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q-fractional functional integro-differential equation with q-integral condition

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Abstract

In this paper, we study the existence and the uniqueness of the solution of the non-local problems of q-fractional functional integro-differential equation with non-local q-integral condition. The continuous dependence of the solution will be studied. Two examples will be given to illustrate results.

Keywords: Functional equations, existence of solutions, continuous dependence, q-calculus.

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1. Introduction

q-calculus, This area of research has several applications, see [2, 10, 11] and references therein. There are several developments and applications of the q-calculus in mathematical physics, the theory of relativity and special functions [1, 4]. In several papers [13, 15], integro-differential equation with infinite-point boundary conditions have been studied. In [5–7] El-Sayed et al. introduced and studied integro-differential equation with infinite-point boundary conditions.

In this paper, we are concerned with q-fractional functional integro-differential equation

$$\frac{dx}{dt} = f(t, x(t), I_q^{\delta} g(t, x(t))), \quad \text{a.e.} \quad t \in (0, 1], \tag{1.1}$$

with the non-local q-integral

$$\int_0^1 x(s) d_q s = x_0. \tag{1.2}$$

The existence and the uniqueness of solution, under certain conditions, will be proved. The continuous dependence of the solution on x_0, will be studied.

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Importance of fractional differential equations appears in many of the physical and engineering phenomena [13]. Problems with non-local conditions and related topics were studied, for example [6, 7, 9]. The attention of researchers subject of q-differential equations appeared in recent years [17]. Noted recently the attention of many researchers is in the field of fractional q-calculus [3, 14, 18, 19].

The paper is organized as follows. In Section 2 some necessary definitions and results are recalled from the literature and the equivalence q-fractional functional integro-differential equation (1.1) with non-local q-integral condition (1.2) is given. In Section 3, we study the existence of continuous solutions of the problem (1.1)-(1.2). In Section 4, we establish the existence of exactly one solution of (1.1)-(1.2). In Section 5, the continuous dependence of the solution is studied. In Section 6, two examples are given to demonstrate the main existence result. Finally, the conclusion of our work is introduced.

2. q-calculus

First, we write the basic definitions of the q-fractional integral and q-derivatives, for more details.

- Let a real parameter $q \in (0,1)$, we define a q-real number $[a]_q$ by

$$[a]_q = \frac{1-q^a}{1-q}, \ a \in \mathbb{R}.$$

- A q-analog of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a;q)_n = \begin{cases} 1, & n=0, \\ \prod_{i=1}^{n-1}(1-aq^i), & n \in \mathbb{N}. \end{cases}$$

- The q-analog of the power $(a-b)^n$ is given by

$$(a-b)^n = \begin{cases} 1, & n=0, \\ \prod_{i=1}^{n-1}(a-bq^i), & n \in \mathbb{N}, \ a,b \in \mathbb{R}. \end{cases}$$

- Also,

$$(a-b)^n = a^n({}^b/_a;q)_n, \ a \neq 0.$$

  Notice that, $\lim_{n\to\infty}(a;q)_n$ exists and we will denote it by $(a;q)_\infty$.
- More generally, for $\lambda \in \mathbb{R}$, $aq^\lambda \neq q^{-n} (n \in \mathbb{N})$, we define

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \quad \text{and} \quad (a-b)^{(\lambda)} = a^\lambda \frac{({}^b/_a;q)_\infty}{(q^\lambda\, {}^b/_a;q)_\infty}.$$

- The q-gamma function is defined by

$$\Gamma_q(t) = \frac{G(q^t)}{(1-q)^{t-1}G(q)}, \ t \in \mathbb{R} - \{0,-1,-2,\ldots\},$$

  where $G(q^t) = \frac{1}{(q^t;q)_\infty}$. Or, equivalently, $\Gamma_q(t) = \frac{1-q^{(t-1)}}{1-q^{t-1}}$, and satisfies $\Gamma_q(t+1) = [t]_q\Gamma_q(t)$.

