1/4}+\cosh z^{1/4}\right].
\end{aligned}$$

The Mittag-Leffler function appears naturally in the solution of fractional order differential and integral equations. In the study of complex systems and super diffusive transport, in particular, fractional generalization of the kinetic equation, random walks, and Levy flights. Several properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found, e.g., in [1, 3, 4, 7, 8, 11, 24]. Observe that MittagLeffler function $E_{\tau,\nu}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler functions as below:

$$E_{\tau,\nu}(z)=z\Gamma(\nu)E_{\tau,\nu}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\nu)}{\Gamma(\tau(n-1)+\nu)}z^n, \tag{1.3}$$

it holds for complex parameters τ,ν, and $z\in\mathbb{C}$. In this paper, we shall restrict our attention to the case of real-valued τ,ν and $z\in E$.

A discrete random variable x is said to have a Borel distribution if it takes the values $1,2,3,\ldots$ with the probabilities $\frac{e^{-\lambda}}{1!},\frac{2\lambda e^{-2\lambda}}{2!},\frac{9\lambda^2e^{-3}}{3!},\ldots$, respectively, where λ is called the parameter.

Elementary distributions such as the Pascal, Poisson, logarithmic, binomial, beta negative binomial and Touchard polynomials have been examined theoretically in Geometric Function Theory. We suggest readers to [5, 19, 27, 31] for a more complete research.

Very recently, Wanas and Khuttar [32] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=\rho)=\frac{(\rho\lambda)^{\rho-1}e^{-\lambda\rho}}{\rho!},\quad \rho=1,2,3,\ldots.$$

Wanas and Khuttar introduced a series $\mathcal{M}_\lambda(z)$ whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{M}_\lambda(z)=z+\sum_{n=2}^{\infty}\frac{[\lambda(n-1)]^{n-2}e^{-\lambda(n-1)}}{(n-1)!}z^n,(0<\lambda\leqslant 1). \tag{1.4}$$

In [16], Murugusundaramoorthy and El-Deeb defined the Mittag-Leffler-type Borel distribution as follows:

$$\mathcal{P}_\lambda(\tau,\nu;\rho)=\frac{(\lambda\rho)^{\rho-1}}{E_{\tau,\nu}(\lambda\rho)\Gamma(\tau\rho+\nu)},\quad \rho=0,1,2,\ldots,$$

where

$$E_{\tau,\nu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\tau n+\nu)},\quad (\tau,\ \nu\in\mathbb{C},\ \Re(\tau)>0,\ \Re(\nu)>0).$$

Thus by using (1.3) and (1.4) and by convolution operator, the Mittag-Leffler-type Borel distribution series defined as below

$$B_\lambda(\tau,\nu)(z) = z + \sum_{n=2}^{\infty} \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2}e^{-\lambda(n-1)}}{(n-1)!E_{\tau,\nu}(\lambda(n-1))\Gamma(\tau(n-1)+\nu)} z^n, (0 < \lambda \leqslant 1).$$

Further, by the convolution operator, we define

$$\begin{aligned} B_\lambda(\tau,\nu)u(z) &= B_\lambda(\tau,\nu)(z) * u(z) = z + \sum_{n=2}^{\infty} \phi(n) a_n z^n, \quad (\tau,\nu \in \mathbb{C}, \Re(\tau) > 0, \Re(\nu) > 0, 0 < \lambda \leqslant 1), \\ \text{where} \quad \phi(n) &= \frac{[\lambda(n-1)]![\lambda(n-1)]^{n-2}e^{-\lambda(n-1)}}{(n-1)!E_{\tau,\nu}(\lambda(n-1))\Gamma(\tau(n-1)+\nu)}, \\ \text{and} \quad \phi(2) &= \frac{\lambda! e^{-\lambda}}{E_{\tau,\nu}(\lambda)\Gamma(\tau+\nu)}. \end{aligned} \tag{1.5}$$

If $u \in T$ is given by (1.1), then we have

$$B_\lambda(\tau,\nu)u(z) = z - \sum_{n=2}^{\infty} \phi(n) a_n z^n,$$

where $\phi(n)$ is given by (1.5).

