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The invariance and formulas for solutions of some fifth-order difference equations

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Abstract

Lie group analysis of the difference equations of the form

$$x_{n+1} = \frac{x_{n-4}x_{n-3}}{x_n(a_n + b_n x_{n-4}x_{n-3}x_{n-2}x_{n-1})},$$

where a_n and b_n are real sequences, is performed and non-trivial symmetries are derived. Furthermore, we find formulas for exact solutions of the equations. This work generalizes a recent result in the literature.

Keywords: Difference equation, symmetry, reduction, group invariant solutions.

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1. Introduction

In recent years, following the work of Sophus Lie [16] on differential equations, various researchers showed interest in symmetries. Lie investigated the group of transformations which leaves the differential equations invariant. The idea of symmetry is also connected to conservation laws and this connection between the two areas has led to greater motivation in researchers, after the work of Noether [21]. It is known that so long as the symmetries and first integrals are related via the invariance condition, one can implement the double reduction of the differential equations [20, 26]. The notion of using symmetries has had its extension to difference equations [17, 18]. On symmetries in difference equations, refer to [7, 8, 10, 11, 15, 22, 25]. Hydon [11] established a symmetry based algorithm that makes solution finding possible. Despite the fact that Hydon [10] emphasized on lower-order difference equations, his procedure works for any order. However, for higher-order equations, computations are cumbersome as such certain assumptions are put in order to lessen the burden of computation.

In this paper, we are inspired by the work of Elsayed [6], who studied the following recursive sequences:

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(\pm 1 \pm x_{n-1}x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, \dots, \tag{1.1}$$

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where the initial conditions are arbitrary real numbers. Clearly, (1.1) are special cases of a more general form

$$x_{n+1} = \frac{x_{n-3}x_{n-4}}{x_n(a_n + b_n x_{n-1}x_{n-2}x_{n-3}x_{n-4})}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where (a_n) and (b_n) are real sequences. Our aim is to utilize symmetry methods to solve this more general difference equation (1.2). Equivalently, we study the forward difference equation

$$u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})}, \tag{1.3}$$

since we follow the notation of [11]. Note that (A_n) and (B_n) are real sequences. For more work on recurrence equations, please see [1–5, 9, 12, 13, 19, 23].

2. Preliminaries

This section provides background to difference equations in the context of Lie symmetry analysis.

Definition 2.1. Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-invariant, and G is called symmetry group of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.

Definition 2.2. Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\mathcal{V} : M \to \mathbb{R}$ is an invariant function for G if and only if

$$X(\mathcal{V}) = 0 \quad \text{for all} \quad x \in M,$$

and every infinitesimal generator X of G.

Definition 2.3. A parameterized set of point transformations,

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon),$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$;
2. $\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0;
3. each $\hat{x}_i$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore
$$\hat{x}_i(x;\, \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, \ldots, p.$$

Assume that the forward rth-order difference equation takes the form

$$u_{n+r} = F(n, u_n, u_{n+1}, \ldots, u_{n+r-1}), \quad n \in D \tag{2.1}$$

for some smooth function F and a regular domain $D \subset \mathbb{Z}$. So as to compute a symmetry group of (2.1), we take into consideration the group of point transformations given as

$$\hat{n} = n, \qquad \hat{u}_n = u_n + \varepsilon Q(n, u_n) + O(\varepsilon^2), \qquad \hat{u}_{n+j} = u_{n+j} + \varepsilon S^j Q(n, u_n) + O(\varepsilon^2), \tag{2.2}$$

where ϵ (ϵ is sufficiently small) is the parameter, $Q = Q(n, u_n)$ is a continuous function, referred to as characteristic and S is the shift operator defined as $S : n \mapsto n + 1$. The criterion of invariance is then

$$\hat{u}_{n+r} = F(\hat{n}, \hat{u}_n, \hat{u}_{n+1}, \ldots, \hat{u}_{n+r-1}), \tag{2.3}$$

which yields the linearized symmetry condition [11]

$$S^r Q - XF = 0, \tag{2.4}$$

by substituting (2.2) in (2.3). Observe that

$$X = Q(n,u_n)\frac{\partial}{\partial u_n} + SQ(n,u_n)\frac{\partial}{\partial u_{n+1}} + \cdots + S^{r-1}Q(n,u_n)\frac{\partial}{\partial u_{n+r-1}},$$

is the corresponding 'prolonged' infinitesimal of the group of transformations (2.2). Upon knowledge of the function(s) Q, one is able to obtain the invariant $\mathcal{V}$ by using the canonical coordinate [14]

$$S_n = \int \frac{du_n}{Q(n,u_n)}. \tag{2.5}$$

Generally, the steps involved are lengthy even though very exact and do not give room to guess work on the perfect choice of invariants.

For more understanding on Lie analysis of differential and difference equations, see [11, 24].