Definition 2.1 ([2])**.** Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\delta \geqslant 0$ is given by

$$(I_q^\delta f)(t) = \begin{cases} f(t), & \delta = 0, \\ \frac{1}{\Gamma_q(\delta)}\int_0^t (t-qs)^{\delta-1}f(s)d_qs = (1-q)^\delta t^\delta \sum_{i=0}^\infty q^i \frac{(q^\delta;q)_i}{(q;q)_i}f(q^it), & \delta > 0. \end{cases}$$

Lemma 2.2 ([16])**.** *For $\delta > 0$, using q-integration by parts, we have*

$$(I_q^\delta 1)(t) = \frac{t^{(\delta)}}{\Gamma_q(\delta+1)}.$$

The equivalence of (1.1)-(1.2) and integral equation is given in the following lemma.

Lemma 2.3. *If the solution of the non-local problem (1.1)-(1.2) exists, then the functional* q*-fractional integro-differential equation (1.1)-(1.2) and the functional* q*-integral equation*

$$x(t)=\frac{1}{\int_0^1 d_q s}[x_0-\int_0^1\int_0^s f(\theta,x(\theta),I_q^\delta g(\theta,x(\theta)))d\theta d_q s]+\int_0^t f(s,x(s),I_q^\delta g(s,x(s)))ds. \tag{2.1}$$

are equivalent.

Proof. Let x be a solution of the non-local problem (1.1)-(1.2), integrating both sides of (1.1) we get

$$x(t)=x(0)+\int_0^t f(s,x(s),I_q^\delta g(s,x(s)))ds. \tag{2.2}$$

Using the non-local q-condition (1.2), we get

$$\int_0^1 x(s)d_q s=\int_0^1 x(0)d_q s+\int_0^1\int_0^s f(\theta,x(\theta),I_q^\delta g(\theta,x(\theta)))d\theta d_q s,$$

then

$$x(0)=\frac{1}{\int_0^1 d_q s}[x_0-\int_0^1\int_0^s f(\theta,x(\theta),I_q^\delta g(\theta,x(\theta)))d\theta d_q s]. \tag{2.3}$$

Using (2.2) and (2.3), we obtain (2.1). To complete the proof, suppose that x satisfies equation (2.1), differentiating (2.1) we obtain

$$\begin{aligned}\frac{dx}{dt}&=\frac{d}{dt}\left\{\frac{1}{\int_0^1 d_q s}[x_0-\int_0^1\int_0^s f(\theta,x(\theta),I_q^\delta g(\theta,x(\theta)))d\theta d_q s]\int_0^t f(s,x(s),I_q^\delta g(s,x(s)))ds\right\}\\&=0+\frac{d}{dt}\int_0^t f(s,x(s),I_q^\delta g(s,x(s)))ds=f(t,x(t),I_q^\delta g(t,x(t)))\end{aligned}$$

and

$$\int_0^1 x(\tau)d_q\tau=\frac{1}{\int_0^1 d_q s}[x_0-\int_0^1\int_0^s f(\theta,x(\theta),I_q^\delta g(\theta,x(\theta)))d\theta d_q s]\int_0^1 d_q\tau+\int_0^1\int_0^\theta f(s,x(s),I_q^\delta g(s,x(s)))ds d_q\tau.$$

Then

$$\int_0^1 x(\tau)d_q\tau=x_0.$$

□

3. Existence of solution

In the following theorem, using Schauder fixed point theorem [8], we establish existence of at least one solution of (1.1)-(1.2).

Theorem 3.1. *Let* $f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ *and* $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ *satisfies Carathéodory condition, if there exist functions* $c_{1,2}\in L^1[0,1]$ *and positive constants* $b_{1,2}>0$ *such that*

$$|f(t,x,y)|\leqslant c_1(t)+b_1|x|+b_1|y|,\ |g(t,x)|\leqslant c_2(t)+b_2|x|,$$

and

$$\sup_{t\in[0,1]}\int_0^t c_1(s)ds\leqslant M_1,\quad \sup_{t\in[0,1]}\int_0^t I_q^\delta c_2(s)ds\leqslant M_2,\quad (2b_1+\frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)})\in[0,1),$$

then the non-local problem (1.1)-(1.2) has at least one solution.