Geometric Function Theory relies heavily on the study of operators. Many differential and integral operators may be written as analytic function convolutions. This method has been discovered to assist future mathematical study and to improve understanding of the geometric and symmetric properties of these operators. The significance of convolution in operator theory may be clearly appreciated from the work in [11]. Furthermore, probability is applied in a wide range of real-life applications, from insurance to forecasting and politics to economic forecasting. We recommend the reader to [34] for other applications. Motivated by all of the preceding studies and the work of Khan et al. [12, 14], who established a class of analytic functions with Mittag-Leffler type Poisson distribution in the Janowski domain, analytic functions with Mittag-Leffler type Borel distribution [1, 16, 17, 28], and work in the publications [6, 13, 25, 27].

Now, by making use of the Mittag-Leffler-type Borel distribution series $B_\lambda(\tau,\nu)$, we define a new subclass of functions belonging to the class $\mathcal{A}$ motivated by Popade et al. [18].

Definition 1.1. For $-1 \leqslant \upsilon < 1$ and $\rho \geqslant 0$, we let $TS(\upsilon,\rho)$ be the subclass of A consisting of functions of the form (1.2) and fulfiling the analytic condition

$$\Re\left\{\frac{z(B_\lambda(\tau,\nu)u(z))'}{B_\lambda(\tau,\nu)u(z)} - \upsilon\right\} \geqslant \rho\left|\frac{z(B_\lambda(\tau,\nu)u(z))'}{B_\lambda(\tau,\nu)u(z)} - 1\right|, \tag{1.6}$$

for $z \in E$.

The aim of present paper is to study the coefficient bounds, growth and distortion theorems, extreme points, radii of close-to-convex, starlikeness and convolution property for the defined class.

2. Coefficient bounds

We get a required and adequate condition for function $u(z)$ in the class $TS(\upsilon,\rho)$ in this section. To find the coefficient estimates for our class, we use the approach proposed by Aqlan et al. [2].

Theorem 2.1. *The function* u *defined by* (1.2) *is in the class* $TS(\upsilon, \rho)$ *if and only if*

$$\sum_{n=2}^{\infty}[n(1+\rho)-(\upsilon+\rho)]\phi(n)|a_n| \leqslant 1-\upsilon, \tag{2.1}$$

where $-1 \leqslant \upsilon < 1, \rho \geqslant 0$. *The result is sharp.*

Proof. We have $f \in TS(\upsilon, \rho)$ if and only if the condition (1.6) satisfied. Upon the fact that

$$\Re(w) > \rho|w-1|+\upsilon \Leftrightarrow \Re\{w(1+\rho e^{i\theta})-\rho e^{i\theta}\} > \upsilon, \ -\pi \leqslant \theta \leqslant \pi.$$

Equation (1.6) may be written as

$$\begin{aligned}&\Re\left\{\frac{z(B_\lambda(\tau,\nu)u(z))'}{B_\lambda(\tau,\nu)u(z)}(1+\rho e^{i\theta})-\rho e^{i\theta}\right\}\\ &\quad = \Re\left\{\frac{z(B_\lambda(\tau,\nu)u(z))'(1+\rho e^{i\theta})-\rho e^{i\theta}B_\lambda(\tau,\nu)u(z)}{B_\lambda(\tau,\nu)u(z)}\right\} > \upsilon.\end{aligned} \tag{2.2}$$

Now, we let

$$E(z) = z(B_\lambda(\tau,\nu)u(z))'(1+\rho e^{i\theta})-\rho e^{i\theta}B_\lambda(\tau,\nu)u(z), \qquad F(z) = B_\lambda(\tau,\nu)u(z).$$

Then (2.2) is equivalent to

$$|E(z)+(1-\upsilon)F(z)| > |E(z)-(1+\upsilon)F(z)|, \text{ for } 0 \leqslant \upsilon < 1.$$

