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(inf, sup)-Hesitant Fuzzy Subalgebras of BCK/BCI-Algebras

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Abstract

In this paper, we introduce the concept of (inf, sup)-hesitant fuzzy subalgebras, which is a general concept of interval-valued fuzzy subalgebras, in BCK/BCI-algebras and investigate its properties. We characterize (inf, sup)-hesitant fuzzy subalgebras in terms of sets, fuzzy sets, hesitant fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, negative fuzzy sets and bipolar fuzzy sets. Furthermore, characterizations of subalgebras, fuzzy subalgebras, anti-fuzzy subalgebras, negative fuzzy subalgebras, Pythagorean fuzzy subalgebras and bipolar fuzzy subalgebras of BCK/BCI-algebras are given in terms of (inf, sup)-hesitant fuzzy subalgebras and interval-valued fuzzy subalgebras.

Keywords: BCK/BCI-algebra, hesitant fuzzy subalgebra, (inf, sup)-hesitant fuzzy subalgebra, interval-valued fuzzy subalgebra, Pythagorean fuzzy subalgebra, bipolar fuzzy subalgebra.

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1. Introduction

Algebraic structures play a prominent role in mathematics with wide applications in many disciplines such as computer science, information science, engineering, etc. BCK-algebras [11] and BCI-algebras [12] are important algebraic structures and inspired the creation of logical algebras such as JU-algebras [2], BRK-algebras [4], BE-algebras [27], IUP-algebras [10], KU-algebras [39], etc. In 1991, Xi [43] applied the concept of fuzzy sets, introduced by Zadeh [46], to BCK-algebras. Later, many algebraists around the world applied and studied the concept of fuzzy sets and its extended, related or general

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concepts to BCK/BCI-algebras, for examples, Hong and Jun [8] introduced and discussed anti-types of fuzzy sets based on subalgebras and ideals of BCK-algebras. Subha and Dhanalakshmi [40] studied Pythagorean fuzzy sets based on subalgebras and ideals of BCK-algebras. Jun [21] introduced and studied interval-valued fuzzy subalgebras and ideals of BCK-algebras. Jun et al. [24, 25] introduced notion of N-subalgebras in BCK/BCI-algebras and investigated its characterizations. Lee [28] introduced the concepts of bipolar fuzzy subalgebras and ideals of BCK/BCI-algebras, and investigated their related properties. Jun and Ahn [23] applied the concept of hesitant fuzzy sets to BCK/BCI-algebras, and introduced hesitant fuzzy subalgebras and ideals of BCK/BCI-algebras. Muhiuddin et al. studied and developed hesitant fuzzy set theory on BCK/BCI-algebras (see [34, 36]).

In 2022, Julatha and Iampan [19] introduced new types of hesitant fuzzy sets on Γ-semigroups related to the infimum and supremum of its images, which are called an inf-hesitant fuzzy Γ-ideal and an $(\sup,\inf)$-hesitant fuzzy Γ-ideal. The new types of hesitant fuzzy sets are generalizations of the concept of interval-valued fuzzy Γ-ideals, and characterized in terms of sets, fuzzy sets, intuitionistic fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Moreover, characterizations of Γ-ideals of Γ-semigroups are discussed by inf-hesitant and $(\sup,\inf)$-hesitant fuzzy Γ-ideals. In the same year, Julatha and Iampan [18] introduced an inf-hesitant fuzzy ideal (right ideal, left ideal, lateral ideal) and an $(\sup,\inf)$-hesitant fuzzy ideal (right ideal, left ideal, lateral ideal), which are generalizations of an interval-valued fuzzy ideal, in a ternary semigroup and given their characterizations in terms of sets, fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets and hesitant fuzzy sets. Also, characterizations of an ideal of a ternary semigroup given by the inf-hesitant and $(\sup,\inf)$-hesitant fuzzy ideals. Some related concepts of hesitant fuzzy sets on algebraic structures, in the meaning of the infimum or supremum of their images, are studied in [7, 14–17, 20, 26, 30, 31, 38]. The researcher who inspired this kind of study and research was Muhiuddin, which can be seen in [32, 33, 35, 37].

As previously stated, it motivated us to study hesitant fuzzy sets on BCK/BCI-algebras in the meaning of infimum and supremum. We introduce the concept of $(\inf,\sup)$-hesitant fuzzy subalgebras of BCK/BCI-algebras and investigate its properties. It is showed that every interval-valued fuzzy subalgebra of a BCK/BCI-algebra is an $(\inf,\sup)$-hesitant fuzzy subalgebra, but the converse is not true. Characterizations of $(\inf,\sup)$-hesitant fuzzy subalgebras are investigated in terms of sets, fuzzy sets, hesitant fuzzy sets, interval-valued fuzzy sets, Pythagorean fuzzy sets, negative fuzzy sets and bipolar fuzzy sets. Moreover, we characterize subalgebras, fuzzy subalgebras, anti-fuzzy subalgebras, negative fuzzy subalgebras, Pythagorean fuzzy subalgebras and bipolar fuzzy subalgebras of BCK/BCI-algebras in terms of $(\inf,\sup)$-hesitant fuzzy subalgebras and interval-valued fuzzy subalgebras.

2. Preliminaries

An algebra $(\mathcal{U};\circledast,0)$ of type $(2,0)$ is called a **BCI-algebra**, where $\mathcal{U}$ is a nonempty set, 0 is a fixed element of $\mathcal{U}$ and $\circledast$ is a binary operation on $\mathcal{U}$ if the following hold:

(I) $(\forall u,v,w \in \mathcal{U})(((u\circledast v)\circledast(u\circledast w))\circledast(w\circledast v)=0)$;
(II) $(\forall u,v \in \mathcal{U})((u\circledast(u\circledast v))\circledast v=0)$;
(III) $(\forall u \in \mathcal{U})(u\circledast u=0)$;
(IV) $(\forall u,v \in \mathcal{U})(u\circledast v=0=v\circledast u \Rightarrow u=v)$.

By a **BCK-algebra** we mean a BCI-algebra $(\mathcal{U};\circledast,0)$ satisfies $0\circledast u=0$ for all $u\in\mathcal{U}$. In a BCK/BCI-algebra $(\mathcal{U};\circledast,0)$, the following are true:

$$(\forall u \in \mathcal{U})(u\circledast 0=u), \qquad (\forall u,v,w \in \mathcal{U})((u\circledast v)\circledast w=(u\circledast w)\circledast v).$$

A nonempty subset $\mathcal{A}$ of a BCK/BCI-algebra $(\mathcal{U};\circledast,0)$ is called a **subalgebra** (SA) of $\mathcal{U}$ if $u\circledast v\in\mathcal{A}$ for all $u,v\in\mathcal{A}$. Then 0 is in a subalgebra $\mathcal{A}$.

We refer the reader to the books [9, 29] and paper [13] for further information regarding BCK/BCI-algebras.

In what follows, let $\mathcal{U}$ denote a BCK/BCI-algebra $(\mathcal{U}, \circledast, 0)$, $\mathbb{R}$ denote the set of all real numbers, and $\mathcal{V}$ denote a nonempty set unless otherwise specified. For an arbitrary element r of $\mathbb{R}$ and arbitrary functions δ and η from $\mathcal{V}$ into $\mathbb{R}$, we define

$$r\delta : \mathcal{V} \to \mathbb{R}, u \mapsto r\delta(u), \qquad r+\delta : \mathcal{V} \to \mathbb{R}, u \mapsto r+\delta(u),$$
$$\delta - r : \mathcal{V} \to \mathbb{R}, u \mapsto \delta(u) - r, \qquad \delta \leqslant \eta \Leftrightarrow (\forall u \in \mathcal{V})(\delta(u) \leqslant \eta(u)).$$

We denote $-\delta$, $r-\delta$ and $\frac{\delta}{r}$ for $(-1)\delta$, $r+(-\delta)$ and $(\frac{1}{r})\delta$ (when $r \neq 0$), respectively.

A **fuzzy set** (FS) [46] in $\mathcal{V}$ is an arbitrary function from $\mathcal{V}$ into $[0,1]$. an FS δ in $\mathcal{U}$ is call a **fuzzy subalgebra** (FSA) [43] of $\mathcal{U}$ if $\delta(u \circledast v) \geqslant \min\{\delta(u), \delta(v)\}$ for all $u, v \in \mathcal{U}$, and called an **anti-fuzzy subalgebra** (AFSA) [8] of $\mathcal{U}$ if $\delta(u \circledast v) \leqslant \max\{\delta(u), \delta(v)\}$ for all $u, v \in \mathcal{U}$. Then the following hold:

(1) if δ is an FSA of $\mathcal{U}$, then $\delta(0) \geqslant \delta(u)$ for all $u \in \mathcal{U}$;
(2) if δ is an AFSA of $\mathcal{U}$, then $\delta(0) \leqslant \delta(u)$ for all $u \in \mathcal{U}$;
(3) δ is both an FSA and an AFSA of $\mathcal{U}$ if and only if δ is a constant function.