Proof. Define the operator A associated with the integral equation (2.1) by

$$Ax(t) = \frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta d_q s] + \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds.$$

Let $Q_r = \{x(t) \in \mathbb{R} : \|x\| \leqslant r\}$, where $r = \frac{|x_0|+2M_1+2b_1M_2}{1-(2b_1+\frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)})}$. Then, for $x \in Q_r$, we have

$$\begin{aligned}
|Ax(t)| &\leqslant \frac{1}{\int_0^1 d_q s}[|x_0| + \int_0^1 \int_0^s |f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))|d\theta d_q s] + \int_0^t |f(s, x(s), I_q^\delta g(s, x(s)))|ds \\
&\leqslant \frac{1}{\int_0^1 d_q s}[|x_0| + \int_0^1 \int_0^s (c_1(\theta) + b_1|x(\theta)| + b_1 I_q^\delta |g(\theta, x(\theta))|)d\theta d_q s] \\
&\quad + \int_0^t (c_1(s) + b_1|x(s)| + b_1 I_q^\delta |g(s, x(s))|)ds \\
&\leqslant \frac{1}{\int_0^1 d_q s}[|x_0| + \int_0^1 (M_1 + b_1 r + b_1 \int_0^s I_q^\delta |c_2(\theta) + b_2|x(\theta)|d_q s)] \\
&\quad + M_1 + b_1 r + b_1 \int_0^t I_q^\delta (c_2(s) + b_2|x(s)|)ds \\
&\leqslant \frac{1}{\int_0^1 d_q s}[|x_0| + \int_0^1 (M_1 + b_1 r + b_1 M_2 + b_1 b_2 r \int_0^s \frac{\theta^\delta}{\Gamma_q(\delta+1)} d\theta)]d_q s \\
&\quad + M_1 + b_1 r + b_1 M_2 + b_1 b_2 r \int_0^t \frac{s^\delta}{\Gamma_q(\delta+1)} ds \\
&= \frac{|x_0|}{\int_0^1 d_q s} + 2M_1 + 2b_1 r + 2b_1 M_2 + \frac{2b_1 b_2 r}{(\delta+1)\Gamma_q(\delta+1)} = r.
\end{aligned}$$

This proves that $A : Q_r \to Q_r$ and the class of functions $\{Ax\}$ is uniformly bounded in Q_r.

Now, let $t_1, t_2 \in (0, 1)$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Ax(t_2) - Ax(t_1)| &= \left| \int_0^{t_2} f(s, x(s), I_q^\delta g(s, x(s)))ds - \int_0^{t_1} f(s, x(s), I_q^\delta g(s, x(s)))ds \right| \\
&\leqslant \int_{t_1}^{t_2} |f(s, x(s), I_q^\delta g(s, x(s))ds)|ds \\
&\leqslant \int_{t_1}^{t_2} (c_1(s) + b_1|x(s)| + b_1 I_q^\delta |g(s, x(s))ds)|ds \\
&\leqslant \int_{t_1}^{t_2} c_1(s)ds + (t_2 - t_1)b_1 r + b_1 \int_{t_1}^{t_2} I_q^\delta c_2(s)ds + b_1 b_2 r \int_{t_1}^{t_2} \frac{s^\delta}{\Gamma_q(\delta+1)} ds.
\end{aligned}$$

This mean that the class of functions $\{Ax\}$ is equi-continuous in Q_r. Let $x_n \in Q_r$, $x_n \to x (n \to \infty)$, then from continuity of the functions f and g, we obtain $f(t, x_n(t), y_n(t)) \to f(t, x(t), y(t))$ and $g(t, x_n(t)) \to g(t, x(t))$ as $n \to \infty$. Also

$$\begin{aligned}
\lim_{n\to\infty} Ax_n(t) = \lim_{n\to\infty} \Big[&\frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1 \int_0^s f(\theta, x_n(\theta), I_q^\delta g(\theta, x(\theta)))d\theta d_q s] \\
&+ \int_0^t f(s, x_n(s), I_q^\delta g(s, x_n(s)))ds \Big].
\end{aligned} \tag{3.1}$$

Using (1.1)-(1.2) and Lebesgue dominated convergence Theorem [12], from (3.1) we obtain

$$\begin{aligned}\lim_{n\to\infty} Ax_n(t) = \Big[\frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1\int_0^s f(\theta, x_n(\theta), I_q^\delta g(\theta, x(\theta)))d\theta d_q s] \\ + \int_0^t \lim_{n\to\infty} f(s, x_n(s), I_q^\delta g(s, x_n(s)))ds\Big] = Ax(t).\end{aligned}$$

Then $Ax_n \to Ax$ as $n \to \infty$. This mean that the operator A is continuous. Then by Schauder fixed point Theorem [8] there exist at least one solution $x \in C[0,1]$ of the functional equation (1.1)-(1.2). □

4. Uniqueness of the solution

In the following theorem, we establish existence of exactly one solution of (1.1)-(1.2).

