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Exact travelling solutions for the sixth-order Boussinesq equation

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Abstract

In this paper, we establish some distinct exact solutions for a nonlinear evolution equation. The sin-cosine method and the rational Exp-Function and the rational hyperbolic function method are used to construct the solitary travelling wave solutions of the sixth-order Boussinesq equation . These solutions may be important of significance for the explanation of some practical physical problem.

Keywords: Traveling wave solutions; sin-cosine method; Exponential rational function method; the rational hyperbolic functions methods, the sixth-order Boussinesq equation

1- Introduction

Nonlinear evolution equations are widely used as models to describe complex physical phenomena and have a significant role in several scientific and engineering fields. These equations appear in solid state physics [1], fluid mechanics [2], chemical kinetics [3], plasma physics [4], population models, nonlinear optics, propagation of fluxons in Josephson junctions and etc... Analytical exact solutions to nonlinear partial

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differential equation play an important role in nonlinear science, since they can provide us much physical information and more inside into the physical aspects of the problem and thus lead to further applications. In recent years, quite a few methods for obtaining explicit travelling and solitary wave solutions of nonlinear evolutions equations have been proposed. A variety of powerful methods, such as inverse scattering method [5,6], bilinear transformation [7], Bäcklund and Darboux transformation [7-11], the tanh-sech method [12,13,14], extended tanh method [15], Exp-function method[16-19], the sine-cosine method [20-22], the Jacobi elliptic function method [23-25], (G'/G)-expansion method [26,27], Li group analysis[28], He's variational iteration method[29], He's homotopy perturbation method[30-32] and homogeneous balance method [33,34] and so on.

The sine–cosine method was developed by Wazwaz [22] and was successfully applied to nonlinear evolution equations [21,35,36,37], to nonlinear equations systems [38].

In this paper we will apply the sine-cosine method , rational exponential function method and rational hyperbolic function method to obtain the exact traveling wave solution of the sixth-order Boussinesq equation

$$u_{tt} - u_{xx} - [15uu_{4x} + 30u_x u_{3x} + 15(u_{2x})^2 + 45u^2 u_{2x} + 90uu_x{}^2 + u_{6x}] = 0, \tag{1.1}$$

Which was recently derived by Wazwaz by generalizing the bilinear forms of the standard Boussinesq equation [39]. Wazwaz obtained the single soliton solutions of the sixth-order Boussinesq equation using the tanh method and multiple-soliton solutions using Hirota bilinear method.

2- Sine–cosine method

Wazwaz has summarized the main steps introduced for using sine–cosine method, as following:

1- We introduce the wave variables $\xi = x - ct$ into the PDE, we get

$$\phi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, u_{xxx} \ldots) = 0. \tag{2.1}$$

where *u(x, t)* is travelling wave solution. This enables us to use the following changes:

$$u(x,t) = U(\xi), \tag{2.2}$$

$$\frac{\partial}{\partial t} = -c\frac{d}{d\xi},, \qquad \frac{\partial^2}{\partial t^2} = c^2\frac{d^2}{d\xi^2}, \qquad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \qquad \frac{\partial^2}{\partial x^2} = \frac{d^2}{d\xi^2}, \ldots \tag{2.3}$$

And so on for the other derivates. Using (2.3) and(2.1) ,the nonlinear PDE (2.1))changes to a nonlinear ODE:

$$\psi(U, -cU', U', c^2U'', U'', -cU', U''' \ldots) = 0, \tag{2.4}$$

2-If all terms of the resulting ODE contain derivatives in ξ, then by integrating this equation, by considering the constant of integration to be zero, we obtain a simplified ODE.