For $E(z)$ and $F(z)$ as above, we have

$$|E(z)+(1-\upsilon)F(z)| \geqslant (2-\upsilon)|z| - \sum_{n=2}^{\infty}[n+1-\upsilon+\rho(n-1)]\phi(n)|a_n||z^n|$$

and similarly

$$|E(z)-(1+\upsilon)F(z)| \leqslant \upsilon|z| - \sum_{n=2}^{\infty}[n-1-\upsilon+\rho(n-1)]\phi(n)|a_n||z^n|.$$

Therefore

$$|E(z)+(1-\upsilon)F(z)|-|E(z)-(1+\upsilon)F(z)| \geqslant 2(1-\upsilon)-2\sum_{n=2}^{\infty}[n-\upsilon+\rho(n-1)]\phi(n)|a_n|$$

or

$$\sum_{n=2}^{\infty}[n-\upsilon+\rho(n-1)]\phi(n)|a_n| \leqslant (1-\upsilon),$$

which yields (2.1). On the other hand, we must have

$$\Re\left\{\frac{z(B_\lambda(\tau,\nu)u(z))'}{B_\lambda(\tau,\nu)u(z)}(1+\rho e^{i\theta})-\rho e^{i\theta}\right\} \geqslant \upsilon.$$

Upon choosing the values of z on the positive real axis where $0 \leqslant |z| = r < 1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{n=2}^{\infty}[n-\upsilon+\rho e^{i\theta}(n-1)]\phi(n)|a_n|\, r^n}{z - \sum_{n=2}^{\infty}\phi(n)|a_n|\, r^n}\right\} \geqslant 0.$$

Since $\Re(-e^{i\theta}) \geqslant -|e^{i\theta}| = -1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\upsilon)r - \sum_{n=2}^{\infty}[n-\upsilon+\rho(n-1)]\phi(n)|a_n|\, r^n}{z - \sum_{n=2}^{\infty}\phi(n)|a_n|\, r^n}\right\} \geqslant 0.$$

Letting $r \to 1^-$, we get the desired result. Finally the result is sharp with the extrenal function u given by

$$u(z) = z - \frac{1-\upsilon}{[n(1+\rho)-(\upsilon+\rho)]\phi(n)}z^n. \tag{2.3}$$

□

3. Growth and distortion theorems

Theorem 3.1. *Let the function u defined by (1.2) be in the class $TS(\upsilon, \rho)$. Then for $|z| = r$*

$$r - \frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}r^2 \leqslant |u(z)| \leqslant r + \frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}r^2. \tag{3.1}$$

Equality holds for the function

$$u(z) = z - \frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}z^2. \tag{3.2}$$

Proof. Since the other inequality can be explained using identical reasoning, we just prove the right hand side inequality in (3.1). In view of Theorem 2.1, we have

$$\sum_{n=2}^{\infty}|a_n| \leqslant \frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}. \tag{3.3}$$

Since,

$$u(z) = z - \sum_{n=2}^{\infty}a_n z^n,$$

$$|u(z)| = \left|z - \sum_{n=2}^{\infty}a_n z^n\right| \leqslant r + \sum_{n=2}^{\infty}|a_n|r^n \leqslant r + r^2\sum_{n=2}^{\infty}|a_n| \leqslant r + \sum_{n=2}^{\infty}\frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}r^2,$$

which yields the right hand side inequality of (3.1) and

$$|u(z)| = \left|z - \sum_{n=2}^{\infty}a_n z^n\right| \geqslant r - \sum_{n=2}^{\infty}|a_n|r^n \leqslant r - r^2\sum_{n=2}^{\infty}|a_n| \geqslant r - \sum_{n=2}^{\infty}\frac{1-\upsilon}{(2-\upsilon+\rho)\phi(2)}r^2,$$

which yields the left hand side inequality of (3.1). □

Next, by using the same technique as in proof of Theorem 3.1, we give the distortion result.