By a **negative fuzzy set** (NFS)[25], we mean a function from $\mathcal{V}$ into $[-1, 0]$. Then

(1) if δ is an FS of $\mathcal{V}$, then $\delta - 1$ and $-\delta$ are NFSs of $\mathcal{V}$;
(2) if δ is a NFS of $\mathcal{V}$, then $\delta + 1$ and $-\delta$ are FSs of $\mathcal{V}$.

A NFS δ in $\mathcal{U}$ is called a **negative fuzzy subalgebra** (NFSA) [25] of $\mathcal{U}$ if $\delta(u \circledast v) \leqslant \max\{\delta(u), \delta(v)\}$ for all $u, v \in \mathcal{U}$.

A **Pythagorean fuzzy set** (PFS) [44, 45] on $\mathcal{V}$ is an object having the form $P = \{(u, \delta(u), \eta(u)) \mid u \in \mathcal{V}\}$ when the functions $\delta : \mathcal{V} \to [0,1]$ denote the degree of membership and $\eta : \mathcal{V} \to [0,1]$ denote the degree of nonmembership, and $0 \leqslant (\delta(u))^2 + (\eta(u))^2 \leqslant 1$ for all $u \in \mathcal{V}$. We denote (δ, η) for the PFS $\{(u, \delta(u), \eta(u)) \mid u \in \mathcal{V}\}$. Then $(\frac{\delta}{1+i}, \frac{\eta}{1+i})$, $(\frac{i+\delta}{1+2j}, \frac{i+\eta}{1+2j})$ and $(\frac{\delta}{1+i}, \frac{\delta}{1+i})$ are PFSs in $\mathcal{V}$ for each FSs δ and η in $\mathcal{V}$ and positive integers i and j such that $i \leqslant j$. Thus the concept of PFSs is an extension of the concept of FSs. A PFS (δ, η) on $\mathcal{U}$ is called a **Pythagorean fuzzy subalgebra** (PFSA) [40] of $\mathcal{U}$ if δ is an FSA and η is an AFSA of $\mathcal{U}$.

A **bipolar fuzzy set** (BFS) [48] in $\mathcal{V}$ is an object having the form $B = \{(u, \delta(u), \eta(u)) \mid u \in \mathcal{V}\}$, where δ is a NFS and η is an FS in $\mathcal{V}$. We denote $\langle \delta, \eta \rangle$ for the BFS $\{(u, \delta(u), \eta(u)) \mid u \in \mathcal{V}\}$. Then we have the following conditions:

(1) $\langle \delta - 1, \delta \rangle$ is a BFS in $\mathcal{V}$ for each FS δ in $\mathcal{V}$;
(2) $\langle \delta - 1, \eta \rangle$ and $\langle \eta - 1, \delta \rangle$ are BFSs in $\mathcal{V}$ for each PFS (δ, η) in $\mathcal{V}$;
(3) $(\frac{1+\delta}{1+i}, \frac{\eta}{1+i})$ and $(\frac{\eta}{1+i}, \frac{1+\delta}{1+i})$ are PFSs in $\mathcal{V}$ for each BFS $\langle \delta, \eta \rangle$ in $\mathcal{V}$ and positive integer i.

Note that the concept of BFSs is an extension of the concept of FSs.

A BFS $B = \langle \delta, \eta \rangle$ in $\mathcal{U}$ is called a **bipolar fuzzy subalgebra** (BFSA) [28] of $\mathcal{U}$ if δ is an NFSA of $\mathcal{U}$ and η is an FSA of $\mathcal{U}$.

By an interval number $\check{t}$ we mean an interval $[t^-, t^+]$, where $0 \leqslant t^- \leqslant t^+ \leqslant 1$. We denote $\mathcal{D}([0,1])$ for the set of all interval numbers. For each elements $\check{s} = [s^-, s^+]$, $\check{t} = [t^-, t^+] \in \mathcal{D}([0,1])$, define the operations $\precsim$, $=$, $\prec$ and rmin as follows:

(1) $\check{s} \precsim \check{t} \Leftrightarrow s^- \leqslant t^-$ and $s^+ \leqslant t^+$;
(2) $\check{s} = \check{t} \Leftrightarrow s^- = t^-$ and $s^+ = t^+$;
(3) $\check{s} \prec \check{t} \Leftrightarrow \check{s} \precsim \check{t}$ and $\check{s} \neq \check{t}$;
(4) $\text{rmin}\{\check{s}, \check{t}\} = [\min\{s^-, t^-\}, \min\{s^+, t^+\}]$.

An **interval-valued fuzzy set** (IvFS) [47] on $\mathcal{V}$ is defined to be a function $\check{\pi} : \mathcal{V} \to \mathcal{D}([0,1])$, where $\check{\pi}(u) = [\check{\pi}^-(u), \check{\pi}^+(u)]$ for all $u \in \mathcal{V}$, $\check{\pi}^-$ and $\check{\pi}^+$ are FSs in $\mathcal{V}$ such that $\check{\pi}^- \leqslant \check{\pi}^+$. Thus the concept of IvFSs is an extension of the concept of FSs. An IvFS $\check{\pi}$ on $\mathcal{U}$ is called an **interval-valued fuzzy subalgebra** (IvFSA) [21] of $\mathcal{U}$ if $\text{rmin}\{\check{\pi}(u), \check{\pi}(v)\} \precsim \check{\pi}(u \circledast v)$ for all $u, v \in \mathcal{U}$.

Remark 2.1. an IvFS $\breve{\pi}$ on $\mathcal{U}$ is an IvFSA of $\mathcal{U}$ if and only if $\breve{\pi}^-$ and $\breve{\pi}^+$ are FSAs of $\mathcal{U}$.

A **hesitant fuzzy set** (HFS) [41, 42] on $\mathcal{V}$ is defined to be a function $\widetilde{\kappa} : \mathcal{V} \to \wp([0,1])$ when $\wp([0,1])$ is the set of all subsets of $[0,1]$. Then $\mathcal{D}([0,1]) \subseteq \wp([0,1])$ and we see that every IvFS on $\mathcal{V}$ is a HFS on $\mathcal{V}$. Thus the concept of HFSs is both a generalization of the concept of IvFSs, and an extension of the concept of FSs. A HFS $\widetilde{\kappa}$ is a **hesitant fuzzy subalgebra** (HFSA) [22, 23] of $\mathcal{U}$ if $\widetilde{\kappa}(u) \cap \widetilde{\kappa}(v) \subseteq \widetilde{\kappa}(u \circledast v)$ for all $u, v \in \mathcal{U}$.

3. (inf, sup)-hesitant fuzzy subalgebras

In this section, we introduce the concept of (inf, sup)-hesitant fuzzy subalgebras of BCK/BCI-algebras, investigate its properties and give its examples. Later, we show that the concept is a general concept of interval-valued fuzzy subalgebras. Finally, we investigate characterizations of the concept of (inf, sup)-hesitant fuzzy subalgebras in terms of sets, fuzzy sets, negative fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets, hesitant fuzzy sets and bipolar fuzzy sets.

For each element $\Theta \in \wp([0,1])$ and HFS $\widetilde{\kappa}$ on $\mathcal{U}$, define the elements $\mathrm{SUP}\,\Theta$ [14, 20] and $\mathrm{INF}\,\Theta$ [15, 16] of $[0,1]$ and the subset $[\mathcal{U}, \widetilde{\kappa}, \Theta]$ of $\mathcal{U}$ as follows:

$$\mathrm{SUP}\,\Theta = \begin{cases} \sup \Theta, & \text{if } \Theta \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \qquad \mathrm{INF}\,\Theta = \begin{cases} \inf \Theta, & \text{if } \Theta \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[\mathcal{U}, \widetilde{\kappa}, \Theta] = \{u \in \mathcal{U} \mid \mathrm{SUP}\,\widetilde{\kappa}(u) \geqslant \mathrm{SUP}\,\Theta, \mathrm{INF}\,\widetilde{\kappa}(u) \geqslant \mathrm{INF}\,\Theta\}.$$

Definition 3.1. A HFS $\widetilde{\kappa}$ on $\mathcal{U}$ is said to be an **(inf, sup)-hesitant fuzzy subalgebra** ((inf, sup)-HFSA) of $\mathcal{U}$ if the set $[\mathcal{U}, \widetilde{\kappa}, \Theta]$ is a SA of $\mathcal{U}$ for all $\Theta \in \wp([0,1])$ when $[\mathcal{U}, \widetilde{\kappa}, \Theta] \neq \emptyset$.