3. Main results

We are studying the equation

$$u_{n+5} = F = \frac{u_n u_{n+1}}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})}. \tag{3.1}$$

Applying condition (2.4) to (3.1), we get

$$\begin{aligned} Q(n+5,F) - Q(n,u_n)\frac{\partial F}{\partial u_n} - Q(n+1,u_{n+1})\frac{\partial F}{\partial u_{n+1}} - Q(n+2,u_{n+2})\frac{\partial F}{\partial u_{n+2}} \\ - Q(n+3,u_{n+3})\frac{\partial F}{\partial u_{n+3}} - Q(n+4,u_{n+4})\frac{\partial F}{\partial u_{n+4}} = 0, \end{aligned}$$

that is,

$$\begin{aligned} Q(n+5,F) &+ \frac{A_n u_n u_{n+1} Q(n+4,u_{n+4})}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})^2} \\ &+ \frac{B_n {u_n}^2 {u_{n+1}}^2 u_{n+2} u_{n+3} Q(n+4,u_{n+4})}{{u_{n+4}}^2 (A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})^2} + \frac{B_n {u_n}^2 {u_{n+1}}^2 u_{n+2} Q(n+3,u_{n+3})}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})^2} \\ &+ \frac{B_n {u_n}^2 {u_{n+1}}^2 u_{n+3} Q(n+2,u_{n+2})}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})^2} - \frac{A_n u_{n+1} Q(n,u_n)}{u_{n+4}(A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})^2} = 0. \end{aligned} \tag{3.2}$$

Eliminating F is achieved by applying implicit differentiation with respect to u_n (regarding u_{n+4} as a function of u_n, u_{n+1}, u_{n+2}, u_{n+3} and u_{n+5}) via the differential operator

$$L = \frac{\partial}{\partial u_n} + \frac{\partial u_{n+4}}{\partial u_n}\frac{\partial}{\partial u_{n+4}} = \frac{\partial}{\partial u_n} - \left[\left(\frac{\partial F}{\partial u_n}\right) \Big/ \left(\frac{\partial F}{\partial u_{n+4}}\right)\right]\frac{\partial}{\partial u_{n+4}}.$$

With some simplification, we get

$$\begin{aligned} (A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})\, Q'(n+4,u_{n+4}) + B_n u_n u_{n+1} u_{n+2} Q(n+3,u_{n+3}) \\ + B_n u_n u_{n+1} u_{n+3} Q(n+2,u_{n+2}) + B_n u_n u_{n+2} u_{n+3} Q(n+2,u_{n+2}) \\ + B_n u_n u_{n+2} u_{n+3} Q(n+1,u_{n+1}) - (A_n + B_n u_n u_{n+1} u_{n+2} u_{n+3})\, Q'(n,u_n) \end{aligned} \tag{3.3}$$

$$+2B_n u_{n+1}u_{n+2}u_{n+3}Q(n,u_n)+\frac{A_n}{u_n}Q\left(n,u_n\right)=0.$$

The symbol $'$ denotes the derivative with respect to the continuous variable. Differentiating (3.3) with respect to u_n twice, keeping u_{n+4} constant, yields

$$\begin{aligned}&-B_n u_n u_{n+1}u_{n+2}u_{n+3}Q'''(n,u_n)-A_nQ'''(n,u_n)+\frac{A_n}{u_n}Q''\left(n,u_n\right)\\&-\frac{2A_n}{{u_n}^2}Q'\left(n,u_n\right)+\frac{2A_n}{{u_n}^3}Q\left(n,u_n\right)=0.\end{aligned}\tag{3.4}$$

The characteristic in (3.4) is a function of u_n only and thus we split (3.4) to get the system

$$\begin{aligned}1:Q'''(n,u_n)-\frac{1}{u_n}Q''\left(n,u_n\right)+\frac{2}{{u_n}^2}Q'\left(n,u_n\right)-\frac{2}{{u_n}^3}Q\left(n,u_n\right)=0,\\u_{n+1}u_{n+2}u_{n+3}:Q'''(n,u_n)=0.\end{aligned}\tag{3.5}$$

One obtains the solution to (3.5) as

$$Q\left(n,u_n\right)=\alpha_n{u_n}^2+\beta_n u_n\tag{3.6}$$

for some arbitrary functions α_n and β_n of n. Substituting (3.6) and its shifts in (3.2), and making a replacement of the expression of u_{n+5} given in (3.1) in the resulting equation leads to

$$\begin{aligned}&B_n u_n u_{n+1}u_{n+2}{u_{n+3}}^2u_{n+4}\alpha_{n+3}+B_n u_n u_{n+1}u_{n+2}u_{n+3}{u_{n+4}}^2\alpha_{n+4}\\&+B_n u_n u_{n+1}{u_{n+2}}^2u_{n+3}u_{n+4}\alpha_{n+2}+B_n u_n u_{n+1}u_{n+2}u_{n+3}u_{n+4}(\beta_{n+2}+\beta_{n+3}\\&+\beta_{n+4}+\beta_{n+5})+A_n{u_{n+4}}^2\alpha_{n+4}-A_n u_n u_{n+4}\alpha_n-A_n u_{n+1}u_{n+4}\alpha_{n+1}\\&-A_n u_{n+4}\left(\beta_n+\beta_{n+1}-\beta_{n+4}-\beta_{n+5}\right)+u_n u_{n+1}\alpha_{n+5}=0.\end{aligned}$$