3- By virtue of the technique of solution, we introduce the anstaz:

$$U(\xi) = u(x,t) = \lambda \sin^{\beta}(\mu\xi), \qquad |\mu\xi| < \frac{\pi}{2} \tag{2.5}$$

Or

$$U(\xi) = u(x,t) = \lambda \cos^{\beta}(\mu\xi), \qquad |\mu\xi| < \frac{\pi}{2\mu} \tag{2.6}$$

Where λ, μ and β are parameters are to be determined later, μ and c are the wave number and the wave speed, respectively, we use:

$$\begin{aligned} &U(\xi) = \lambda \sin^{\beta}(\mu\xi), \\ &U^{n}(\xi) = \lambda^{n} \sin^{n\beta}(\mu\xi), \\ &(U^{n})_{\xi} = n\mu\beta\lambda^{n}\cos(\mu\xi)\sin^{n\beta-1}(\mu\xi), \\ &(U^{n})_{\xi\xi} = -n^{2}\mu^{2}\beta^{2}\lambda^{n}\sin^{n\beta}(\mu\xi) + n\mu^{2}\lambda^{n}\beta(n\beta-1)\sin^{n\beta-2}(\mu\xi), \\ &(U^{n})_{\xi\xi\xi} = n\lambda^{n}\mu^{3}\beta(n\beta^{2}-3\beta+2)\sin^{n\beta-3}(\mu\xi)\cos^{3}(\mu\xi) \\ &\qquad + \lambda^{n}\mu^{3}n\beta(3\beta n-2)\sin^{n\beta-1}(\mu\xi)\cos(\mu\xi), \end{aligned} \tag{2.7}$$

and the derivatives of Eq. (2.6) becomes:

$$\begin{aligned} &U(\xi) = \lambda \cos^{\beta}(\mu\xi), \\ &U^{n}(\xi) = \lambda^{n} \cos^{n\beta}(\mu\xi), \\ &(U^{n})_{\xi} = -n\mu\beta\lambda^{n}\sin(\mu\xi)\cos^{n\beta-1}(\mu\xi), \\ &(U^{n})_{\xi\xi} = -n^{2}\mu^{2}\beta^{2}\lambda^{n}\cos^{n\beta}(\mu\xi) + n\mu^{2}\lambda^{n}\beta(n\beta-1)\cos^{n\beta-2}(\mu\xi), \\ &(U^{n})_{\xi\xi\xi} = n\lambda^{n}\mu^{3}\beta(-n^{2}\beta^{2}+3n\beta-2)\cos^{n\beta-3}(\mu\xi)\sin^{3}(\mu\xi) \\ &\qquad + \lambda^{n}\mu^{3}n\beta(3\beta n-2)\sin^{n\beta-1}(\mu\xi)\sin(\mu\xi), \end{aligned} \tag{2.8}$$

and so on for the other derivatives.

4- We substitute Eq. (2.7) or (2.8) into the reduced equation obtained above in (2.4), balance the terms of the cosine functions when (2.8) is used, or balance the terms of the sine functions when (2.7) is used, and solving the resulting system of algebraic equations by using the computerized symbolic calculations. We next collect all terms with same power in $\cos^{k}(\mu\xi)$ or $\sin^{k}(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknowns μ, β and λ. We obtained all possible value of the parameters μ, β and λ.

3. The rational function in exp (ξ) method

The exp-function method was first proposed by He and Wu in 2006[17] and systematically studied in [18,19], and was successfully applied to KdV equation with variable coefficients [40], to high-dimensional nonlinear evolution equation [41], to burgers and combine KdV-mKdV (extended KdV) [42] equations, etc. In this section, we shall seek a rational function type of solution for a given partial equation, in terms of exp (ξ), of the following form [43]:

$$U = \sum_{k=0}^{m} \frac{a_k}{(1+e^{\xi})^{k}}, \tag{3.1}$$

Where $a_0, a_1, \ldots, a_m$ are some constants to be determined from the solution of (2.4).

Differentiating (3.1) with respect to ξ, introducing the result into Eq. (2.4), and setting the coefficients of the same power of e^{ξ} equal to zero, we obtain algebraic equations. The rational function solution of the equation (2.1) can be solved by obtaining $a_0, a_1, \ldots, a_m$ from this system.

4. The rational hyperbolic functions methods

The rational hyperbolic functions methods [44, 45] can be expressed in the form

$$U(\xi) = \frac{a_0 + b_0 f^{\,n}(\mu\xi)}{1 + a_1 f^{\,n}(\mu\xi)}, \qquad \xi = x - ct\ ,\ \ n = 1,2, \tag{4.1}$$

Where $f(\mu\xi)$ takes anyone of the hyperbolic functions and a_0, a_1, b_0, c and μ are parameters that will be determined. The rational hyperbolic functions methods can be applied directly as assumed before. We then collect the coefficients of the resulting hyperbolic functions and setting it equal to zero, and solving the resulting equations to determine the parameters a_0, a_1, b_0, c and μ.