For any HFS $\widetilde{\kappa}$ on $\mathcal{V}$, define the FSs $F^{\widetilde{\kappa}}$ and $F_{\widetilde{\kappa}}$ in $\mathcal{V}$ by $F^{\widetilde{\kappa}}(u) = \mathrm{SUP}\,\widetilde{\kappa}(u)$, and $F_{\widetilde{\kappa}}(u) = \mathrm{INF}\,\widetilde{\kappa}(u)$ for all $u \in \mathcal{V}$. A HFS $\widetilde{\varepsilon}$ on $\mathcal{V}$ is called a **supremum complement** of $\widetilde{\kappa}$ on $\mathcal{V}$ if $\mathrm{SUP}\,\widetilde{\varepsilon}(u) = (1 - F^{\widetilde{\kappa}})(u)$ for all $u \in \mathcal{V}$ and called an **infimum complement** [15, 19] of $\widetilde{\kappa}$ on $\mathcal{V}$ if $\mathrm{INF}\,\widetilde{\varepsilon}(u) = (1 - F_{\widetilde{\kappa}})(u)$ for all $u \in \mathcal{V}$. The set of all supremum complements of $\widetilde{\kappa}$ is denoted by $\mathrm{SC}(\widetilde{\kappa})$ and the set of all infimum complements of $\widetilde{\kappa}$ is denoted by $\mathrm{IC}(\widetilde{\kappa})$. Define the HFSs $\widetilde{\kappa}^{\pm}$ and $\widetilde{\kappa}^{\mp}$ on $\mathcal{V}$ by $\widetilde{\kappa}^{\pm}(u) = \{(1 - F_{\widetilde{\kappa}})(u)\}$ and $\widetilde{\kappa}^{\mp}(u) = \{(1 - F^{\widetilde{\kappa}})(u)\}$ for all $u \in \mathcal{V}$. Then $\widetilde{\kappa}^{\pm} \in \mathrm{IC}(\widetilde{\kappa})$ and $\widetilde{\kappa}^{\mp} \in \mathrm{SC}(\widetilde{\kappa})$. Moreover, $F_{\widetilde{\kappa}^{\pm}} = F_{\widetilde{\varepsilon}} = 1 - F_{\widetilde{\kappa}}$ for each $\widetilde{\varepsilon} \in \mathrm{IC}(\widetilde{\kappa})$ and $F^{\widetilde{\kappa}^{\mp}} = F^{\widetilde{\tau}} = 1 - F^{\widetilde{\kappa}}$ for each $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\kappa})$. Next, we investigate characterizations of (inf, sup)-HFSAs of BCK/BCI-algebras in terms of SAs, FSAs, AFSAs and NFSAs.

Lemma 3.2. *The following are equivalent for a HFS $\widetilde{\kappa}$ on $\mathcal{U}$.*

(1) *$\widetilde{\kappa}$ is an (inf, sup)-HFSA of $\mathcal{U}$.*
(2) *$F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$.*
(3) *$-F_{\widetilde{\kappa}}$ and $-F^{\widetilde{\kappa}}$ are NFSAs of $\mathcal{U}$.*
(4) *$F_{\widetilde{\kappa}^{\pm}}$ and $F^{\widetilde{\kappa}^{\mp}}$ are AFSAs of $\mathcal{U}$.*
(5) *$F_{\widetilde{\kappa}^{\pm}} - 1$ and $F^{\widetilde{\kappa}^{\mp}} - 1$ are NFSAs of $\mathcal{U}$.*
(6) *$F_{\widetilde{\varepsilon}}$ and $F^{\widetilde{\tau}}$ are AFSAs of $\mathcal{U}$ for each $\widetilde{\varepsilon} \in \mathrm{IC}(\widetilde{\kappa})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\kappa})$.*
(7) *$F_{\widetilde{\varepsilon}} - 1$ and $F^{\widetilde{\tau}} - 1$ are NFSAs of $\mathcal{U}$ for each $\widetilde{\varepsilon} \in \mathrm{IC}(\widetilde{\kappa})$ and $\widetilde{\tau} \in \mathrm{SC}(\widetilde{\kappa})$.*
(8) *$[\mathcal{U}, \widetilde{\kappa}, \check{t}]$ is a SA of $\mathcal{U}$ for each $\check{t} \in \mathcal{D}([0,1])$ such that $[\mathcal{U}, \widetilde{\kappa}, \check{t}] \neq \emptyset$.*

Proof. (1)⇒(8), (6)⇒(4), (7)⇒(5), and (5)⇔(3). They are clear.

(8)⇒(2). Let $u, v \in \mathcal{U}$ and choose

$$\check{t} := \{m \in [0,1] \mid \min\{\mathrm{INF}\,\widetilde{\kappa}(u), \mathrm{INF}\,\widetilde{\kappa}(v)\} \leqslant m \leqslant \min\{\mathrm{SUP}\,\widetilde{\kappa}(u), \mathrm{SUP}\,\widetilde{\kappa}(v)\}\}.$$

Then $\check{t} \in \mathcal{D}([0,1])$ and $u, v \in [\mathcal{U}, \widetilde{\kappa}, \check{t}]$. By the assumption (8), we get $u \circledast v \in [\mathcal{U}, \widetilde{\kappa}, \check{t}]$. Thus

$$F^{\widetilde{\kappa}}(u \circledast v) = \operatorname{SUP} \widetilde{\kappa}(u \circledast v) \geqslant t^{+} = \min\{\operatorname{SUP} \widetilde{\kappa}(u), \operatorname{SUP} \widetilde{\kappa}(v)\} = \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\},$$
$$F_{\widetilde{\kappa}}(u \circledast v) = \operatorname{INF} \widetilde{\kappa}(u \circledast v) \geqslant t^{-} = \min\{\operatorname{INF} \widetilde{\kappa}(u), \operatorname{INF} \widetilde{\kappa}(v)\} = \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\}.$$

Hence $F^{\widetilde{\kappa}}$ and $F_{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$.

(2)⇒(1). Let $\Theta \in \wp([0,1])$ and $u, v \in [\mathcal{U}, \widetilde{\kappa}, \Theta]$. Then $\operatorname{SUP} \widetilde{\kappa}(u) \geqslant \operatorname{SUP} \Theta$, $\operatorname{SUP} \widetilde{\kappa}(v) \geqslant \operatorname{SUP} \Theta$, $\operatorname{INF} \widetilde{\kappa}(u) \geqslant \operatorname{INF} \Theta$ and $\operatorname{INF} \widetilde{\kappa}(v) \geqslant \operatorname{INF} \Theta$. By the assumption (2), we have

$$\operatorname{SUP} \widetilde{\kappa}(u \circledast v) = F^{\widetilde{\kappa}}(u \circledast v) \geqslant \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\} = \min\{\operatorname{SUP} \widetilde{\kappa}(u), \operatorname{SUP} \widetilde{\kappa}(v)\} \geqslant \operatorname{SUP} \Theta,$$
$$\operatorname{INF} \widetilde{\kappa}(u \circledast v) = F_{\widetilde{\kappa}}(u \circledast v) \geqslant \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\} = \min\{\operatorname{INF} \widetilde{\kappa}(u), \operatorname{INF} \widetilde{\kappa}(v)\} \geqslant \operatorname{INF} \Theta.$$

Hence $[\mathcal{U}, \widetilde{\kappa}, \Theta]$ is a SA of $\mathcal{U}$.

(3)⇒(2). By the assumption (3), we have

$$-F^{\widetilde{\kappa}}(u \circledast v) \leqslant \max\{-F^{\widetilde{\kappa}}(u), -F^{\widetilde{\kappa}}(v)\} \quad \text{and} \quad -F_{\widetilde{\kappa}}(u \circledast v) \leqslant \max\{-F_{\widetilde{\kappa}}(u), -F_{\widetilde{\kappa}}(v)\}$$

for all $u, v \in \mathcal{U}$. Then, for all $u, v \in \mathcal{U}$, we get

$$F^{\widetilde{\kappa}}(u \circledast v) = -(-F^{\widetilde{\kappa}}(u \circledast v)) \geqslant -(\max\{-F^{\widetilde{\kappa}}(u), -F^{\widetilde{\kappa}}(v)\}) = \min\{-(-F^{\widetilde{\kappa}}(u)), -(-F^{\widetilde{\kappa}}(v))\} = \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\},$$
$$F_{\widetilde{\kappa}}(u \circledast v) = -(-F_{\widetilde{\kappa}}(u \circledast v)) \geqslant -(\max\{-F_{\widetilde{\kappa}}(u), -F_{\widetilde{\kappa}}(v)\}) = \min\{-(-F_{\widetilde{\kappa}}(u)), -(-F_{\widetilde{\kappa}}(v))\} = \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\}.$$

Hence $F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$.

(2)⇒(6). By the assumption (2), we have $F^{\widetilde{\kappa}}(u \circledast v) \geqslant \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\}$ and $F_{\widetilde{\kappa}}(u \circledast v) \geqslant \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\}$ for all $u, v \in \mathcal{U}$. Then

$$F^{\widetilde{\tau}}(u \circledast v) = 1 - F^{\widetilde{\kappa}}(u \circledast v) \leqslant 1 - \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\} = \max\{1 - F^{\widetilde{\kappa}}(u), 1 - F^{\widetilde{\kappa}}(v)\} = \max\{F^{\widetilde{\tau}}(u), F^{\widetilde{\tau}}(v)\},$$
$$F_{\widetilde{\varepsilon}}(u \circledast v) = 1 - F_{\widetilde{\kappa}}(u \circledast v) \leqslant 1 - \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\} = \max\{1 - F_{\widetilde{\kappa}}(u), 1 - F_{\widetilde{\kappa}}(v)\} = \max\{F_{\widetilde{\varepsilon}}(u), F_{\widetilde{\varepsilon}}(v)\},$$

for all $\widetilde{\tau} \in SC(\widetilde{\kappa})$, $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$ and $u, v \in \mathcal{U}$. Hence $F_{\widetilde{\varepsilon}}$ and $F^{\widetilde{\tau}}$ are AFSAs of $\mathcal{U}$ for all $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$ and $\widetilde{\tau} \in SC(\widetilde{\kappa})$.