Now equate coefficients of all powers of shifts of u_n to zero, i.e.,

$$\begin{aligned}&u_n u_{n+1}u_{n+2}{u_{n+3}}^2u_{n+4} &&: \alpha_{n+3}=0,\\&u_n u_{n+1}u_{n+2}u_{n+3}{u_{n+4}}^2 &&: \alpha_{n+4}=0,\\&u_n u_{n+1}{u_{n+2}}^2u_{n+3}u_{n+4} &&: \alpha_{n+2}=0,\\&u_n u_{n+1}u_{n+2}u_{n+3}u_{n+4} &&: \beta_{n+2}+\beta_{n+3}+\beta_{n+4}+\beta_{n+5}=0,\\&{u_{n+4}}^2 &&: \alpha_{n+4}=0,\\&u_n u_{n+4} &&: \alpha_n=0,\\&u_{n+1}u_{n+4} &&: \alpha_{n+1}=0,\\&u_{n+4} &&: \beta_n+\beta_{n+1}-\beta_{n+4}-\beta_{n+5}=0,\\&u_n u_{n+1} &&: \alpha_{n+5}=0.\end{aligned}$$

So the system above is reduced to

$$\alpha_n=0,\quad \beta_n+\beta_{n+1}+\beta_{n+2}+\beta_{n+3}=0.\tag{3.7}$$

The three independent solutions of the linear third-order difference equation (3.7) are given by $(-1)^n$, β^n, and $\bar{\beta}^n$, where $\beta=\exp\{i\pi/2\}$ and $\bar{\beta}$ denotes its complex conjugate. The characteristics are then given by

$$Q_1(n,u_n)=(-1)^n u_n,\quad Q_2(n,u_n)=\beta^n u_n,\quad \text{and}\quad Q_3(n,u_n)=\bar{\beta}^n u_n,$$

and therefore, the symmetry operators admitted by (3.1) are given by

$$X_1 = \sum_{j=0}^{4} (-1)^{n+j} u_{n+j} \frac{\partial}{\partial_{u_{n+j}}}, \qquad X_2 = \sum_{j=0}^{4} \beta^{n+j} u_{n+j} \frac{\partial}{\partial_{u_{n+j}}}, \qquad X_3 = \sum_{j=0}^{4} \bar{\beta}^{n+j} u_{n+j} \frac{\partial}{\partial_{u_{n+j}}}.$$

One can choose any one of the characteristics to write the canonical coordinate. We select Q_2. Thus

$$S_n = \int \frac{du_n}{Q_2(n, u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n|$$

and we use relation (3.7) to derive the invariant function $\tilde{V}_n$ as follows:

$$\tilde{V}_n = S_n \beta^n + S_{n+1} \beta^{n+1} + S_{n+2} \beta^{n+2} + S_{n+3} \beta^{n+3}. \tag{3.8}$$

Actually,

$$\begin{aligned} X_1(\tilde{V}_n) &= (-1)^n + (-1)^{n+1} + (-1)^{n+2} + (-1)^{n+3} = 0, \\ X_2(\tilde{V}_n) &= \beta^n + \beta^{n+1} + \beta^{n+2} + \beta^{n+3} = 0, \\ X_3(\tilde{V}_n) &= \bar{\beta}^n + \bar{\beta}^{n+1} + \bar{\beta}^{n+2} + \bar{\beta}^{n+3} = 0. \end{aligned}$$

For the sake of simplicity, we utilize

$$|V_n| = \exp(-\tilde{V}_n) \tag{3.9}$$

instead. In other words, $V_n = \pm 1/(u_n u_{n+1} u_{n+2} u_{n+3})$. One can show via (3.1) and (3.9) that

$$V_{n+2} = A_n V_n \pm B_n. \tag{3.10}$$

By utilizing the plus sign (one is allowed to choose), the solution of (3.10) can be presented in closed form as follows:

$$V_{2n+j} = V_j \left(\prod_{k_1=0}^{n-1} A_{2k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{2l+j} \prod_{k_2=l+1}^{n-1} A_{2k_2} \right), \quad j = 0, 1.$$

From the above equation, obtaining the solution of (3.1) is easier. We first use (2.5) to get

$$|u_n| = \exp(\beta_n S_n).$$

Secondly, we use (3.8) to get

$$\begin{aligned} |u_n| = \exp \Bigg[& \beta^n c_1 + \bar{\beta}^n c_2 + (-1)^n c_3 - \left(\frac{1}{4} - \frac{i}{4} \right) \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} |\tilde{V}_{k_1}| \\ & - \left(\frac{1}{4} + \frac{i}{4} \right) \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} |\tilde{V}_{k_2}| - \frac{1}{2} \sum_{k_3=0}^{n-1} (-1)^{n-k_3} |\tilde{V}_{k_3}| \Bigg]. \end{aligned}$$

Finally, invoking (3.9) yields

$$\begin{aligned} |u_n| = \exp \Bigg[& \beta^n c_1 + \bar{\beta}^n c_2 + (-1)^n c_3 + \left(\frac{1}{4} - \frac{i}{4} \right) \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \ln |V_{k_1}| \\ & + \left(\frac{1}{4} + \frac{i}{4} \right) \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \ln |V_{k_2}| + \frac{1}{2} \sum_{k_3=0}^{n-1} (-1)^{n-k_3} \ln |V_{k_3}| \Bigg] \end{aligned} \tag{3.11}$$

$$= \exp\left(H_n + \frac{1}{2}\sum_{k=0}^{n-1}\left[\sqrt{2}\cos\left(\frac{\pi(2k-2n+1)}{4}\right) + (-1)^{k-n}\right]\ln|V_k|\right),$$

where $H_n = \beta^n c_1 + \bar{\beta}^n c_2 + (-1)^n c_3$. Replacing n with $4n+j$ for $j = 0, 1, 2, 3$ yields

$$|u_{4n+j}| = \exp\left[H_j + \frac{1}{2}\sum_{k=0}^{n-1}\left(\sqrt{2}\cos\left(\frac{\pi(2k-2j+1)}{4}\right) + (-1)^{k-j}\right)\ln|V_k|\right]. \quad (3.12)$$