5. Exact travelling solutions for the sixth-order Boussinesq equation

5.1. Application of Sine–cosine method

As described in Section 2, we make the transformation:

$$u(x,t) = U(\xi), \qquad \xi = x - ct\ , \tag{5.1}$$

Substituting Eq. (5.1) into (1.1) yields an ODE:

$$c^2U'' - U'' - 15UU^{(4)} - 30U'U''' - 15U''^2 - 45U^2U'' - 90UU'^2 - U^{(6)} = 0, \tag{5.2}$$

Integrating (5.2) twice and using the constants of integration to be zero, we find:

$$c^2U - U - 15UU'' - 15U^3 - U^{(4)} = 0, \tag{5.3}$$

Substituting Eq. (2.6) and Eq. (2.8) into (3.3) and rewriting the equation in terms of the sine function gives:

$$\begin{aligned}
&(-\lambda\beta^4\mu^4 - \lambda + c^2\lambda)\sin^{\beta}(\mu\xi) + 15\lambda^2\beta^2\mu^2\sin^{2\beta}(\mu\xi) - 15\lambda^3\sin^{3\beta}(\mu\xi)\\
&+(-4\lambda\beta\mu^4 + 8\lambda\beta^2\mu^4 - 6\lambda\beta^3\mu^4 + 2\lambda\beta^4\mu^4)\sin^{\beta-2}(\mu\xi)\\
&+(6\lambda\beta\mu^4 - \lambda\beta^4\mu^4 - 11\lambda\beta^2\mu^4 + 6\lambda\beta^3\mu^4)\sin^{\beta-4}(\mu\xi)\\
&+(15\lambda^2\beta\mu^2 - 15\lambda^2\beta^2\mu^2)\sin^{2\beta-2}(\mu\xi) = 0
\end{aligned} \tag{5.4}$$

Balancing the terms of the sine functions, we have:

$$\beta(\beta-1)(\beta-2)(\beta-3) \neq 0\ , \tag{5.5a}$$

$$\beta - 4 = 3\beta \qquad \Rightarrow \beta = -2\ , \tag{5.5b}$$

Substituting Eq. (5.5b) into Eq.(5.4) and Equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

$$\sin^{-6}(\mu\xi):\quad -90\lambda^2\mu^2 - 120\lambda\mu^4 - 15\lambda^3 = 0, \tag{5.6a}$$

$$\sin^{-4}(\mu\xi):\quad 60\lambda\mu^2(\lambda + 2\mu^2) = 0, \tag{5.6b}$$

$$\sin^{-4}(\mu\xi): \quad \lambda(-16\mu^4 + c^2 - 1) = 0, \tag{5.6c}$$

Solving (5.6a)-(5.6c) with the aid of Maple, we obtain:

$$\lambda = -2\mu^2 \ , \qquad c = \sqrt{1+16\mu^4} \tag{5.7}$$

The results in Eq. (5.7) can be easily obtained if we also use the sine method (2.5). Combining (5.7) with (2.6),the following solutions will be obtained:

$$u_1(x,t) = -\frac{2\mu^2}{\cos^2(\mu(x-\sqrt{1+16\mu^4}t))}, \tag{5.8}$$

Eq.(4.8) will satisfy Eq.(1.1).

5.2. Application of rational exponential function method

Now we shall seek a rational function type of solution to the sixth-order Boussinesq equation, in terms of exp (ξ) in the form :

$$u = U(\xi), \qquad \xi = \alpha(x - \beta t) \ , \tag{5.9}$$

Substituting Eq. (5.9) into Eq. (1.1) yields an ODE:

$$\beta^2 U'' - U'' - 15\alpha^2 UU^{(4)} - 30\alpha^2 U'U''' - 15\alpha^2 U''^2 - 45U^2 U'' - 90UU'^2 - \alpha^4 U^{(6)} = 0, \tag{5.10}$$