Theorem 3.2. *Let the function u defined by (1.2) be in the class $TS(\upsilon, \rho)$. Then for $|z| = r$*

$$1 - \frac{(1-\upsilon)}{(2-\upsilon+\rho)\phi(2)}r \leqslant |u'(z)| \leqslant 1 + \frac{(1-\upsilon)}{(2-\upsilon+\rho)\phi(2)}r.$$

Equality holds for the function given by (3.2).

Proof. Since $f \in TS(\upsilon, \rho)$ by Theorem 2.1, we have that

$$[2(1+\rho)-(\upsilon+\rho)]\,\varphi(2)\sum_{n=2}^{\infty} na_n \leqslant [n(1+\rho)-(\upsilon+\rho)]\,\varphi(n)|a_n| \leqslant 1-\upsilon$$

or

$$\sum_{n=2}^{\infty} n|a_n| \leqslant \frac{(1-\upsilon)}{(2-\upsilon+\rho)\varphi(2)}\ .$$

Thus from (3.3), we obtain

$$|u'(z)| \leqslant 1 + r\sum_{n=2}^{\infty} n|a_n| \leqslant 1 + \frac{(1-\upsilon)}{(2-\upsilon+\rho)\varphi(2)}r,$$

which is right hand inequality of Theorem 3.2.

On the other hand, similarly

$$|u'(z)| \geqslant 1 - \frac{(1-\upsilon)}{(2-\upsilon+\rho)\varphi(2)}r$$

and thus proof is completed. □

Theorem 3.3. *If* $u \in TS(\upsilon, \rho)$, *then* $u \in TS(\gamma)$, *where*

$$\gamma = 1 - \frac{(n-1)(1-\upsilon)}{[n-\upsilon+\rho(n-1)]\varphi(n)-(1-\upsilon)}.$$

Equality holds for the function given by (3.2).

Proof. It is sufficient to show that (2.1) implies

$$\sum_{n=2}^{\infty}(n-\gamma)|a_n| \leqslant 1-\gamma,$$

that is

$$\frac{n-\gamma}{1-\gamma} \leqslant \frac{[n-\upsilon+\rho(n-1)]\varphi(n)}{(1-\upsilon)},$$

then

$$\gamma \leqslant 1 - \frac{(n-1)(1-\upsilon)}{[n-\upsilon+\rho(n-1)]\varphi(n)-(1-\upsilon)}.$$

The above inequality holds true for $n \in \mathbb{N}_0, n \geqslant 2, \rho \geqslant 0$, and $0 \leqslant \upsilon < 1$. □

4. Extreme points

Theorem 4.1. *Let* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\varphi(n)}z^n, \tag{4.1}$$

for $n = 2, 3, \dots$. *Then* $u(z) \in TS(\upsilon, \rho)$ *if and only if* $u(z)$ *can be expressed in the form* $u(z) = \sum_{n=1}^{\infty} \zeta_n u_n(z)$, *where* $\zeta_n \geqslant 0$ *and* $\sum_{n=1}^{\infty} \zeta_n = 1$.

Proof. Suppose $u(z)$ can be expressed as in (4.1). Then

$$\begin{aligned}
u(z) &= \sum_{n=1}^{\infty} \zeta_n u_n(z) = \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n u_n(z) \\
&= \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n \left\{ z - \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} z^n \right\} \\
&= \zeta_1 z + \sum_{n=2}^{\infty} \zeta_n z - \sum_{n=2}^{\infty} \zeta_n \left\{ \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} z^n \right\} \\
&= z - \sum_{n=2}^{\infty} \zeta_n \left\{ \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} z^n \right\}.
\end{aligned}$$

Thus

$$\sum_{n=2}^{\infty} \zeta_n \left(\frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} \right) \left(\frac{[n(\rho+1)-(\upsilon+\rho)]\phi(n)}{1-\upsilon} \right) = \sum_{n=2}^{\infty} \zeta_n = \sum_{n=1}^{\infty} \zeta_n - \zeta_1 = 1 - \zeta_1 \leqslant 1.$$

So, by Theorem 2.1, $u \in TS(\upsilon, \rho)$.