Set $j = 0$ in (3.12) to get

$$|u_{4n}| = \exp(H_0)\prod_{s=0}^{n-1}\left|\frac{V_{4s}}{V_{4s+1}}\right|.$$

But substituting $n = 0$ in (3.11) leads to $|u_0| = \exp(H_0)$. Furthermore, using (3.1) and (3.9), it can be shown that there is no need of absolute values. Hence

$$u_{4n} = u_0\prod_{s=0}^{n-1}\frac{V_{4s}}{V_{4s+1}} = u_0\prod_{s=0}^{n-1}\frac{V_0\left(\prod_{k_1=0}^{2s-1}A_{2k_1}\right) + \sum_{l=0}^{2s-1}\left(B_{2l}\prod_{k_2=l+1}^{2s-1}A_{2k_2}\right)}{V_1\left(\prod_{k_1=0}^{2s-1}A_{2k_1+1}\right) + \sum_{l=0}^{2s-1}\left(B_{2l+1}\prod_{k_2=l+1}^{2s-1}A_{2k_2}\right)}$$

$$= \frac{u_4^n}{u_0^{n-1}}\prod_{s=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2s-1}A_{2k_1}\right) + u_0u_1u_2u_3\sum_{l=0}^{2s-1}\left(B_{2l}\prod_{k_2=l+1}^{2s-1}A_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s-1}A_{2k_1+1}\right) + u_1u_2u_3u_4\sum_{l=0}^{2s-1}\left(B_{2l+1}\prod_{k_2=l+1}^{2s-1}A_{2k_2+1}\right)}.$$

For $j = 1$, we find that

$$u_{4n+1} = u_1\prod_{s=0}^{n-1}\frac{V_{4s+1}}{V_{4s+2}},$$

so that

$$u_{4n+1} = u_1\prod_{s=0}^{n-1}\frac{V_1\left(\prod_{k_1=0}^{2s-1}A_{2k_1+1}\right) + \sum_{l=0}^{n-1}\left(B_{2l+1}\prod_{k_2=l+1}^{2s-1}A_{2k_2+1}\right)}{V_0\left(\prod_{k_1=0}^{2s}A_{2k_1}\right) + \sum_{l=0}^{2s}\left(B_{2l}\prod_{k_2=l+1}^{2s}A_{2k_2}\right)}$$

$$= \frac{u_0^n u_1}{u_4^n}\prod_{s=0}^{n-1}\frac{\left(\prod_{k_1=0}^{2s-1}A_{2k_1+1}\right) + u_1u_2u_3u_4\sum_{l=0}^{2s-1}\left(B_{2l+1}\prod_{k_2=l+1}^{2s-1}A_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{2s}A_{2k_1}\right) + u_0u_1u_2u_3\sum_{l=0}^{2s}\left(B_{2l}\prod_{k_2=l+1}^{2s}A_{2k_2}\right)}.$$

For $j = 2$, we have

$$u_{4n+2} = u_2\prod_{s=0}^{n-1}\frac{V_{4s+2}}{V_{4s+3}},$$

which evaluates to

$$u_{4n+2} = u_2\prod_{s=0}^{n-1}\frac{V_0\left(\prod_{k_1=0}^{2s}A_{2k_1}\right) + \sum_{l=0}^{2s}\left(B_{2l}\prod_{k_2=l+1}^{2s}A_{2k_2}\right)}{V_1\left(\prod_{k_1=0}^{2s}A_{2k_1+1}\right) + \sum_{l=0}^{2s}\left(B_{2l+1}\prod_{k_2=l+1}^{2s}A_{2k_2+1}\right)}$$

$$= \frac{u_4^n u_2}{u_0^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s} A_{2k_1}\right) + u_0u_1u_2u_3 \sum_{l=0}^{2s} \left(B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s} A_{2k_1+1}\right) + u_1u_2u_3u_4 \sum_{l=0}^{2s} \left(B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}\right)}.$$

Finally, for $j = 3$, we obtain

$$u_{4n+3} = u_3 \prod_{s=0}^{n-1} \frac{V_{4s+3}}{V_{4s+4}},$$

so that

$$u_{4n+3} = u_3 \prod_{s=0}^{n-1} \frac{V_1 \left(\prod_{k_1=0}^{2s} A_{2k_1+1}\right) + \sum_{l=0}^{2s} \left(B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}\right)}{V_0 \left(\prod_{k_1=0}^{2s+1} A_{2k_1}\right) + \sum_{l=0}^{2s+1} \left(B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2}\right)}$$

$$= \frac{u_0^n u_3}{u_4^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s} A_{2k_1+1}\right) + u_1u_2u_3u_4 \sum_{l=0}^{2s} \left(B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{2s+1} A_{2k_1}\right) + u_0u_1u_2u_3 \sum_{l=0}^{2s+1} \left(B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2}\right)}.$$