Integrating (5.10) twice and using the constants of integration to be zero, we find:

$$\beta^2 U - U - 15\alpha^2 UU'' - 15U^3 - \alpha^4 U^{(4)} = 0, \tag{5.11}$$

By use of the exp-function method, we may choose the solution of (5.11) in the form:

$$U = a_0 + \frac{a_1}{1+e^{\xi}} + \frac{a_2}{(1+e^{\xi})^2}, \tag{5.12}$$

Differentiating (5.12) with respect to ξ, introducing the result into Eq. (5.11), and setting the coefficients of the same power of e^{ξ} equal to zero, we obtain these algebraic equations:

$$\begin{aligned}
&-90a_1a_2a_0 - 45a_1^2a_0 + \beta^2 a_2 - 15a_1^3 - a_1 + \beta^2 a_0 - 45a_1^2a_2 - 15a_2^3 - 45a_2^2a_0 - a_0 \\
&-15a_0^3 - 45a_1a_2^2 + \beta^2 a_1 - 45a_1a_0^2 - 45a_2a_0^2 = 0 \\[1em]
&-90a_2^2a_0 - 90a_1^2a_2 - 90a_0^3 + 15\alpha^2 a_0a_1 + 30\alpha^2 a_0a_2 - 5a_1 + 45\alpha^2 a_1a_2 + 30\alpha^2 a_2^2 \\
&-225a_1a_0^2 + 4\beta^2 a_2 - 180a_2a_0^2 - 270a_1a_2a_0 + 15\alpha^2 a_1^2 - 45a_1a_2^2 + 5\beta^2 a_1 - 180a_1^2a_0 \\
&-45a_1^3 + \alpha^4 a_1 - 6a_0 - 4a_2 + 2\alpha^4 a_2 + 6\beta^2 a_0 = 0 \\[1em]
&15\alpha^2a_1^2 - 10a_1 - 60\alpha^2a_2^2 + 15\beta^2a_0 - 270a_1^2a_0 - 45a_2^2a_0 - 15a_0 - 270a_2a_0^2 - 6a_2 \\
&+10\beta^2a_1 - 225a_0^3 + 6\beta^2a_2 - 45a_1^2a_2 - 30\alpha^2a_1a_2 - 36\alpha^4a_2 - 10\alpha^4a_1 - 450a_1a_0^2 \\
&-270a_1a_2a_0 - 45a_1^3 + 30\alpha^2a_0a_1 = 0 \\[1em]
&20\beta^2a_0 - 20a_0 - 180a_1^2a_0 - 450a_1a_0^2 - 10a_1 - 75\alpha^2a_1a_2 - 15\alpha^2a_1^2 - 90a_1a_2a_0 \\
&+66\alpha^4a_2 - 180a_2a_0^2 - 15a_1^3 - 300a_0^3 - 90\alpha^2a_0a_2 + 4\beta^2a_2 + 10\beta^2a_1 - 4a_2 = 0
\end{aligned} \tag{5.13}$$

$$15\alpha^2 a_1^{\ 2} + 5\beta^2 a_1 - 60\alpha^2 a_0 a_2 + 15\beta^2 a_0 + \beta^2 a_2 - a_2 - 15a_0 - 5a_1 - 45a_2 a_0^{\ 2} - 30\alpha^2 a_0 a_1$$
$$+10\alpha^4 a_1 - 45a_1^{\ 2} a_0 - 225a_1 a_0^{\ 2} - 225a_0^{\ 3} = 0$$

$$6\beta^2 a_0 - 6a_0 - a_1 - \alpha^4 a_1 - 90a_0^{\ 3} - 45a_1 a_0^{\ 2} - 15\alpha^2 a_0 a_1 = 0$$

$$\beta^2 a_0 - 15a_0^{\ 3} - a_0 = 0$$

With the aid of *Maple*, the solutions of these algebraic equations are found to be:

$$a_0 = 0, \quad a_1 = 2\alpha^2, \qquad a_2 = -2\alpha^2, \ \beta = \sqrt{1+\alpha^4}, \tag{5.14a}$$

And

$$a_0 = \left(-\frac{1}{4} + \frac{1}{60}\sqrt{105}\right)\alpha^2, \quad a_1 = 2\alpha^2, \qquad a_2 = -2\alpha^2, \quad \beta = \sqrt{1 - \frac{1}{2}\alpha^4 - \frac{15}{2}\left(-\frac{1}{4} + \frac{1}{60}\sqrt{105}\right)\alpha^4}, \tag{5.14b}$$