(6)⇒(7). By the assumption (6), we have $F^{\widetilde{\tau}}(u \circledast v) \leqslant \max\{F^{\widetilde{\tau}}(u), F^{\widetilde{\tau}}(v)\}$ and $F_{\widetilde{\varepsilon}}(u \circledast v) \leqslant \max\{F_{\widetilde{\varepsilon}}(u), F_{\widetilde{\varepsilon}}(v)\}$ for all $\widetilde{\tau} \in SC(\widetilde{\kappa})$, $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$ and $u, v \in \mathcal{U}$. Then

$$\begin{aligned}
(F^{\widetilde{\tau}} - 1)(u \circledast v) &= F^{\widetilde{\tau}}(u \circledast v) - 1 \\
&\leqslant \max\{F^{\widetilde{\tau}}(u), F^{\widetilde{\tau}}(v)\} - 1 = \max\{F^{\widetilde{\tau}}(u) - 1, F^{\widetilde{\tau}}(v) - 1\} = \max\{(F^{\widetilde{\tau}} - 1)(u), (F^{\widetilde{\tau}} - 1)(v)\}, \\
(F_{\widetilde{\varepsilon}} - 1)(u \circledast v) &= F_{\widetilde{\varepsilon}}(u \circledast v) - 1 \\
&\leqslant \max\{F_{\widetilde{\varepsilon}}(u), F_{\widetilde{\varepsilon}}(v)\} - 1 = \max\{F_{\widetilde{\varepsilon}}(u) - 1, F_{\widetilde{\varepsilon}}(v) - 1\} = \max\{(F_{\widetilde{\varepsilon}} - 1)(u), (F_{\widetilde{\varepsilon}} - 1)(v)\}
\end{aligned}$$

for all $\widetilde{\tau} \in SC(\widetilde{\kappa})$, $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$ and $u, v \in \mathcal{U}$. Thus $F_{\widetilde{\varepsilon}} - 1$ and $F^{\widetilde{\tau}} - 1$ are NFSAs of $\mathcal{U}$ for all $\widetilde{\tau} \in SC(\widetilde{\kappa})$ and $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$.

(4)⇒(5). It is similar to prove (6)⇒(7) and we omit the details. □

Example 3.3. *Let $\mathcal{U} = \{0, s, t, u, v\}$ be a BCI-algebra (see [9, 22]) with the following Cayley table:*

⊛	0	s	t	u	v
0	0	0	t	u	v
s	s	0	t	u	v
t	t	t	0	v	u
u	u	u	v	0	t
v	v	v	u	t	0

We define a HFS $\widetilde{\kappa}$ on $\mathcal{U}$ by $\widetilde{\kappa}(0) = \{0.6, 0.9\}, \widetilde{\kappa}(s) = (0.6, 0.7), \widetilde{\kappa}(t) = [0.5, 0.7], \widetilde{\kappa}(u) = \{0\}, \widetilde{\kappa}(v) = \emptyset$. Then $F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$ and by Lemma 3.2, we obtain that $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$. However, we see that $\widetilde{\kappa}$ is not a HFSA of $\mathcal{U}$ because $\widetilde{\kappa}(t) \cap \widetilde{\kappa}(t) = [0.5, 0.7] \nsubseteq \{0.6, 0.9\} = \widetilde{\kappa}(t \circledast t)$, and $\widetilde{\kappa}$ is not an IvFSA of $\mathcal{U}$ because it is not an IvFS.

Example 3.4. *Let $\mathcal{U} = \{0, t, u, v\}$ be a BCK-algebra (see [22, 29]) with the following Cayley table:*

$\circledast$	0	t	u	v
0	0	0	0	0
t	t	0	t	0
u	u	u	0	0
v	v	u	t	0

Define a HFS $\widetilde{\kappa}$ on $\mathcal{U}$ by $\widetilde{\kappa}(0) = [0.7, 1], \widetilde{\kappa}(t) = (0.6, 0.7] \cup \{0.9\}, \widetilde{\kappa}(u) = \{0.4, 0.6, 0.8\}$ and $\widetilde{\kappa}(v) = [0, 0.7]$. It is routine to verify that $F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$. By using Lemma 3.2, we obtain that $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$. However, we see that $\widetilde{\kappa}$ is not a HFSA of $\mathcal{U}$ because $\widetilde{\kappa}(v) \cap \widetilde{\kappa}(t) = (0.6, 0.7] \nsubseteq \{0.4, 0.6, 0.8\} = \widetilde{\kappa}(v \circledast t)$, and $\widetilde{\kappa}$ is not an IvFSA of $\mathcal{U}$ because it is not an IvFS.

Proposition 3.5. *Every IvFSA of $\mathcal{U}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*

Proof. It follows from Remark 2.1 and Lemma 3.2. □

The converse of the above proposition is not true, generally, as we see in Example 3.4. Then by Example 3.4 and Proposition 3.5, we see that an (inf, sup)-HFSA of a BCK/BCI-algebra $\mathcal{U}$ is a general concept of an IvFSA of $\mathcal{U}$.

Theorem 3.6. *The following are equivalent for an IvFS $\check{\pi}$ on $\mathcal{U}$.*

(1) *$\check{\pi}$ is an IvFSA of $\mathcal{U}$.*
(2) *$\check{\pi}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(3) *$[\mathcal{U}, \check{\pi}, \check{t}]$ is a SA of $\mathcal{U}$ for each $\check{t} \in \mathcal{D}([0, 1])$ such that $[\mathcal{U}, \check{\pi}, \check{t}] \neq \emptyset$.*

Proof. It follows from Remark 2.1, Lemma 3.2, and Proposition 3.5. □

Theorem 3.7. *The following are equivalent for a HFS $\widetilde{\kappa}$ on $\mathcal{U}$.*

(1) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(2) *If $\check{\pi}$ is an IvFS on $\mathcal{U}$ such that $\check{\pi}^- = F_{\widetilde{\kappa}}$ and $\check{\pi}^+ = F^{\widetilde{\kappa}}$, then $\check{\pi}$ is an IvFSA of $\mathcal{U}$.*
(3) *$\widetilde{\varepsilon}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each HFS $\widetilde{\varepsilon}$ on $\mathcal{U}$ such that $F_{\widetilde{\varepsilon}} = F_{\widetilde{\kappa}}$ and $F^{\widetilde{\varepsilon}} = F^{\widetilde{\kappa}}$.*

Proof. It follows from Lemma 3.2 and Theorem 3.6. □

For each HFS $\widetilde{\kappa}$ on $\mathcal{V}$ and element Θ of $\wp([0, 1])$, define the HFS $\mathcal{H}^{\widetilde{\kappa}}_{\Theta}$ on $\mathcal{V}$ as follows:

$$\mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u) = \{m \in \Theta \mid \tfrac{1+F^{\widetilde{\kappa}}}{2}(u) \geqslant m \geqslant \tfrac{F_{\widetilde{\kappa}^{\pm}}}{2}(u)\} \text{ for all } u \in \mathcal{V}.$$

We denote $\mathcal{H}^{\widetilde{\kappa}}$ for $\mathcal{H}^{\widetilde{\kappa}}_{[0,1]}$. Then $\mathcal{H}^{\widetilde{\kappa}}(u) \neq \emptyset$ and $\mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u) \subseteq \mathcal{H}^{\widetilde{\kappa}}_{\Psi}(u) \subseteq \mathcal{H}^{\widetilde{\kappa}}(u)$ when $u \in \mathcal{V}$ and $\Theta \subseteq \Psi \subseteq [0, 1]$.

Theorem 3.8. *The following are equivalent for a HFS $\widetilde{\kappa}$ on $\mathcal{U}$.*

(1) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(2) *$\mathcal{H}^{\widetilde{\kappa}}$ is a HFSA of $\mathcal{U}$.*
(3) *$\mathcal{H}^{\widetilde{\kappa}}_{\Theta}$ is a HFSA of $\mathcal{U}$ for each $\Theta \in \wp([0, 1])$.*

Proof.

(1)⇒(3). Let $u, v \in \mathcal{U}$, $\Theta \in \wp([0, 1])$ and $m \in \mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u) \cap \mathcal{H}^{\widetilde{\kappa}}_{\Theta}(v)$. Then $m \in \Theta$, $\frac{1+F^{\widetilde{\kappa}}}{2}(u) \geqslant m \geqslant \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u)$ and $\frac{1+F^{\widetilde{\kappa}}}{2}(v) \geqslant m \geqslant \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)$. By the assumption (1) and Lemma 3.2, we get $F^{\widetilde{\kappa}}(u \circledast v) \geqslant \min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\}$ and $F_{\widetilde{\kappa}^{\pm}}(u \circledast v) \leqslant \max\{F_{\widetilde{\kappa}^{\pm}}(u), F_{\widetilde{\kappa}^{\pm}}(v)\}$. Thus

$$\frac{1+F^{\widetilde{\kappa}}}{2}(u \circledast v) \geqslant \min\{\frac{1+F^{\widetilde{\kappa}}}{2}(u), \frac{1+F^{\widetilde{\kappa}}}{2}(v)\} \geqslant m \geqslant \max\{\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u), \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)\} \geqslant \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u \circledast v),$$

which implies that $m \in \mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u \circledast v)$. Hence $\mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u) \cap \mathcal{H}^{\widetilde{\kappa}}_{\Theta}(v) \subseteq \mathcal{H}^{\widetilde{\kappa}}_{\Theta}(u \circledast v)$. Therefore, we conclude that $\mathcal{H}^{\widetilde{\kappa}}_{\Theta}$ is a HFSA of $\mathcal{U}$ for each $\Theta \in \wp([0, 1])$.