Hence, the solution to (1.2) is given by

$$x_{4n-4} = \frac{x_0^n}{x_{-4}^{n-1}} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s-1} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s-1} \left(b_{2l} \prod_{k_2=l+1}^{2s-1} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s-1} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s-1} \left(b_{2l+1} \prod_{k_2=l+1}^{2s-1} a_{2k_2+1}\right)},$$

which can be rearranged as

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n} \prod_{s=0}^{n} \frac{\left(\prod_{k_1=0}^{2s-1} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s-1} \left(b_{2l} \prod_{k_2=l+1}^{2s-1} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s-1} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s-1} \left(b_{2l+1} \prod_{k_2=l+1}^{2s-1} a_{2k_2+1}\right)}.$$

The term $s = 0$ in the product (indexed by s) is equal to 1 using the facts that $\sum_{i=0}^{-1} a_i = 0$ and $\prod_{j=0}^{-1} a_j = 1$. As a result, we can still rewrite the solution as

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s+1} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s+1} \left(b_{2l} \prod_{k_2=l+1}^{2s+1} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s+1} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s+1} \left(b_{2l+1} \prod_{k_2=l+1}^{2s+1} a_{2k_2+1}\right)}. \tag{3.13}$$

Furthermore, observe that

$$x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s-1} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s-1} \left(b_{2l+1} \prod_{k_2=l+1}^{2s-1} a_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{2s} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s} \left(b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}\right)}, \tag{3.14}$$

$$x_{4n-2} = \frac{x_0^n x_{-2}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s} \left(b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}\right)}{\left(\prod_{k_1=0}^{2s} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s} \left(b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}\right)}, \tag{3.15}$$

$$x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n} \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{2s} a_{2k_1+1}\right) + x_{-3}x_{-2}x_{-1}x_0 \sum_{l=0}^{2s} \left(b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}\right)}{\left(\prod_{k_1=0}^{2s+1} a_{2k_1}\right) + x_{-4}x_{-3}x_{-2}x_{-1} \sum_{l=0}^{2s+1} \left(b_{2l} \prod_{k_2=l+1}^{2s+1} a_{2k_2}\right)}, \tag{3.16}$$

as long as any of the denominators does not vanish.

In the following sections, we look at some special cases.

4. The case a_n, b_n are 1-periodic

In this case $a_n = a$ and $b_n = b$, where $a, b \in \mathbb{R}$.

4.1. *The case* $a \neq 1$

From (3.13)-(3.16), the solution is given by

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{a^{2s+2} + bx_{-4}x_{-3}x_{-2}x_{-1}\frac{1-a^{2s+2}}{1-a}}{a^{2s+2} + bx_{-3}x_{-2}x_{-1}x_0\frac{1-a^{2s+2}}{1-a}}, \qquad x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n} \prod_{s=0}^{n-1} \frac{a^{2s} + bx_{-3}x_{-2}x_{-1}x_0\frac{1-a^{2s}}{1-a}}{a^{2s+1} + bx_{-4}x_{-3}x_{-2}x_{-1}\frac{1-a^{2s+1}}{1-a}},$$

$$x_{4n-2} = \frac{x_0^n x_{-2}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{a^{2s+1} + bx_{-4}x_{-3}x_{-2}x_{-1}\frac{1-a^{2s+1}}{1-a}}{a^{2s+1} + bx_{-3}x_{-2}x_{-1}x_0\frac{1-a^{2s+1}}{1-a}}, \qquad x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n} \prod_{s=0}^{n-1} \frac{a^{2s+1} + bx_{-3}x_{-2}x_{-1}x_0\frac{1-a^{2s+1}}{1-a}}{a^{2s+2} + bx_{-4}x_{-3}x_{-2}x_{-1}\frac{1-a^{2s+2}}{1-a}},$$

where $x_{-4}, x_0 \neq 0$ and for all $(i, s) \in \{0, 1\} \times \{0, 1, 2, 3, \dots, n-1\}$,

$$(1-a)a^{2s+i} + (1-a^{2s+i})bx_{-3}x_{-2}x_{-1}x_0 \neq 0$$

and

$$(1-a)a^{2s+1+i} + (1-a^{2s+1+i})bx_{-4}x_{-3}x_{-2}x_{-1} \neq 0.$$

4.1.1. *The case* $a = -1$

In this case, the solution which for $b = \pm 1$ appears in [6] (see Theorems 3 and 8), is given by

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n}, \qquad x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n}(-1 + bx_{-4}x_{-3}x_{-2}x_{-1})^{-n},$$

$$x_{4n-2} = \frac{x_0^n x_{-2}}{x_{-4}^n}\left(\frac{-1 + bx_{-4}x_{-3}x_{-2}x_{-1}}{-1 + bx_{-3}x_{-2}x_{-1}x_0}\right)^n, \qquad x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n}(-1 + bx_{-3}x_{-2}x_{-1}x_0)^n,$$

where $x_{-4}, x_0 \neq 0$, $bx_{-4}x_{-3}x_{-2}x_{-1} \neq 1$ and $bx_{-3}x_{-2}x_{-1}x_0 \neq 1$.