Conversely, we suppose $u \in TS(\upsilon, \rho)$. Since

$$|a_n| \leqslant \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)}, \quad n \geqslant 2.$$

We may set

$$\zeta_n = \frac{[n(\rho+1)-(\upsilon+\rho)]\phi(n)}{1-\upsilon} |a_n|, \quad n \geqslant 2$$

and $\zeta_1 = 1 - \sum_{n=2}^{\infty} \zeta_n$. Then

$$\begin{aligned}
u(z) = z - \sum_{n=2}^{\infty} a_n z^n &= z - \sum_{n=2}^{\infty} \zeta_n \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} z^n \\
&= z - \sum_{n=2}^{\infty} \zeta_n [z - u_n(z)] \\
&= z - \sum_{n=2}^{\infty} \zeta_n z + \sum_{n=2}^{\infty} \zeta_n u_n(z) = \zeta_1 u_1(z) + \sum_{n=2}^{\infty} \zeta_n u_n(z) = \sum_{n=1}^{\infty} \zeta_n u_n(z).
\end{aligned}$$

□

Corollary 4.2. *The extreme points of* $TS(\upsilon, \rho)$ *are the functions* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{1-\upsilon}{[n(\rho+1)-(\upsilon+\rho)]\phi(n)} z^n, \; n \geqslant 2.$$

5. Radii of Close-to-convexity, starlikeness, and convexity

A function $u \in TS(\upsilon, \rho)$ is said to be close-to-convex of order δ if it satisfies

$$\Re\{u'(z)\} > \delta, \quad (0 \leqslant \delta < 1; \; z \in E).$$

Also A function $u \in TS(\upsilon, \rho)$ is said to be starlike of order δ if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta, \quad (0 \leqslant \delta < 1;\ z \in E).$$

Further a function $u \in TS(\upsilon, \rho)$ is said to be convex of order δ if and only if $zu'(z)$ is starlike of order δ that is if

$$\Re\left\{1 + \frac{zu'(z)}{u(z)}\right\} > \delta, \quad (0 \leqslant \delta < 1;\ z \in E).$$

Theorem 5.1. *Let $u \in TS(\upsilon, \rho)$. Then u is close-to-convex of order δ in $|z| < R_1$, where*

$$R_1 = \inf_{k \geqslant 2}\left[\frac{(1-\delta)[n-\upsilon+\rho(n-1)]\phi(n)}{n(1-\upsilon)}\right]^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

Proof. It is sufficient to show that $|u'(z) - 1| \leqslant 1 - \delta$, for $|z| < R_1$. We have

$$|u'(z) - 1| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leqslant \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus $|u'(z) - 1| \leqslant 1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta}|a_n||z|^{n-1} \leqslant 1. \tag{5.1}$$

But Theorem 2.1 confirms that

$$\sum_{n=2}^{\infty} \frac{[n(\rho+1) - (\upsilon+\rho)]\phi(n)}{1-\upsilon}|a_n| \leqslant 1. \tag{5.2}$$

Hence (5.1) will be true if

$$\frac{n|z|^{n-1}}{1-\delta} \leqslant \frac{[n(\rho+1) - (\upsilon+\rho)]\phi(n)}{1-\upsilon}.$$

We obtain

$$|z| \leqslant \left[\frac{(1-\delta)[n-\upsilon+\rho(n-1)]\phi(n)}{n(1-\upsilon)}\right]^{\frac{1}{n-1}}, n \geqslant 2$$

as required. □

Theorem 5.2. *Let $u \in TS(\upsilon, \rho)$. Then u is starlike of order δ in $|z| < R_2$, where*

$$R_2 = \inf_{k \geqslant 2}\left[\frac{(1-\delta)[n-\upsilon+\rho(n-1)]\phi(n)}{(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function u is given by (2.3).