Substituting Eq. (5.14a) and Eq. (5.143b) in Eq. (5.12) ,we obtain two exact solutions for Eq. (1.1) of the form:

$$u_3 = \frac{2\alpha^2}{1 + e^{\left(\alpha\left(x - \sqrt{1+\alpha^4}\,t\right)\right)}} - \frac{2\alpha^2}{\left(1 + e^{\left(\alpha\left(x + \sqrt{1+\alpha^4}\,t\right)\right)}\right)^2}, \tag{5.15a}$$

And

$$u_4 = \left(-\frac{1}{4} + \frac{1}{60}\sqrt{105}\right)\alpha^2 + \frac{2\alpha^2}{1 + e^{\left(\alpha\left(x - \sqrt{1 - \frac{1}{2}\alpha^4 - \frac{15}{2}\left(-\frac{1}{4} + \frac{1}{60}\sqrt{105}\right)\alpha^4}\,t\right)\right)}}$$
$$- \frac{2\alpha^2}{\left(1 + e^{\left(\alpha\left(x - \sqrt{1 - \frac{1}{2}\alpha^4 - \frac{15}{2}\left(-\frac{1}{4} + \frac{1}{60}\sqrt{105}\right)\alpha^4}\,t\right)\right)}\right)^2}, \tag{5.15b}$$

5.3.Application of rational hyperbolic functions methods

We next substitute the rational hyperbolic method (4.1) for $n = 1$and $f\ (\mu\xi) = \sinh(\mu\xi)$ into Eq.(5.3) and Collect the coefficients of the same power of resulting hyperbolic function equal to zero, the following algebraic system will be obtained:

$$c^2 a_0 - a_0 - 30\mu^2 a_1^2 a_0^2 - 15a_0^3 + 24b_0\mu^4 a_1^3 + 30\mu^2 a_0 b_0 a_1 - 24a_1^4\mu^4 a_0 - 8a_1^2\mu^4 a_0$$
$$+8b_0\mu^4 a_1 = 0$$
$$-4a_0 a_1 - b_0 - 45a_0^2 b_0 + 20a_1^3\mu^4 a_0 + 30\mu^2 b_0^2 a_1 + 4c^2 a_0 a_1 - 30\mu^2 a_1^3 a_0^2$$
$$-30a_0^3 a_1 + c^2 b_0 - 20b_0\mu^4 a_1^2 + a_1\mu^4 a_0 - 15\mu^2 a_0 b_0 - b_0\mu^4 + 15\mu^2 a_0^2 a_1 = 0$$

$$-11a_1^2\mu^4a_0-45a_0b_0^2-20a_1^4\mu^4a_0+4c^2b_0a_1+30\mu^2b_0^2a_1^2-30\mu^2b_0a_1^3a_0$$
$$-90a_0^2b_0a_1-6a_1^2a_0+6c^2a_1^2a_0-15\mu^2b_0^2-4b_0a_1+15\mu^2a_0b_0a_1+11b_0\mu^4a_1$$
$$+20b_0\mu^4a_1^3-15a_0^3a_1^2=0$$

(5.16)

$$-15\mu^2a_1^3a_0^2-11b_0\mu^4a_1^2+6c^2b_0a_1^2+4c^2a_0a_1^3-45a_0^2b_0a_1^2-15b_0^3$$
$$+11a_1^3\mu^4a_0-90a_0b_0^2a_1-4a_0a_1^3-6b_0a_1^2+15\mu^2b_0a_1^2a_0=0$$

$$b_0\mu^4a_1^3-a_1^4\mu^4a_0-4b_0a_1^3-30b_0^3a_1-15\mu^2b_0a_1^3a_0+15\mu^2b_0^2a_1^2$$
$$-a_0a_1^4+4c^2b_0a_1^3+c^2a_0a_1^4-45a_0b_0^2a_1^2=0$$

$$-b_0a_1^4+c^2b_0a_1^4-15b_0^3a_1^2=0$$

Solving the above system, we the following results will be obtained:

$$a_0=\mu^2,\quad a_1=I,\quad b_0=0,\ c=\sqrt{\mu^4+1}, \tag{5.17a}$$

And

$$a_0=\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)\mu^2,\quad a_1=I,$$

$$b_0=\frac{\frac{1}{2}I\mu^2\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)}{\frac{13}{14}-\frac{1}{4}\sqrt{105}},\ c=\sqrt{1-\frac{15}{2}\mu^4\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)+7\mu^4}, \tag{5.17b}$$