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in t for any $x, y \in \mathbb{R}$ and continuous in x, y for all $t \in [0,1]$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and continuous in x for all $t \in [0,1]$. If there exists $b_{1,2} > 0$ with*

$$|f(t,x,y) - f(t,u,v)| \leqslant b_1|x-u| + b_1|y-v| \quad \textit{and} \quad |g(t,x) - g(t,u)| \leqslant b_2|x-u|,$$

then the solution of the non-local problem (1.1)-(1.2) is unique.

Proof. Let x, y be two the solution of (1.1)-(1.2), then

$$\begin{aligned}|x(t) - y(t)| &= |\frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1\int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta)))d\theta d_q s] + \int_0^t f(s, x(s), I_q^\delta g(s, x(s)))ds \\ &\quad - \frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1\int_0^s f(\theta, y(\theta), I_q^\delta g(\theta, y(\theta)))d\theta d_q s] - \int_0^t f(s, y(s), I_q^\delta g(s, y(s)))ds| \\ &\leqslant \frac{1}{\int_0^1 d_q s}\int_0^1\int_0^s |f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta))) - f(\theta, y(\theta), I_q^\delta g(\theta, y(\theta)))|d\theta d_q s \\ &\quad + \int_0^t |f(s, x(s), I_q^\delta g(s, x(s))) - f(s, y(s), I_q^\delta g(s, y(s)))|ds \\ &\leqslant \frac{1}{\int_0^1 d_q s}\int_0^1\int_0^s |(b_1\|x-y\| + b_1 I_q^\delta|g(\theta, x(\theta)) \\ &\quad - g(\theta, y(\theta))|)d\theta d_q s + \int_0^t |(b_1\|x-y\| + b_1 I_q^\delta|g(s, x(s)) - g(s, y(s))|)ds \\ &\leqslant 2b_1\|x-y\| + \frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)}\|x-y\| = (2b_1 + \frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)})\|x-y\|.\end{aligned}$$

Hence

$$(1 - 2b_1 + \frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)})\|x-y\| \leqslant 0.$$

since $(2b_1 + \frac{2b_1b_2}{(\delta+1)\Gamma_q(\delta+1)}) < 1$, then $x = y$ and the solution of the non-local problem (1.1)-(1.2) is unique. □

5. Continuous dependence

In the following, we give the definition of solutions continuously depend on initial data.

Definition 5.1. The solution $x \in C[0,1]$ of the non-local problem (1.1)-(1.2) depends continuously on x_0, if

$$\forall \epsilon > 0, \quad \exists \quad \delta_1(\epsilon) \quad \text{s.t.} \quad |x_0 - x_0^*| < \delta_1 \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the non-local problem

$$\frac{dx^*}{dt} = f(t, x^*(t), I_q^\delta g(t, x^*(t))), \quad \text{a.e.} \quad t \in (0,1], \tag{5.1}$$

with the non-local q-condition

$$\int_0^1 x^*(s) d_q s = x_0^*. \tag{5.2}$$

Theorem 5.2. *Let the assumptions of Theorem 4.1 be satisfied, then the solution of the non-local problem (1.1)-(1.2) depends continuously on x_0.*

Proof. Let x, x^* be two solutions of the non-local problem (1.1)-(1.2) and (5.1)-(5.2), respectively. Then