Proof. We must show that $\left|\frac{zu'(z)}{u(z)} - 1\right| \leqslant 1 - \delta$, for $|z| < R_2$. We have

$$\left|\frac{zu'(z)}{u(z)} - 1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}}\right| \leqslant \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}} \leqslant 1 - \delta. \tag{5.3}$$

Hence (5.3) holds true if

$$\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1} \leqslant (1-\delta)\left(1-\sum_{n=2}^{\infty}|a_n||z|^{n-1}\right)$$

or equivalently,

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}|a_n||z|^{n-1} \leqslant 1. \tag{5.4}$$

Hence, by using (5.2) and (5.4) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leqslant \frac{[n(\rho+1)-(\upsilon+\rho)]\phi(n)}{1-\upsilon}$$
$$\Rightarrow |z| \leqslant \left[\frac{(1-\delta)[n-\upsilon+\rho(n-1)]\phi(n)}{(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}, n \geqslant 2,$$

which completes the proof. □

By using the same approach in the proof of Theorem 5.2, we can show that $\left|\frac{zu''(z)}{u'(z)}-1\right| \leqslant 1-\delta$, for $|z| < R_3$, with the aid of Theorem 2.1. Thus we have the assertion of the following Theorem 5.3.

Theorem 5.3. *Let $u \in TS(\upsilon,\rho)$. Then u is convex of order δ in $|z| < R_3$, where*

$$R_3 = \inf_{k\geqslant 2}\left[\frac{(1-\delta)[n-\upsilon+\rho(n-1)]\phi(n)}{n(n-\delta)(1-\upsilon)}\right]^{\frac{1}{n-1}}.$$

The result is sharp with the extremal function u given by (2.3).

6. Inclusion theorem involving modified Hadamard products

For functions

$$u_j(z) = z - \sum_{n=2}^{\infty}|a_{n,j}|z^n,\ j = 1,2 \tag{6.1}$$

in the class A, we define the modified Hadamard product $(u_1 * u_2)(z)$ of $u_1(z)$ and $u_2(z)$ given by

$$(u_1 * u_2)(z) = z - \sum_{n=2}^{\infty}|a_{n,1}||a_{n,2}|z^n.$$

We can prove the following.

Theorem 6.1. *Let the function u_j, $j = 1,2$, given by (6.1) be in the class $TS(\upsilon,\rho)$. Then $(u_1 * u_2)(z) \in TS(\upsilon,\rho,\lambda,t,\xi)$, where*

$$\xi = 1 - \frac{(1-\upsilon)^2}{(n+1)(2-\upsilon)(2-\upsilon+\rho)(1+\lambda)-(1-\upsilon)^2}.$$

Proof. Employing the approach used earlier by Schild and Silverman [23], we need to find the biggest ξ such that

$$\sum_{n=2}^{\infty}\frac{[n-\xi+\rho(n-1)]\phi(n)}{1-\xi}|a_{n,1}||a_{n,2}| \leqslant 1.$$

Since $u_j \in TS(\upsilon,\rho),\ j=1,2$, then we have

$$\sum_{n=2}^{\infty} \frac{[n-\upsilon+\rho(n-1)]\phi(n)}{1-\upsilon}|a_{n,1}| \leqslant 1 \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{[n-\upsilon+\rho(n-1)]\phi(n)}{1-\upsilon}|a_{n,2}| \leqslant 1,$$

by the Cauchy-Schwartz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[n-\upsilon+\rho(n-1)]\phi(n)}{1-\upsilon}\sqrt{|a_{n,1}||a_{n,2}|} \leqslant 1.$$

Thus it is sufficient to show that

$$\frac{[n-\xi+\rho(n-1)]\phi(n)}{1-\xi}|a_{n,1}||a_{n,2}| \leqslant \frac{[n-\upsilon+\rho(n-1)]\phi(n)}{1-\upsilon}\sqrt{|a_{n,1}||a_{n,2}|}, \quad n \geqslant 2,$$

that is

$$\sqrt{|a_{n,1}||a_{n,2}|} \leqslant \frac{(1-\xi)[n-\upsilon+\rho(n-1)]}{1-\upsilon)[n-\xi+\rho(n-1)]}.$$