(3)⇒(2). It is clear.

(2)⇒(1). Let $u, v \in \mathcal{U}$. Then $\max\{\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u), \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)\}, \min\{\frac{1+F^{\widetilde{\kappa}}}{2}(u), \frac{1+F^{\widetilde{\kappa}}}{2}(v)\} \in \mathcal{H}^{\widetilde{\kappa}}(u) \cap \mathcal{H}^{\widetilde{\kappa}}(v)$.

By the assumption (2), we have $\max\{\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u), \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)\}, \min\{\frac{1+F^{\widetilde{\kappa}}}{2}(u), \frac{1+F^{\widetilde{\kappa}}}{2}(v)\} \in \mathcal{H}^{\widetilde{\kappa}}(u \circledast v)$. Thus $\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u \circledast v) \leqslant \max\{\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u), \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)\}$ and $\frac{1+F^{\widetilde{\kappa}}}{2}(u \circledast v) \geqslant \min\{\frac{1+F^{\widetilde{\kappa}}}{2}(u), \frac{1+F^{\widetilde{\kappa}}}{2}(v)\}$. Since $F^{\widetilde{\kappa}} = 2(\frac{1+F^{\widetilde{\kappa}}}{2}) - 1$ and $F_{\widetilde{\kappa}} = 1 - 2(\frac{F_{\widetilde{\kappa}^{\pm}}}{2})$, we obtain that

$$\begin{aligned}\min\{F^{\widetilde{\kappa}}(u), F^{\widetilde{\kappa}}(v)\} &= \min\{2(\frac{1+F^{\widetilde{\kappa}}}{2}(u)) - 1, 2(\frac{1+F^{\widetilde{\kappa}}}{2}(v)) - 1\} \\ &= 2(\min\{\frac{1+F^{\widetilde{\kappa}}}{2}(u), \frac{1+F^{\widetilde{\kappa}}}{2}(v)\}) - 1 \leqslant 2(\frac{1+F^{\widetilde{\kappa}}}{2}(u \circledast v)) - 1 = F^{\widetilde{\kappa}}(u \circledast v),\end{aligned}$$

and

$$\begin{aligned}\min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\} &= \min\{1 - 2(\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u)), 1 - 2(\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v))\} \\ &= 1 - 2(\max\{\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u), \frac{F_{\widetilde{\kappa}^{\pm}}}{2}(v)\}) \leqslant 1 - 2(\frac{F_{\widetilde{\kappa}^{\pm}}}{2}(u \circledast v)) = F_{\widetilde{\kappa}}(u \circledast v).\end{aligned}$$

Hence $F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$. By Lemma 3.2, we conclude that $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$. □

Lemma 3.9. *An FS δ in $\mathcal{U}$ is an AFSA of $\mathcal{U}$ if and only if $1 - \delta$ is an FSA of $\mathcal{U}$.*

Proof. Let δ be an AFSA of $\mathcal{U}$. Then

$$1 - \delta(u \circledast v) \geqslant 1 - \max\{\delta(u), \delta(v)\} = \min\{1 - \delta(u), 1 - \delta(v)\}$$

for all $u, v \in \mathcal{U}$. Hence $1 - \delta$ is an FSA of $\mathcal{U}$.

Conversely, assume that $1 - \delta$ is an FSA of $\mathcal{U}$. Then

$$\begin{aligned}\delta(u \circledast v) = 1 - (1-\delta)(u \circledast v) &\leqslant 1 - \min\{(1-\delta)(u), (1-\delta)(v)\} \\ &= \max\{1 - (1-\delta)(u), 1 - (1-\delta)(v)\} = \max\{\delta(u), \delta(v)\}\end{aligned}$$

for all $u, v \in \mathcal{U}$. Hence δ is an AFSA of $\mathcal{U}$. □

Next, we characterize $(\inf, \sup)$-HFSAs of BCK/BCI-algebras in terms of PFSs.

Theorem 3.10. *The following are equivalent for a HFS $\widetilde{\kappa}$ on $\mathcal{U}$.*

(1) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(2) *$(F_{\widetilde{\kappa}}, F^{\widetilde{\kappa}^{\mp}})$ is a PFSA of $\mathcal{U}$.*
(3) *$(F_{\widetilde{\kappa}}, F^{\widetilde{\tau}})$ is a PFSA of $\mathcal{U}$ for each $\widetilde{\tau} \in SC(\widetilde{\kappa})$.*
(4) *$(\frac{i+F^{\widetilde{\kappa}}}{1+2j}, \frac{i+F_{\widetilde{\kappa}^{\pm}}}{1+2j})$ is a PFSA of $\mathcal{U}$ for each positive integers i and j such that $i \leqslant j$.*
(5) *$(\frac{i+F^{\widetilde{\kappa}}}{1+2j}, \frac{i+F_{\widetilde{\varepsilon}}}{1+2j})$ is a PFSA of $\mathcal{U}$ for each $\widetilde{\varepsilon} \in IC(\widetilde{\kappa})$ and positive integers i and j such that $i \leqslant j$.*

Proof.

(1)⇒(3). The proof follows from Lemma 3.2.

(3)⇒(2) and (5)⇒(4). The proofs are clear.

(2)⇒(1). Assume that (2) holds. Then $F^{\widetilde{\kappa}^{\mp}}$ is an AFSA of $\mathcal{U}$ and $F_{\widetilde{\kappa}}$ is an FSA of $\mathcal{U}$. By Lemma 3.9, we obtain that $F^{\widetilde{\kappa}} = 1 - F^{\widetilde{\kappa}^{\mp}}$ is an FSA of $\mathcal{U}$. Thus $F_{\widetilde{\kappa}}$ and $F^{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$. Therefore, it follows from Lemma 3.2 that $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.

(1)⇒(5). Assume that (1) holds. Let i and j be positive integers such that $i \leqslant j$ and $\tilde{\varepsilon} \in IC(\tilde{\kappa})$. By Lemma 3.2, we have $F^{\tilde{\kappa}}$ is an FSA and $F_{\tilde{\varepsilon}}$ is an AFSA of $\mathcal{U}$. Then

$$\frac{i+F^{\tilde{\kappa}}(u \circledast v)}{1+2j} \geqslant \frac{i+\min\{F^{\tilde{\kappa}}(u), F^{\tilde{\kappa}}(v)\}}{1+2j} = \min\{\frac{i+F^{\tilde{\kappa}}(u)}{1+2j}, \frac{i+F^{\tilde{\kappa}}(v)}{1+2j}\},$$
$$\frac{i+F_{\tilde{\varepsilon}}(u \circledast v)}{1+2j} \leqslant \frac{i+\max\{F_{\tilde{\varepsilon}}(u), F_{\tilde{\varepsilon}}(v)\}}{1+2j} = \max\{\frac{i+F_{\tilde{\varepsilon}}(u)}{1+2j}, \frac{i+F_{\tilde{\varepsilon}}(v)}{1+2j}\}$$

for all $u, v \in \mathcal{U}$. Thus $\frac{i+F^{\tilde{\kappa}}}{1+2j}$ is an FSA and $\frac{i+F_{\tilde{\varepsilon}}}{1+2j}$ is an AFSA of $\mathcal{U}$. Hence the PFS $(\frac{i+F^{\tilde{\kappa}}}{1+2j}, \frac{i+F_{\tilde{\varepsilon}}}{1+2j})$ is a PFSA of $\mathcal{U}$.