4.2. *The case* $a = 1$

From (3.13)-(3.16), the solution, which for $b = \pm 1$ appears in [6] (see Theorems 1 and 6), is given by

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{1 + (2s+2)bx_{-4}x_{-3}x_{-2}x_{-1}}{1 + (2s+2)bx_{-3}x_{-2}x_{-1}x_0}, \qquad x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n} \prod_{s=0}^{n-1} \frac{1 + 2sbx_{-3}x_{-2}x_{-1}x_0}{1 + (2s+1)bx_{-4}x_{-3}x_{-2}x_{-1}},$$

$$x_{4n-2} = \frac{x_0^n x_{-2}}{x_{-4}^n} \prod_{s=0}^{n-1} \frac{1 + (2s+1)bx_{-4}x_{-3}x_{-2}x_{-1}}{1 + (2s+1)bx_{-3}x_{-2}x_{-1}x_0}, \qquad x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n} \prod_{s=0}^{n-1} \frac{1 + (2s+1)bx_{-3}x_{-2}x_{-1}x_0}{1 + (2s+2)bx_{-4}x_{-3}x_{-2}x_{-1}},$$

where $x_{-4}, x_0 \neq 0$, $2jbx_{-4}x_{-3}x_{-2}x_{-1} \neq -1$, $(2j-1)bx_{-4}x_{-3}x_{-2}x_{-1} \neq -1$, $2jbx_{-3}x_{-2}x_{-1}x_0 \neq -1$, and $(2j-1)bx_{-3}x_{-2}x_{-1}x_0 \neq -1$ for all $j = 1, 2, 3, \dots, n$.

5. The case a_n, b_n are 2-periodic

We assume that $\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_0, a_1, \ldots$ and $\{b_n\}_{n=0}^{\infty} = b_0, b_1, b_0, b_1, \ldots$. Then, from (3.13)-(3.16), we have

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n}\prod_{s=0}^{n-1}\frac{a_0^{2s+2}+b_0x_{-4}x_{-3}x_{-2}x_{-1}\sum\limits_{l=0}^{2s+1}a_0^l}{a_1^{2s+2}+b_1x_{-3}x_{-2}x_{-1}x_0\sum\limits_{l=0}^{2s+1}a_1^l}, \quad x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n}\prod_{s=0}^{n-1}\frac{a_1^{2s}+b_1x_{-3}x_{-2}x_{-1}x_0\sum\limits_{l=0}^{2s-1}a_1^l}{a_0^{2s+1}+b_0x_{-4}x_{-3}x_{-2}x_{-1}\sum\limits_{l=0}^{2s}a_0^l},$$

$$x_{4n-2} = \frac{x_0^n x_{-2}}{x_{-4}^n}\prod_{s=0}^{n-1}\frac{a_0^{2s+1}+b_0x_{-4}x_{-3}x_{-2}x_{-1}\sum\limits_{l=0}^{2s}a_0^l}{a_1^{2s+1}+b_1x_{-3}x_{-2}x_{-1}x_0\sum\limits_{l=0}^{2s}a_1^l}, \quad x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n}\prod_{s=0}^{n-1}\frac{a_1^{2s+1}+b_1x_{-3}x_{-2}x_{-1}x_0\sum\limits_{l=0}^{2s}a_1^l}{a_0^{2s+2}+b_0x_{-4}x_{-3}x_{-2}x_{-1}\sum\limits_{l=0}^{2s+1}a_0^l},$$

as long as $x_{-4}, x_0 \neq 0$ and for all $(i, s) \in \{0,1\} \times \{0,1,2,\ldots,n-1\}$, $a_0^{2s+1+i} + b_0x_{-4}x_{-3}x_{-2}x_{-1}\sum\limits_{l=0}^{2s+i} a_0^l \neq 0$ and $a_1^{2s+1+i} + b_1x_{-3}x_{-2}x_{-1}x_0\sum\limits_{l=0}^{2s+i} a_1^l \neq 0$.

5.1. *The case* $a_0 = 1$ *and* $a_1 = -1$

The solution is given by

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n}\prod_{s=0}^{n-1}(1+(2s+2)b_0x_{-4}x_{-3}x_{-2}x_{-1}),$$

$$x_{4n-3} = \frac{x_{-4}^n x_{-3}}{x_0^n}\prod_{s=0}^{n-1}\frac{1}{1+(2s+1)b_0x_{-4}x_{-3}x_{-2}x_{-1}},$$

$$x_{4n-2} = x_{-2}\left(\frac{x_0}{x_{-4}(-1+b_1x_{-3}x_{-2}x_{-1}x_0)}\right)^n\prod_{s=0}^{n-1}(1+(2s+1)b_0x_{-4}x_{-3}x_{-2}x_{-1}),$$

$$x_{4n-1} = x_{-1}\left(\frac{x_{-4}(-1+b_1x_{-3}x_{-2}x_{-1}x_0)}{x_0}\right)^n\prod_{s=0}^{n-1}\frac{1}{1+(2s+2)b_0x_{-4}x_{-3}x_{-2}x_{-1}},$$

where $x_{-4}, x_0 \neq 0$, $b_1x_{-3}x_{-2}x_{-1}x_0 \neq 1$ and $jb_0x_{-4}x_{-3}x_{-2}x_{-1} \neq -1$ for all $j = 1,2,3,\ldots,2n$.