Note that

$$\sqrt{|a_{n,1}||a_{n,2}|} \leqslant \frac{(1-\upsilon)}{[n-\upsilon+\rho(n-1)]\phi(n)}.$$

Consequently, we need only to prove that

$$\frac{(1-\upsilon)}{[n-\upsilon+\rho(n-1)]\phi(n)} \leqslant \frac{(1-\xi)[n-\upsilon+\rho(n-1)]}{1-\upsilon)[n-\xi+\rho(n-1)]}, \ n \geqslant 2,$$

or equivalently

$$\xi \leqslant 1 - \frac{(n-1)(1+\rho)(1-\upsilon)^2}{[n-\upsilon+\rho(n-1)]^2\phi(n)-(1-\upsilon)^2}, \ n \geqslant 2.$$

Since

$$A(k) = 1 - \frac{(n-1)(1+\rho)(1-\upsilon)^2}{[n-\upsilon+\rho(n-1)]^2\phi(n)-(1-\upsilon)^2}, \ n \geqslant 2$$

is an increasing function of $n, n \geqslant 2$, letting $n=2$ in last equation, we obtain

$$\xi \leqslant A(2) = 1 - \frac{(1+\rho)(1-\upsilon)^2}{[2-\upsilon+\rho]^2\phi(n)-(1-\upsilon)^2}.$$

Finally, by taking the function given by (3.2), we can see that the result is sharp. □

7. Convolution and integral operators

Let $u(z)$ be defined by (1.2) and suppose that $g(z) = z - \sum_{n=2}^{\infty} |b_n|z^n$. Then the Hadamard product (or convolution) of $u(z)$ and $g(z)$ is defined by

$$u(z) * g(z) = (u * g)(z) = z - \sum_{n=2}^{\infty} |a_n||b_n|z^n.$$

Theorem 7.1. *Let* $u \in TS(\upsilon,\rho)$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n|z^n, 0 \leqslant |b_n| \leqslant 1$. *Then* $u * g \in TS(\upsilon,\rho)$.

Proof. In view of Theorem 2.1, we have

$$\sum_{n=2}^{\infty}[n-\upsilon+\rho(n-1)]\phi(n)|a_n||b_n| \leqslant \sum_{n=2}^{\infty}[n-\upsilon+\rho(n-1)]\phi(n)|a_n| \leqslant (1-\upsilon).$$

□

Theorem 7.2. *Let* $u \in TS(\upsilon,\rho)$ *and* σ *be real number such that* $\sigma > -1$. *Then the function*

$$M(z) = \frac{\sigma+1}{z^{\sigma}}\int_0^z t^{\sigma-1}u(t)dt$$

also belongs to the class $TS(\upsilon,\rho)$.

Proof. From the representation of $M(z)$, it follows that

$$M(z) = z - \sum_{n=2}^{\infty}|A_n|z^n, \text{ where } A_n = \left(\frac{\sigma+1}{\sigma+n}\right)|a_n|.$$

Since $\sigma > -1$, then $0 \leqslant A_n \leqslant |a_n|$, which in view of Theorem 2.1, $M \in TS(\upsilon,\rho)$. □

8. Conclusion

In this paper, by making use of the well-known Borel distribution series, a new class of analytic functions was systematically defined. Then, for this newly defined functions class, we studied well-known results, such as coefficient estimates, growth and distortion properties, radii of starlike and convexity and convolution properties. Furthermore, we believe that this study will motivate a number of researchers to extend this idea to meromorphic functions, bi-univalent functions, harmonic functions, q-calculus and (p,q)-calculus. One may also apply this idea to the use sine domain, cosine domain and petal shaped domain. We hope that this distribution series play a significant role in several branches of Mathematics, Science, and Technology.

Acknowledgment

The authors are grateful to the editor and reviewer for their remarks which improve the previous version of the paper.

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