(4)⇒(1) . Assume that (4) holds. We choose $i = 1 = j$ and then the $(\frac{1+F^{\tilde{\kappa}}}{3}, \frac{1+F_{\tilde{\kappa}^{\pm}}}{3})$ is a PFSA of $\mathcal{U}$. Thus $\frac{1+F^{\tilde{\kappa}}}{3}$ is an FSA and $\frac{1+F_{\tilde{\kappa}^{\pm}}}{3}$ is an AFSA of $\mathcal{U}$. Since $F^{\tilde{\kappa}} = 3(\frac{1+F^{\tilde{\kappa}}}{3}) - 1$ and $F_{\tilde{\kappa}^{\pm}} = 3(\frac{1+F_{\tilde{\kappa}^{\pm}}}{3}) - 1$, we get

$$\begin{aligned} F^{\tilde{\kappa}}(u \circledast v) = 3(\frac{1+F^{\tilde{\kappa}}(u \circledast v)}{3}) - 1 &\geqslant 3(\frac{1+\min\{F^{\tilde{\kappa}}(u), F^{\tilde{\kappa}}(v)\}}{3}) - 1 \\ &= \min\{3(\frac{1+F^{\tilde{\kappa}}(u)}{3}) - 1, 3(\frac{1+F^{\tilde{\kappa}}(v)}{3}) - 1\} = \min\{F^{\tilde{\kappa}}(u), F^{\tilde{\kappa}}(v)\}, \\ F_{\tilde{\kappa}^{\pm}}(u \circledast v) = 3(\frac{1+F_{\tilde{\kappa}^{\pm}}(u \circledast v)}{3}) - 1 &\leqslant 3(\frac{1+\max\{F_{\tilde{\kappa}^{\pm}}(u), F_{\tilde{\kappa}^{\pm}}(v)\}}{3}) - 1 \\ &= \max\{3(\frac{1+F_{\tilde{\kappa}^{\pm}}(u)}{3}) - 1, 3(\frac{1+F_{\tilde{\kappa}^{\pm}}(v)}{3}) - 1\} = \max\{F_{\tilde{\kappa}^{\pm}}(u), F_{\tilde{\kappa}^{\pm}}(v)\} \end{aligned}$$

for all $u, v \in \mathcal{U}$, which imply that $F^{\tilde{\kappa}}$ is an FSA and $F_{\tilde{\kappa}^{\pm}}$ is an AFSA of $\mathcal{U}$. By Lemma 3.9, we have $F_{\tilde{\kappa}} = 1 - F_{\tilde{\kappa}^{\pm}}$ is an FSA of $\mathcal{U}$. It follows from Lemma 3.2 that $\tilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$. □

Lemma 3.11. *A NFS δ in $\mathcal{U}$ is an NFSA of $\mathcal{U}$ if and only if $-\delta$ is an FSA of $\mathcal{U}$.*

Proof. Let δ be an NFSA of $\mathcal{U}$. Then $\delta(u \circledast v) \leqslant \max\{\delta(u), \delta(v)\}$ for all $u, v \in \mathcal{U}$. Thus

$$-\delta(u \circledast v) \geqslant -(\max\{\delta(u), \delta(v)\}) = \min\{-\delta(u), -\delta(v)\}$$

for all $u, v \in \mathcal{U}$. Hence $-\delta$ is an FSA of $\mathcal{U}$.

Conversely, assume that $-\delta$ is an FSA of $\mathcal{U}$. Then, for all $u, v \in \mathcal{U}$, we have

$$\delta(u \circledast v) = -(-\delta(u \circledast v)) \leqslant -(\min\{-\delta(u), -\delta(v)\}) = \max\{-(-\delta(u)), -(-\delta(v))\} = \max\{\delta(u), \delta(v)\}.$$

Hence δ is an NFSA of $\mathcal{U}$. □

In the following theorem, we characterize $(\inf, \sup)$-HFSAs of BCK/BCI-algebras in terms of BFSs.

Theorem 3.12. *The following are equivalent for a HFS $\tilde{\kappa}$ on $\mathcal{U}$.*

(1) *$\tilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(2) *$\langle -F_{\tilde{\kappa}}, F^{\tilde{\kappa}} \rangle$ is a BFSA of $\mathcal{U}$.*
(3) *$\langle -F^{\tilde{\kappa}}, F_{\tilde{\kappa}} \rangle$ is a BFSA of $\mathcal{U}$.*
(4) *$\langle F_{\tilde{\varepsilon}} - 1, F^{\tilde{\kappa}} \rangle$ is a BFSA of $\mathcal{U}$ for each $\tilde{\varepsilon} \in IC(\tilde{\kappa})$.*
(5) *$\langle F^{\tilde{\tau}} - 1, F_{\tilde{\kappa}} \rangle$ is a BFSA of $\mathcal{U}$ for each $\tilde{\tau} \in SC(\tilde{\kappa})$.*

Proof.

(1)⇒(4) and (1)⇒(5). The proofs follow from Lemma 3.2.

(4)⇒(2) and (5)⇒(3). The proofs are obvious.

(2)⇒(1). Assume that (2) holds. Then $-F_{\tilde{\kappa}}$ is an NFSA of $\mathcal{U}$ and $F^{\tilde{\kappa}}$ is an FSA of $\mathcal{U}$. By Lemma 3.11, we get $F_{\tilde{\kappa}} = -(-F_{\tilde{\kappa}})$ is an FSA of $\mathcal{U}$. Thus $F_{\tilde{\kappa}}$ and $F^{\tilde{\kappa}}$ are FSAs of $\mathcal{U}$. Therefore, it follows from Lemma 3.2 that $\tilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.

(3)⇒(1). It is similar to prove that (2)⇒(1) and we omit the details. □

4. Characterizing SAs, FSAs, AFSAs, NFSAs, PFSAs, and BFSAs by $(\inf, \sup)$-HFSAs and IvFSAs

In this section, we characterize SAs, FSAs, AFSAs, NFSAs, PFSAs, and BFSAs of BCK/BCI-algebras by $(\inf, \sup)$-HFSAs and IvFSAs.

For any subset $\mathcal{A}$ of $\mathcal{V}$ and $\Theta, \Psi \in \wp([0,1])$, define a map $\mathcal{C}(\mathcal{A}, \Theta, \Psi)$ [18, 19] as follows:

$$\mathcal{C}(\mathcal{A}, \Theta, \Psi) \colon \mathcal{V} \to \wp([0,1]), u \mapsto \begin{cases} \Psi, & \text{if } u \in \mathcal{A}, \\ \Theta, & \text{otherwise.} \end{cases}$$

We denote $\mathcal{CI}(\mathcal{A})$ for $\mathcal{C}(\mathcal{A}, [0,0], [1,1])$ and is called the **characteristic interval-valued fuzzy set** of $\mathcal{A}$ on $\mathcal{U}$, and denote $\mathcal{CH}(\mathcal{A})$ for $\mathcal{C}(\mathcal{A}, \emptyset, [0,1])$ and is called the **characteristic hesitant fuzzy set** of $\mathcal{A}$ on $\mathcal{U}$. In the following theorem, we give the equivalent condition that a nonempty subset of a BCK/BCI-algebra is a SA in terms of $(\inf, \sup)$-HFSAs.

Theorem 4.1. *Let $\mathcal{A}$ be a nonempty subset of $\mathcal{U}$ and $\Theta, \Psi \in \wp([0,1])$ such that $\mathrm{INF}\,\Theta \leqslant \mathrm{INF}\,\Psi$, $\mathrm{SUP}\,\Theta < \mathrm{SUP}\,\Psi$ or $\mathrm{INF}\,\Theta < \mathrm{INF}\,\Psi$, $\mathrm{SUP}\,\Theta \leqslant \mathrm{SUP}\,\Psi$. Then $\mathcal{A}$ is a SA of $\mathcal{U}$ if and only if $\mathcal{C}(\mathcal{A}, \Theta, \Psi)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*

Proof.

(⇒). Let $\mathcal{A}$ be a SA of $\mathcal{U}$ and $u, v \in \mathcal{U}$. If $u \notin \mathcal{A}$ or $v \notin \mathcal{A}$, then

$$F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) \geqslant \mathrm{SUP}\,\Theta = \min\{F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\},$$
$$F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) \geqslant \mathrm{INF}\,\Theta = \min\{F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\}.$$

On the other hand, let $u, v \in \mathcal{A}$. Since $\mathcal{A}$ is a SA of $\mathcal{U}$, we get $u \circledast v \in \mathcal{A}$. Then

$$F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) = \mathrm{SUP}\,\Psi = \min\{F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\},$$
$$F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) = \mathrm{INF}\,\Psi = \min\{F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\}.$$

Therefore, we conclude that $F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}$ and $F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}$ are FSAs of $\mathcal{U}$. It follows from Lemma 3.2 that $\mathcal{C}(\mathcal{A}, \Theta, \Psi)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.