5.2. *The case* $a_0 = -1$ *and* $a_1 = 1$

In this case, we obtain

$$x_{4n} = \frac{x_0^{n+1}}{x_{-4}^n}\prod_{s=0}^{n-1}\frac{1}{1+(2s+2)b_1x_{-3}x_{-2}x_{-1}x_0},$$

$$x_{4n-3} = x_{-3}\left(\frac{x_{-4}}{x_0(-1+b_0x_{-4}x_{-3}x_{-2}x_{-1})}\right)^n\prod_{s=0}^{n-1}(1+2sb_1x_{-3}x_{-2}x_{-1}x_0),$$

$$x_{4n-2} = x_{-2}\left(\frac{x_0(-1+b_0x_{-4}x_{-3}x_{-2}x_{-1})}{x_{-4}}\right)^n\prod_{s=0}^{n-1}\frac{1}{1+(2s+1)b_1x_{-3}x_{-2}x_{-1}x_0},$$

$$x_{4n-1} = \frac{x_{-4}^n x_{-1}}{x_0^n}\prod_{s=0}^{n-1}(1+(2s+1)b_1x_{-3}x_{-2}x_{-1}x_0),$$

where $x_{-4}, x_0 \neq 0$, $b_0x_{-4}x_{-3}x_{-2}x_{-1} \neq 1$ and $jb_1x_{-3}x_{-2}x_{-1}x_0 \neq -1$ for all $j = 1,2,3,\ldots,2n$.

6. Numerical examples

In this section, we plot some graphs that illustrate the behavior of the solutions.

Example 6.1. Figure 1 is the graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(1+8u_n u_{n+1} u_{n+2} u_{n+3})}$ with $u_0 = 1, u_1 = 0.96, u_2 = 0.71, u_3 = 8$, and $u_4 = 0.46$.

Example 6.2. Figure 2 is the graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(1+0.15u_n u_{n+1} u_{n+2} u_{n+3})}$ with $u_0 = 0.46, u_1 = 0.44, u_2 = 0.36, u_3 = 0.98$, and $u_4 = 0.6$.

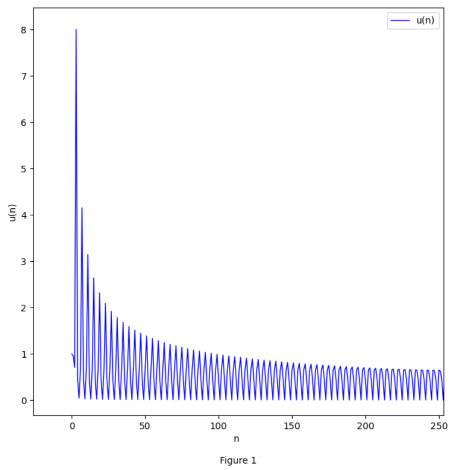

Figure 1: Graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(1+8u_n u_{n+1} u_{n+2} u_{n+3})}$.

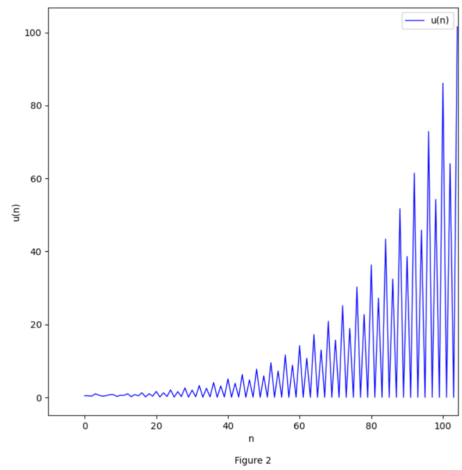

Figure 2: Graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(1+0.15u_n u_{n+1} u_{n+2} u_{n+3})}$.

Example 6.3. Figure 3 is the graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(7+3u_n u_{n+1} u_{n+2} u_{n+3})}$ with $u_0 = 2, u_1 = 0.1, u_2 = 0.96, u_3 = 6$, and $u_4 = 0.26$.

Example 6.4. Figure 4 is the graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(0.93+0.86u_n u_{n+1} u_{n+2} u_{n+3})}$ with $u_0 = 0.79, u_1 = 0.22, u_2 = 0.05, u_3 = 0.74$, and $u_4 = 0.17$.

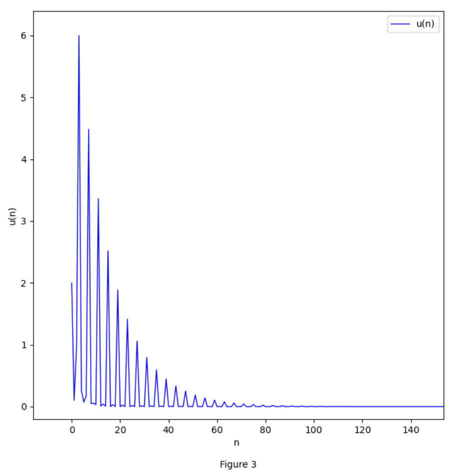

Figure 3: Graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(7+3u_n u_{n+1} u_{n+2} u_{n+3})}$.

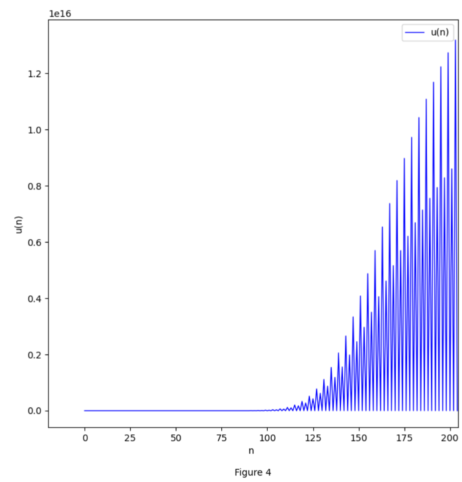

Figure 4: Graph of $u_{n+5} = \frac{u_n u_{n+1}}{u_{n+4}(0.93+0.86u_n u_{n+1} u_{n+2} u_{n+3})}$.