Substituting Eq. (5.17a) and Eq. (5.17b) in Eq. (4.1) along with $n=2$, and $f(\mu\xi)=\sinh(\mu\xi)$, we obtain two exact solutions of Eq. (1.1):

$$u_5=\frac{\mu^2}{\left(1+I\sinh\left(\mu\left(x-\sqrt{\mu^4+1}t\right)\right)\right)}, \tag{5.18a}$$

And

$$u_6=\frac{\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)\mu^2}{1+I\sinh\left(\mu\left(x-\sqrt{1-\frac{15}{2}\mu^4\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)+7\mu^4}t\right)\right)}$$
$$+\frac{\frac{\frac{1}{2}I\mu^2\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)}{\frac{13}{14}-\frac{1}{4}\sqrt{105}}\sinh\left(\mu\left(x-\sqrt{1-\frac{15}{2}\mu^4\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)+7\mu^4}t\right)\right)}{1+I\sinh\left(\mu\left(x-\sqrt{1-\frac{15}{2}\mu^4\left(\frac{3}{4}-\frac{1}{60}\sqrt{105}\right)+7\mu^4}t\right)\right)}, \tag{5.18b}$$

If we choose $n=2$ and $f(\mu\xi)=\sinh(\mu\xi)$,and consider the rational of hyperbolic method solution of the form (4.1) and insert them into Eq.(5.3), and Collect the coefficients of the same power of resulting hyperbolic function equal to zero, the following algebraic system will be resulted:

$$\begin{aligned}
&-24a_1^2\mu^4a_0+c^2a_0+30\mu^2a_1a_0^2-15a_0^3-30\mu^2a_0b_0-8b_0\mu^4+24b_0\mu^4a_1\\
&+8a_1\mu^4a_0-a_0=0\\[1em]
&60\mu^2a_1a_0^2+c^2b_0+240a_1^3\mu^4a_0+90\mu^2a_0b_0a_1-4a_1a_0-45a_0^2b_0-b_0+200b_0\mu^4a_1\\
&-30a_0^3a_1-240b_0\mu^4a_1^2-30\mu^2b_0^2-60\mu^2a_0b_0+4c^2a_0a_1-200a_1^2\mu^4a_0-60\mu^2a_1^2a_0^2\\
&+16a_1\mu^4a_0-16b_0\mu^4=0\\[1em]
&440a_1^3\mu^4a_0-90a_0^2b_0a_1-60\mu^2b_0^2-45a_0b_0^2+6c^2a_1^2a_0+60\mu^2a_0b_0a_1-120a_1^4\mu^4a_0\\
&-4b_0a_1-90\mu^2a_1^3a_0^2-6a_1^2a_0-176a_1^2\mu^4a_0+60\mu^2b_0^2a_1+176b_0\mu^4a_1+4c^2b_0a_1\\
&-440b_0\mu^4a_1^2+120b_0\mu^4a_1^3-15a_0^3a_1^2+30\mu^2b_0a_1^2a_0=0\\
&-4a_0a_1^3-90\mu^2b_0a_1^3a_0+176a_1^3\mu^4a_0-45a_0^2b_0a_1^2+6c^2b_0a_1^2-15b_0^3+4c^2a_0a_1^3\\
&-60\mu^2a_1^3a_0^2+90\mu^2b_0^2a_1^2-6b_0a_1^2-176b_0\mu^4a_1^2-90a_0b_0^2a_1+60\mu^2b_0a_1^2a_0\\
&-120a_1^4\mu^4a_0+120b_0\mu^4a_1^3=0\\[1em]
&c^2a_0a_1^4-4b_0a_1^3+16b_0\mu^4a_1^3-45a_0b_0^2a_1^2-60\mu^2b_0a_1^3a_0-16a_1^4\mu^4a_0\\
&+60\mu^2b_0^2a_1^2+4c^2b_0a_1^3-a_0a_1^4-30b_0^3a_1=0\\[1em]
&-b_0a_1^4+c^2b_0a_1^4-15b_0^3a_1^2=0
\end{aligned} \tag{5.19}$$