(⇐). Let $u, v \in \mathcal{A}$. Then $\mathcal{C}(\mathcal{A}, \Theta, \Psi)(u) = \Psi = \mathcal{C}(\mathcal{A}, \Theta, \Psi)(v)$. In the case where $\mathrm{INF}\,\Theta \leqslant \mathrm{INF}\,\Psi$ and $\mathrm{SUP}\,\Theta < \mathrm{SUP}\,\Psi$, then by using Lemma 3.2, we get

$$F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) \geqslant \min\{F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F^{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\} = \mathrm{SUP}\,\Psi > \mathrm{SUP}\,\Theta.$$

Thus $\mathcal{C}(\mathcal{A}, \Theta, \Psi)(u \circledast v) = \Psi$ which implies that $u \circledast v \in \mathcal{A}$. In the case where $\mathrm{INF}\,\Theta < \mathrm{INF}\,\Psi$ and $\mathrm{SUP}\,\Theta \leqslant \mathrm{SUP}\,\Psi$, then by using Lemma 3.2, we have

$$F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u \circledast v) \geqslant \min\{F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(u), F_{\mathcal{C}(\mathcal{A},\Theta,\Psi)}(v)\} = \mathrm{INF}\,\Psi > \mathrm{INF}\,\Theta.$$

Thus $\mathcal{C}(\mathcal{A}, \Theta, \Psi)(u \circledast v) = \Psi$ which implies that $u \circledast v \in \mathcal{A}$. Hence $\mathcal{A}$ is a SA of $\mathcal{U}$. □

Theorem 4.2. *The following are equivalent for a nonempty subset $\mathcal{A}$ of $\mathcal{U}$.*

(1) *$\mathcal{A}$ is a SA of $\mathcal{U}$.*
(2) *$\mathcal{CI}(\mathcal{A})$ is an IvFSA of $\mathcal{U}$.*
(3) *$\mathcal{CI}(\mathcal{A})$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(4) *$\mathcal{CH}(\mathcal{A})$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.*
(5) *$\mathcal{C}(\mathcal{A}, \check{s}, \check{t})$ is an IvFSA of $\mathcal{U}$ when $\check{s}, \check{t} \in \mathcal{D}([0,1])$ and $\check{s} \prec \check{t}$.*

Proof. It follows from Theorems 3.6 and 4.1. □

For each positive integer i and FS δ in $\mathcal{V}$, we define the HFS $\mathcal{H}(\delta, i)$ and the IvFS $\mathcal{I}(\delta, i)$ on $\mathcal{V}$ by:

$$\mathcal{H}(\delta, i) : \mathcal{V} \to \wp([0,1]), u \mapsto \{\frac{\delta}{1+i}(u), \frac{i+\delta}{1+i}(u)\}$$

and

$$\mathcal{I}(\delta, i) : \mathcal{V} \to \mathcal{D}([0,1]), u \mapsto \{m \in [0,1] \mid \frac{\delta}{1+i}(u) \leqslant m \leqslant \frac{i+\delta}{1+i}(u)\}.$$

Then the following are true.

(1) $\mathcal{H}(\delta, i)(u) \subseteq \mathcal{I}(\delta, i)(u)$ for all $u \in \mathcal{V}$.
(2) $F^{\mathcal{H}(\delta,i)} = \frac{i+\delta}{1+i} = F^{\mathcal{I}(\delta,i)}$ and $F_{\mathcal{H}(\delta,i)} = \frac{\delta}{1+i} = F_{\mathcal{I}(\delta,i)}$.
(3) If δ is a NFS in $\mathcal{V}$, then $\mathcal{H}(-\delta, i)$ is a HFS on $\mathcal{V}$ and $\mathcal{I}(-\delta, i)$ is an IvFS on $\mathcal{V}$.

Next, we characterize FSAs (in Theorem 4.3), AFSAs (in Theorem 4.4) and NFSAs (in Theorem 4.5) of $\mathcal{U}$ in terms of $(\inf, \sup)$-HFSAs and IvFSAs of $\mathcal{U}$.

Theorem 4.3. *The following are equivalent for an FS δ in $\mathcal{U}$.*

(1) *δ is an FSA of $\mathcal{U}$.*
(2) *$\mathcal{I}(\delta, i)$ is an IvFSA of $\mathcal{U}$ for each positive integer i.*
(3) *$\mathcal{I}(\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(4) *$\mathcal{H}(\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(5) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each HFS $\widetilde{\kappa}$ on $\mathcal{U}$ and positive integer i such that $F_{\widetilde{\kappa}} = \frac{\delta}{1+i}$ and $F^{\widetilde{\kappa}} = \frac{i+\delta}{1+i}$.*

Proof. By Theorems 3.6 and 3.7, the conditions (2)-(5) are equivalent. We will prove that the conditions (1) and (5) are equivalent.

Assume that (1) is true. Let i be a positive integer and $\widetilde{\kappa}$ be a HFS on $\mathcal{U}$ such that $F_{\widetilde{\kappa}} = \frac{\delta}{1+i}$ and $F^{\widetilde{\kappa}} = \frac{i+\delta}{1+i}$. Then $\delta(u \circledast v) \geqslant \min\{\delta(u), \delta(v)\}$ for all $u, v \in \mathcal{U}$. Thus

$$F_{\widetilde{\kappa}}(u \circledast v) = \frac{\delta(u \circledast v)}{1+i} \geqslant \frac{\min\{\delta(u), \delta(v)\}}{1+i} = \min\{\frac{\delta(u)}{1+i}, \frac{\delta(v)}{1+i}\} = \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\},$$
$$F^{\widetilde{\kappa}}(u \circledast v) = \frac{i+\delta(u \circledast v)}{1+i} \geqslant \frac{i+\min\{\delta(u), \delta(v)\}}{1+i} = \min\{\frac{i+\delta(u)}{1+i}, \frac{i+\delta(v)}{1+i}\} = \min\{F_{\widetilde{\kappa}}(u), F_{\widetilde{\kappa}}(v)\}$$

for all $u, v \in \mathcal{U}$, which imply that $F^{\widetilde{\kappa}}$ and $F_{\widetilde{\kappa}}$ are FSAs of $\mathcal{U}$. It follows from Lemma 3.2 that $\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$.

Conversely, assume that (5) is true. Let i be a positive integer and $\widetilde{\kappa}$ be a HFS on $\mathcal{U}$ such that $F_{\widetilde{\kappa}} = \frac{\delta}{1+i}$ and $F^{\widetilde{\kappa}} = \frac{i+\delta}{1+i}$. By Lemma 3.2, we have that $F_{\widetilde{\kappa}} = \frac{\delta}{1+i}$ is an FSA of $\mathcal{U}$. Since $\delta = (1+i)(\frac{\delta}{1+i})$, we have

$$\delta(u \circledast v) = (1+i)(\frac{\delta}{1+i}(u \circledast v)) \geqslant (1+i)(\min\{\frac{\delta}{1+i}(u), \frac{\delta}{1+i}(v)\}) = \min\{\delta(u), \delta(v)\}$$

for all $u, v \in \mathcal{U}$. Hence δ is a SA of $\mathcal{U}$. □

Theorem 4.4. *The following are equivalent for a FS δ in $\mathcal{U}$.*

(1) *δ is an AFSA of $\mathcal{U}$.*
(2) *$\mathcal{I}(1-\delta, i)$ is an IvFSA of $\mathcal{U}$ for each positive integer i.*
(3) *$\mathcal{I}(1-\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(4) *$\mathcal{H}(1-\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(5) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each HFS $\widetilde{\kappa}$ on $\mathcal{U}$ and positive integer i such that $F_{\widetilde{\kappa}} = \frac{1-\delta}{1+i}$ and $F^{\widetilde{\kappa}} = 1 + \frac{-\delta}{1+i}$.*

Proof. It follows from Lemma 3.9 and Theorem 4.3. □

Theorem 4.5. *The following are equivalent for a NFS δ in $\mathcal{U}$.*

(1) *δ is an NFSA of $\mathcal{U}$.*
(2) *$\mathcal{I}(-\delta, i)$ is an IvFSA of $\mathcal{U}$ for each positive integer i.*
(3) *$\mathcal{I}(-\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(4) *$\mathcal{H}(-\delta, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(5) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each HFS $\widetilde{\kappa}$ on $\mathcal{U}$ and positive integer i such that $F_{\widetilde{\kappa}} = \frac{-\delta}{1+i}$ and $F^{\widetilde{\kappa}} = \frac{i-\delta}{1+i}$.*

Proof. It follows from Lemma 3.11 and Theorem 4.3. □

For each positive integer i and PFS $P = (\delta, \eta)$ in $\mathcal{V}$, define the HFS $\mathcal{H}(P, i)$ and the IvFS $\mathcal{I}(P, i)$ on $\mathcal{V}$ as follows:

$$\mathcal{H}(P,i)(u) = \{\frac{\delta}{1+i}(u), 1 - \frac{\eta}{1+i}(u)\}, \qquad \mathcal{I}(P,i)(u) = \{m \in [0,1] \mid \frac{\delta}{1+i}(u) \leqslant m \leqslant 1 - \frac{\eta}{1+i}(u)\}$$

for all $u \in \mathcal{V}$. Note that $F^{\mathcal{H}(P,i)} = 1 - \frac{\eta}{1+i} = F^{\mathcal{I}(P,i)}$ and $F_{\mathcal{H}(P,i)} = \frac{\delta}{1+i} = F_{\mathcal{I}(P,i)}$. In the following theorem, we characterize PFSAs of $\mathcal{U}$ by $(\inf, \sup)$-HFSAs and IvFSAs of $\mathcal{U}$.