7. Conclusion

Our work in this paper was twofold. First, we found non-trivial Lie symmetry generators of the difference equations (1.2). Second, we derived explicit formulas for solutions of difference equations in question. Consequently, this generalized what Elsayed found where the values of a_n and b_n were only confined to ± 1. We showed that in those particular cases, our results yielded Elsayed's results.

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References

[1] M. B. Almatrafi, *Analysis of Solutions of Some Discrete Systems of Rational Difference Equations*, J. Comput. Anal. Appl., **29** (2021), 355–368. 1
[2] M. B. Almatrafi, *Exact solutions and stability of sixth order difference equations*, Electron. J. Math. Anal. Appl., **10** (2022), 209–225.
[3] M. B. Almatrafi, M. M. Alzubaidi, *Stability analysis for a rational difference equation*, Arab J. Basic Appl. Sci., **27** (2020), 114–120.
[4] M. B. Almatrafi, M. M. Alzubaidi, *The solution and dynamic behaviour of some difference equations of seventh order*, J. Appl. Nonlinear Dyn., **10** (2021), 709–719.
[5] M. B. Almatrafi, E. M. Elsayed, F. Alzahrani, *The Solution and Dynamic Behavior of Some Difference Equations of Fourth Order*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., **29** (2022), 33–50. 1
[6] E. M. Elsayed, *Expression and behavior of the solutions of some rational recursive sequences*, Math. Methods Appl. Sci., **2016** (2016), 5682–5694. 1, 4.1.1, 4.2
[7] M. Folly-Gbetoula, *Symmetry, reductions and exact solutions of the difference equation* $u_{n+2} = (au_n)/(1 + bu_n u_{n+1})$, J. Difference Equ. Appl., **23** (2017), 1017–1024. 1
[8] M. K. Folly-Gbetoula, A. H. Kara, *Symmetries, conservation laws, and integrability of difference equations*, Adv. Difference Equ., **2014** (2014), 14 pages. 1
[9] M. Folly-Gbetoula, D. Nyirenda, *On some sixth-order rational recursive sequences*, J. Comput. Anal. Appl., **27** (2019), 1057–1069. 1
[10] P. E. Hydon, *Symmetries and first integrals of ordinary difference equations*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., **456** (2000), 2835–2855. 1
[11] P. E. Hydon, *Difference Equations by Differential Equation Methods*, Cambridge University Press, Cambridge, (2014). 1, 1, 2, 2
[12] T. F. Ibrahim, A. Q. Khan, *Forms of solutions for some two-dimensional systems of rational Partial Recursion Equations*, Math. Probl. Eng., **2021** (2021), 10 pages. 1
[13] T. F. Ibrahim, A. Q. Khan, A. Ibrahim, *Qualitative Behavior of a Nonlinear Generalized Recursive Sequence with Delay*, Math. Problems Eng., **2021** (2021), 8 pages. 1
[14] N. Joshi, P. J. Vassiliou, *The existence of Lie Symmetries for First-Order Analytic Discrete Dynamical Systems*, J. Math. Anal. Appl., **195** (1995), 872–887. 2
[15] D. Levi, L. Vinet, P. Winternitz, *Lie group formalism for difference equations*, J. Phys. A, **30** (1997), 633–649. 1
[16] S. Lie, *Classification und Integration von gewohnlichen Differentialgleichungen zwischen xy, die eine Gruppe von Transformationen gestatten I*, Math. Ann., **22** (1888), 213–253. 1
[17] S. Maeda, *Canonical structure and symmetries for discrete systems*, Math. Japon., **25** (1980), 405–420. 1
[18] S. Maeda, *The similarity method for difference equations*, IMA J. Appl. Math., **38** (1987), 129–134. 1
[19] N. Mnguni, M. Folly-Gbetoula, *Invariance analysis of a third-order difference equation with variable coefficients*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms, **25** (2018), 63–73. 1
[20] R. Morris, A. H. Kara, A. Chowdhury, A. Biswas, *Soliton solutions, conservation laws, and reductions of certain classes of nonlinearwave equations*, Zeitschrift für NaturforschungSection A, **67** (2012), 613–620. 1
[21] E. Noether, *Invariante variations probleme*, Math. Phys. Kl., **2** (1918), 235–257. 1
[22] D. Nyirenda, M. Folly-Gbetoula, *Invariance analysis and exact solutions of some sixth-order difference equations*, J. Nonlinear Sci. Appl., **10** (2017), 6262–6273. 1
[23] B. Oğul, D. Simşek, T. F. Ibrahim, *Solution of the Rational Difference Equation*, Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications & Algorithms, **28** (2021), 125–141. 1
[24] P. J. Olver, *Application of Lie Group to Differential Equation*, Springer-Verlag, New York, (1986). 2
[25] G. R. W. Quispel, R. Sahadevan, *Lie symmetries and the integration of difference equations*, Phys. Lett. A, **184** (1993), 64–70. 1
[26] W. Sarlet, F. Cantrijn, *Generalizations of Noether's theorem in classical mechanics*, SIAM Rev., **23** (1981), 467–494. 1