$$\begin{aligned}
|x(t) - x^*(t)| &= |\frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1 \int_0^s f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta))) d\theta d_q s] + \int_0^t f(s, x(s), I_q^\delta g(s, x(s))) ds \\
&\quad - \frac{1}{\int_0^1 d_q s}[x_0 - \int_0^1 \int_0^s f(\theta, x^*(\theta), I_q^\delta g(\theta, x^*(\theta))) d\theta d_q s] - \int_0^t f(s, x^*(s), I_q^\delta g(s, x^*(s))) ds| \\
&\leqslant \frac{|x_0 - x_0^*|}{\int_0^1 d_q s} + \frac{1}{\int_0^1 d_q s} \int_0^1 \int_0^s |f(\theta, x(\theta), I_q^\delta g(\theta, x(\theta))) - f(\theta, y(\theta), I_q^\delta g(\theta, y(\theta)))| d\theta d_q s \\
&\quad + \int_0^t |f(s, x(s), I_q^\delta g(s, x(s))) - f(s, y(s), I_q^\delta g(s, y(s)))| ds \\
&\leqslant \frac{|x_0 - x_0^*|}{\int_0^1 d_q s} + \frac{1}{\int_0^1 d_q s} \int_0^1 \int_0^s |(b_1 \|x - y\| + b_1 I_q^\delta |g(\theta, x(\theta)) \\
&\quad - g(\theta, y(\theta))|) d\theta d_q s + \int_0^t |(b_1 \|x - y\| + b_1 I_q^\delta |g(s, x(s)) - g(s, y(s))|) ds \\
&\leqslant \frac{\delta_1}{\int_0^1 d_q s} + 2b_1 \|x - y\| + \frac{2b_1 b_2}{(\delta+1)\Gamma_q(\delta+1)} \|x - y\| = \frac{\delta_1}{\int_0^1 d_q s} + (2b_1 + \frac{2b_1 b_2}{(\delta+1)\Gamma_q(\delta+1)}) \|x - y\|.
\end{aligned}$$

Hence

$$\|x - x^*\| \leqslant \frac{\delta_1}{(\int_0^1 d_q s)[1 - (2b_1 + \frac{2b_1 b_2}{(\delta+1)\Gamma_q(\delta+1)})]} = \epsilon.$$

This mean that the solution of the non-local problem (1.1)-(1.2) depends continuously on x_0. □

6. Examples

In this section we offer some examples to illustrate our results

Example 6.1. Consider the following nonlinear integro-differential equation

$$\frac{dx}{dt} = t^3 e^{-t} + \frac{\ln(1 + x(t))}{3 + t^2} + I_{0.5}^{\frac{2}{5}} \frac{1}{9}(\cos(3t + 3) + t^5 \cos x(t) + e^{-t} x(t)), \quad \text{a.e.} \quad t \in (0,1], \tag{6.1}$$

with q-condition

$$\int_0^1 x(s)ds = x_0. \tag{6.2}$$

Set

$$f(t, x(t), I_q^\delta g(t, x(t))) = t^3 e^{-t} + \frac{\ln(1 + x(t))}{3 + t^2} + \frac{1}{9} I_{0.5}^{\frac{2}{5}} (\cos(3t + 3) + t^5 \cos x(t) + e^{-t} x(t)).$$

Then

$$|f(t, x(t), I_q^\delta g(t, x(t)))| \leqslant t^3 e^{-t} + \frac{1}{3}(|x| + \frac{1}{3} I_{0.5}^{\frac{2}{5}} \frac{1}{3} |(\cos(3t + 3) + t^5 \cos x(t) + e^{-t} x(t))|),$$

and also

$$|g(t, x(t))| \leqslant \frac{1}{3} |\cos(3t + 3)| + \frac{1}{3} |x(t)|.$$

It is clear that the assumptions of Theorem 3.1 are satisfied with

$$c_1(t) = t^3 e^{-t} \in L^1[0, 1], \quad c_2(t) = \frac{1}{2} |\cos(3t + 3)| \in L^1[0, 1],$$

$$b_1 = \frac{1}{3}, \quad b_2 = \frac{1}{3}, \quad 2b_1 + \frac{2b_1 b_2}{(\delta + 1)\Gamma_q(\delta + 1)} = \frac{2}{3} + \frac{\frac{4}{9}}{\frac{7}{5}\Gamma_{0.5}(\frac{7}{5})} = 0.8390712688 < 1,$$

by applying to Theorem 3.1, the given non-local problem (6.1)-(6.2) has a continuous solution.

Example 6.2. Consider the following nonlinear integro-differential equation

$$\frac{dx}{dt} = t^3 + t + 1 + \frac{x(t)}{\sqrt{t + 3}} + I_{0.5}^{\frac{1}{8}} \frac{1}{4} (\sin^2(3t + 3) + \frac{t x(t)}{2^t (1 + x(t))}), \quad \text{a.e.} \quad t \in (0, 1], \tag{6.3}$$

with q-condition

$$\int_0^1 x(s)ds = x_0. \tag{6.4}$$

Set

$$f(t, x(t), I_q^\delta g(t, x(t))) = t^3 + t + 1 + \frac{x(t)}{\sqrt{2t + 4}} + \frac{1}{4} I_q^\delta (\sin^2(3t + 3) + \frac{t x(t)}{2^t (1 + x(t))}).$$