With the aid of *Maple*, the solutions of these algebraic equations are found to be:

$$\begin{aligned}
&a_0=2\mu^2,\quad a_1=1,\\
&b_0=0,\ c=\sqrt{16\mu^4+1},
\end{aligned} \tag{5.20a}$$

And

$$\begin{aligned}
&a_0=2\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)\mu^2,\quad a_1=1,\\
&b_0=\frac{4}{15}\frac{\mu^2\sqrt{105}}{\frac{7}{2}-\frac{1}{2}\sqrt{105}},\ c=\sqrt{1-60\mu^4\left(\frac{1}{2}-\frac{1}{30}\sqrt{105}\right)+52\mu^4},
\end{aligned} \tag{5.20b}$$

Substituting Eq. (5.20a) and Eq. (5.20b) in Eq. (4.1) along with $n=2$, and $f(\mu\xi)=\sinh(\mu\xi)$,we obtain another two exact solutions of Eq. (1.1):

$$u_7 = \frac{2\mu^2}{1 + I\,\sinh^2\left(\mu\left(x - \sqrt{16\mu^4 + 1}t\right)\right)}, \tag{5.21a}$$

And

$$u_8 = \frac{2\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right)\mu^2}{1 + \sinh^2\left(\mu\left(x - \sqrt{1 - 60\mu^4\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right) + 52\mu^4 t}\right)\right)} +$$

$$\frac{\left(\frac{4}{15}\frac{\mu^2\sqrt{105}}{\frac{7}{2} - \frac{1}{2}\sqrt{105}}\right)\sinh^2\left(\mu\left(x - \sqrt{1 - 60\mu^4\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right) + 52\mu^4 t}\right)\right)}{1 + \sinh^2\left(\mu\left(x - \sqrt{1 - 60\mu^4\left(\frac{1}{2} - \frac{1}{30}\sqrt{105}\right) + 52\mu^4 t}\right)\right)} \tag{5.21b}$$

We now consider the rational hyperbolic function of the form (4.1) with $n = 2$ and $f\,(\mu\xi) = \tanh(\mu\xi)$:

$$U\,(\xi) = \frac{a_0 + b_0 \tanh^2(\mu\xi)}{1 + a_1 \tanh^2(\mu\xi)} \tag{5.22}$$

Substituting Eq.(5.22) into Eq.(4.1) and Collecting the coefficients of the same power of resulting hyperbolic function equal to zero, the following algebraic system will be resulted:

$$-24a_1^2\mu^4 a_0 - a_0 - 15a_0^3 - 30\mu^2 a_0 b_0 + c^2 a_0 + 30\mu^2 a_1 a_0^2 - 16a_1\mu^4 a_0$$
$$+16b_0\mu^4 + 24b_0\mu^4 a_1 = 0$$

$$c^2 b_0 + 136a_1\mu^4 a_0 - 120\mu^2 a_1 a_0^2 + 120\mu^2 a_0 b_0 - 60\mu^2 a_1^2 a_0^2 + 90\mu^2 a_0 b_0 a_1$$
$$-30\mu^2 b_0^2 - b_0 - 240a_1^2\mu^4 b_0 - 30a_0^3 a_1 - 4a_1 a_0 - 45a_0^2 b_0 + 240a_1^3\mu^4 a_0$$
$$+4c^2 a_0 a_1 - 136b_0\mu^4 + 400a_1^2\mu^4 a_0 - 400b_0\mu^4 a_1 = 0$$