Theorem 4.6. *The following are equivalent for a PFS $P = (\delta, \eta)$ in $\mathcal{U}$.*

(1) *P is a PFSA of $\mathcal{U}$.*
(2) *$\mathcal{I}(P, i)$ is an IvFSA of $\mathcal{U}$ for each positive integer i.*
(3) *$\mathcal{I}(P, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(4) *$\mathcal{H}(P, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.*
(5) *$\widetilde{\kappa}$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each HFS $\widetilde{\kappa}$ on $\mathcal{U}$ and positive integer i such that $F_{\widetilde{\kappa}} = \frac{\delta}{1+i}$ and $F^{\widetilde{\kappa}} = 1 - \frac{\eta}{1+i}$.*

Proof. By Theorem 3.7, the conditions (2)-(5) are equivalent. We will prove that the conditions (1) and (4) are equivalent. Now, assume that (1) holds. Then δ is an FSA and η is an AFSA of $\mathcal{U}$. By Lemma 3.9, we get $1 - \eta$ is an FSA of $\mathcal{U}$. Thus

$$\begin{aligned}
\frac{\delta(u \circledast v)}{1+i} &\geqslant \frac{\min\{\delta(u), \delta(v)\}}{1+i} = \min\{\frac{\delta(u)}{1+i}, \frac{\delta(v)}{1+i}\}, \\
1 - \frac{\eta(u \circledast v)}{1+i} &= \frac{i + (1-\eta)(u \circledast v)}{1+i} \\
&\geqslant \frac{i + \min\{(1-\eta)(u), (1-\eta)(v)\}}{1+i} \\
&= \min\{\frac{i + (1-\eta)(u)}{1+i}, \frac{i + (1-\eta)(v)}{1+i}\} = \min\{1 - \frac{\eta(u)}{1+i}, 1 - \frac{\eta(v)}{1+i}\}
\end{aligned}$$

for each $u, v \in \mathcal{U}$ and positive integer i. Hence $F_{\mathcal{H}(P,i)} = \frac{\delta}{1+i}$ and $F^{\mathcal{H}(P,i)} = 1 - \frac{\eta}{1+i}$ are FSAs of $\mathcal{U}$ for each positive integer i. Therefore, it follows from Lemma 3.2 that $\mathcal{H}(P, i)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.

Conversely, assume that (4) holds. Then $\mathcal{H}(P, 1)$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$. By Lemma 3.2, we get $\frac{\delta}{2} = F_{\mathcal{H}(P,1)}$ and $\frac{2-\eta}{2} = F^{\mathcal{H}(P,1)}$ are FSAs of $\mathcal{U}$. Since $\delta = 2(\frac{\delta}{2})$ and $\eta = 2 - 2(\frac{2-\eta}{2})$, we have

$$\begin{aligned}
\delta(u \circledast v) &= 2(\frac{\delta(u \circledast v)}{2}) \geqslant 2(\frac{\min\{\delta(u), \delta(v)\}}{2}) = \min\{\delta(u), \delta(v)\}, \\
\eta(u \circledast v) &= 2 - 2(\frac{2-\eta}{2}(u \circledast v)) \\
&\leqslant 2 - 2(\min\{\frac{2-\eta}{2}(u), \frac{2-\eta}{2}(v)\}) = \max\{2 - 2(\frac{2-\eta}{2}(u)), 2 - 2(\frac{2-\eta}{2}(v))\} = \max\{\eta(u), \eta(v)\}
\end{aligned}$$

for each $u, v \in \mathcal{U}$. Thus δ is an FSA and η is an AFSA of $\mathcal{U}$, which imply that P is a PFSA of $\mathcal{U}$. □

For each positive integer i and BFS $B = \langle \delta, \eta \rangle$ on $\mathcal{V}$, define the HFS $\mathcal{H}\langle B, i\rangle$ and the IvFS $\mathcal{I}\langle B, i\rangle$ on $\mathcal{V}$ as follows:

$$\mathcal{H}\langle B, i\rangle(u) = \{\frac{-\delta}{1+i}(u), \frac{i+\eta}{1+i}(u)\}, \qquad \mathcal{I}\langle B, i\rangle(u) = \{m \in [0,1] \mid \frac{-\delta}{1+i}(u) \leqslant m \leqslant \frac{i+\eta}{1+i}(u)\}$$

for all $u \in \mathcal{V}$. Note that $F^{\mathcal{H}\langle B,i\rangle} = \frac{i+\eta}{1+i} = F^{\mathcal{I}\langle B,i\rangle}$ and $F_{\mathcal{H}\langle B,i\rangle} = \frac{-\delta}{1+i} = F_{\mathcal{I}\langle B,i\rangle}$. In the following theorem, characterizations of BFSAs of $\mathcal{U}$ are given in terms of $(\inf, \sup)$-HFSAs and IvFSAs of $\mathcal{U}$.

Theorem 4.7. *The following are equivalent for a BFS $B = \langle \delta, \eta \rangle$ in $\mathcal{U}$.*

(1) B *is a BFSA of* $\mathcal{U}$.
(2) $\mathcal{I}\langle B, i\rangle$ *is an IvFSA of* $\mathcal{U}$ *for each positive integer* i.
(3) $\mathcal{I}\langle B, i\rangle$ *is an* $(\inf, \sup)$*-HFSA of* $\mathcal{U}$ *for each positive integer* i.
(4) $\mathcal{H}\langle B, i\rangle$ *is an* $(\inf, \sup)$*-HFSA of* $\mathcal{U}$ *for each positive integer* i.
(5) $\widetilde{\kappa}$ *is an* $(\inf, \sup)$*-HFSA of* $\mathcal{U}$ *for each HFS* $\widetilde{\kappa}$ *on* $\mathcal{U}$ *and positive integer* i *such that* $F_{\widetilde{\kappa}} = \frac{-\delta}{1+i}$ *and* $F^{\widetilde{\kappa}} = \frac{i+\eta}{1+i}$.

Proof. By Theorem 3.7, the conditions (2)-(5) are equivalent. We will prove that the conditions (1) and (4) are equivalent. Assume that (1) holds. Then δ is an NFSA and η is an FSA of $\mathcal{U}$. By Lemma 3.11, we have $-\delta$ is an FSA of $\mathcal{U}$. Thus, for each elements $u, v \in \mathcal{U}$ and positive integer i, we get

$$F_{\mathcal{H}\langle B,i\rangle}(u \circledast v) = \frac{-\delta}{1+i}(u \circledast v) \geqslant \min\{\frac{-\delta}{1+i}(u), \frac{-\delta}{1+i}(v)\} = \min\{F_{\mathcal{H}\langle B,i\rangle}(u), F_{\mathcal{H}\langle B,i\rangle}(v)\},$$
$$F^{\mathcal{H}\langle B,i\rangle}(u \circledast v) = \frac{i+\eta}{1+i}(u \circledast v) \geqslant \min\{\frac{i+\eta}{1+i}(u), \frac{i+\eta}{1+i}(v)\} = \min\{F^{\mathcal{H}\langle B,i\rangle}(u), F^{\mathcal{H}\langle B,i\rangle}(v)\}.$$

Hence $F_{\mathcal{H}\langle B,i\rangle}$ and $F^{\mathcal{H}\langle B,i\rangle}$ are FSAs of $\mathcal{U}$ for each positive integer i. It follows from Lemma 3.2 that $\mathcal{H}\langle B, i\rangle$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ for each positive integer i.

Conversely, assume that (4) holds. Then $\mathcal{H}\langle B, 1\rangle$ is an $(\inf, \sup)$-HFSA of $\mathcal{U}$ and thus $\frac{\delta}{2} = -F_{\mathcal{H}\langle B,i\rangle}$ is an NFSA and $\frac{1+\eta}{2} = F^{\mathcal{H}\langle B,i\rangle}$ is an FSA of $\mathcal{U}$ because of Lemma 3.2. Hence

$$\delta(u \circledast v) = 2(\frac{\delta}{2}(u \circledast v)) \leqslant 2(\max\{\frac{\delta}{2}(u), \frac{\delta}{2}(v)\}) = \max\{\delta(u), \delta(v)\},$$
$$\eta(u \circledast v) = 2(\frac{1+\eta}{2}(u \circledast v)) - 1 \geqslant \min\{2(\frac{1+\eta}{2}(u)) - 1, 2(\frac{1+\eta}{2}(v)) - 1\} = \min\{\eta(u), \eta(v)\}$$

for each $u, v \in \mathcal{U}$, which imply that δ is an NFSA and η is an FSA of $\mathcal{U}$. Therefore B is a BFSA of $\mathcal{U}$. □

5. Conclusions

In present paper, we have introduced the concept of $(\inf, \sup)$-HFSAs, which is a general concept of IvFSAs, in BCK/BCI-algebras, and discussed some of its properties. As important study results, $(\inf, \sup)$-HFSAs have been characterized in terms of sets, FSs, HFSs, IvFSs, NFSs, PFSs and BFSs. Furthermore, we use concepts of $(\inf, \sup)$-HFSAs and IvFSAs to investigate characterizations of SAs, FSAs, AFSAs, PFSAs, NFSAs and BFSAs.

The following are objectives for study and research in BCK/BCI-algebras and other algebras:

1. to introduce and study $(\inf, \sup)$-type of HFSs based on H-ideals and p-ideals of BCK/BCI-algebras;
2. to introduce and study $(\inf, \sup)$-type of HFSs based on ideals, interior ideals and bi-ideals of semigroups, ternary semigroups, Γ-semigroups and LA-semigroups;
3. to introduce and study $(\inf, \sup)$-type of HFSs based on substructures of BE-algebras, KU-algebras, JU-algebras and IUP-algebras [2, 10, 27, 39];
4. to extend this study to GE-algebras, G-algebras and BRK-algebras defined by Bandaru [4–6];
5. to apply this study to the concept of rough sets according to Ansari's study [1, 3].

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