$$6c^2 a_1^2 a_0 - 6a_1^2 a_0 - 4b_0 a_1 - 1016a_1^2\mu^4 a_0 - 90\mu^2 a_0 b_0 - 45a_0 b_0^2 + 120a_1^3\mu^4 b_0$$
$$+30\mu^2 b_0 a_1^2 a_0 - 90a_0^2 b_0 a_1 - 880a_1^3\mu^4 a_0 + 120\mu^2 b_0^2 - 120a_1^4\mu^4 a_0 + 60\mu^2 b_0^2 a_1$$
$$+4c^2 b_0 a_1 - 240a_1\mu^4 a_0 - 15a_0^3 a_1^2 + 90\mu^2 a_1 a_0^2 + 1016b_0\mu^4 a_1 - 90\mu^2 a_1^3 a_0^2$$
$$+880a_1^2\mu^4 b_0 - 120\mu^2 a_0 b_0 a_1 + 240b_0\mu^4 = 0$$

$$60\mu^2 b a_1^2 a_0^2 - 90\mu^2 b_0 a_1^3 a_0 + 6c^2 b_0 a_1^2 - 120\mu^2 b_0 a_1^2 a_0 + 1016a_1^3\mu^3 a_0 - 45a_0^2 b_0 a_1^2$$
$$-15b_0^3 + 30\mu^2 a_0 b_0 a_1 + 240a_1^4\mu^4 a_0 + 90\mu^2 b_0^2 a_1^2 - 120b_0\mu^4 - 6a_1^2 b_0 + 880a_1^2\mu^4 a_0$$
$$-4a_1^3 a_0 + 4c^2 a_0 a_1^3 - 90a_0 b_0^2 a_1 + 120\mu^2 a_1^3 a_0^2 + 120a_1\mu^4 a_0 - 240a_1^3\mu^4 b_0 - 90\mu^2 b_0^2$$
$$-880b_0\mu^4 a_1 - 1016a_1^2\mu^4 b_0 = 0 \tag{5.23}$$

$$4c^2b_0a_1^3+90\mu^2b_0a_1^2a_0-136a_1^4\mu^4a_0-60\mu^2b_0^2a_1+136a_1^3\mu^4b_0-4a_1^3b_0$$
$$-120\mu^2b_0^2a_1^2-45a_0b_0^2a_1^2-240a_1^2\mu^4a_1-400a_1^3\mu^4a_0+c^2a_0a_1^4-30b_0^3a_1$$
$$+120\mu^2b_0a_1^3a_0+400a_1^2\mu^4b_0-30\mu^2a_1^3a_0^2-a_1^4a_0=0$$

$$30\mu^2b_0^2a_1^2-b_0a_1^4-30\mu^2b_0a_1^3a_0-16a_1^3\mu^4b_0-24a_1^2\mu^4b_0-15b_0^3a_1^2$$
$$+16a_1^4\mu^4a_0+24a_1^3\mu^4a_0+c^2b_0a_1^4=0$$

With the aid of *Maple*, the solutions of these algebraic equations are found to be:

$$a_0=2\mu^2,\quad a_1=0,$$
$$b_0=-2\mu^2,\ c=\sqrt{16\mu^4+1}, \tag{5.24a}$$

And

$$a_0=2(\frac{1}{2}-\frac{1}{30}\sqrt{105})\mu^2,\quad a_1=0,$$
$$b_0=-2\mu^2,\ c=\sqrt{1-60\mu^4(\frac{1}{2}-\frac{1}{30}\sqrt{105})+52\mu^4}, \tag{5.24b}$$

Substituting Eq. (5.24a) and Eq. (5.24b) in Eq. (5.22),we obtain another two exact solutions of Eq. (1.1):

$$u_7=2\mu^2-2\mu^2\tanh^2(\mu(x-\sqrt{16\mu^4+1}t)), \tag{5.25a}$$

And

$$u_7=2(\frac{1}{2}-\frac{1}{30}\sqrt{105})\mu^2-2\mu^2\tanh^2(\mu(x-\sqrt{1-60\mu^4(\frac{1}{2}-\frac{1}{30}\sqrt{105})+52\mu^4}t)), \tag{5.25b}$$

5- Conclusions

In summary, we have applied the sin-cosine method along with rational exponential function and rational hyperbolic function method to obtain travelling wave solution for the sixth-order Boussinesq equation . In fact, the present methods are readily applicable to a large variety of such nonlinear equations. It is shown that The hyperbolic ansatze handled Gardner equation effectively and the sin-cosine method and the rational exponential function method are powerful and straightforward solution method to find closed-form, periodic and non-periodic analytical expressions for travelling waves of nonlinear wave and evolution equations.

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