Then

$$|f(t, x(t), I_q^\delta g(t, x(t)))| \leqslant t^3 + t + 1 + \frac{1}{3}|x| + \frac{1}{3} I_q^\delta \frac{3}{4} |(\sin^2(3t + 3) + \frac{t x(t)}{2^t (1 + x(t))}|,$$

and also

$$|g(s, x(s))| \leqslant \frac{3}{4} |(\sin^2(3s + 3)| + \frac{3}{8} |x(s)|.$$

It is clear that the assumptions of Theorem 3.1 are satisfied with

$$c_1(t) = t^3 + t + 1 \in L^1[0, 1], \quad c_2(t) = \frac{3}{4} |(\sin^2(3s + 3)| \in L^1[0, 1],$$

$$b_1 = \frac{1}{3}, \quad b_2 = \frac{3}{8}, \quad 2b_1 + \frac{2b_1 b_2}{(\delta + 1)\Gamma_q(\delta + 1)} = \frac{2}{3} + \frac{\frac{1}{4}}{\frac{9}{8}\Gamma_{0.5}(\frac{9}{8})} = 0.89864714178 < 1,$$

by applying to Theorem 3.1, the given non-local problem (6.3)-(6.4) has a continuous solution.

7. Conclusion

In this work, the existence of continuous solution using Schauder fixed point Theorem, its uniqueness, and the continuous dependence of the q-fractional functional integro-differential equation on an initial data have been studied. Some examples are introduced to illustrate the benefits of our results. In the future, generalization of q-fractional functional differential equation in time scale can be examined.

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References

[1] T. Abdeljawad, D. Baleanu, *Caputo q-fractional initial value problems and a q-analogue mittag-leffler function*, Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 4682–4688. 1

[2] R. P. Agarwal, *Certain fractional q-integrals and q-derivatives*, Proc. Cambridge Philos. Soc., **66** (1969), 365–370. 1, 2.1

[3] A. Boutiara, M. Benbachir, M. K. A. Kaabar, F. Martinez, M. E. Samei, M. Kaplan, *Explicit iteration and unbounded solutions for fractional q-difference equations with boundary conditions on an infinite interval*, J. Inequal. Appl., **2022** (2022), 27 pages. 1

[4] R. I. Butt, T. Abdeljawad, M. A. Alqudah, M. ur Rehman, *Ulam stability of caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach*, J. Inequal. Appl., **2019** (2019), 13 pages. 1

[5] A. M. A. El-Sayed, R. G. Ahmed, *Existence of Solutions for a Functional Integro-Differential Equation with Infinite Point and Integral Conditions*, Int. J. Appl. Comput. Math., **5** (2019), 15 pages. 1

[6] A. M. A. El-Sayed, R. G. Ahmed, *Solvability of a coupled system of functional integro-differential equations with infinite point and Riemann-Stieltjes integral conditions*, Appl. Math. Comput., **370** (2020), 18 pages. 1

[7] A. El-Sayed, R. Gamal, *Infinite point and Riemann-Stieltjes integral conditions for an integro-differential equation*, Nonlinear Anal. Model. Control, **24** (2019), 733–754. 1, 1

[8] K. Goeble, W. A. Kirk, *Topics in metric fixed point theory*, Cambirdge Universty Press, Cambirdge, (1990). 3, 3

[9] T. S. Hassan, R. G. Ahmed, A. M. A. El-Sayed, R. A. El-Nabulsi, O. Moaaz, M. B. Mesmouli, *Solvability of a State-Dependence Functional Integro-Differential Inclusion with Delay non-local Condition*, Mathematics, **10** (2022), 11 pages. 1

[10] M. Jleli, M. Mursaleen, B. Samet, *Q-integral equations of fractional orders*, Electron. J. Differential Equations, **2016** (2016), 14 pages. 1

[11] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2001). 1

[12] A. N. Kolomogorov, S. V. Fomin, *Inroductory real analysis*, Dover Publications, New York, (1975). 3

[13] X. Q. Liu, L. S. Liu, Y. H. Wu, *Existence of positive solutions for a singular nonlinear fractional differential equation with integral boundary conditions involving fractional derivatives*, Bound. Value Probl., **2018** (2018), 21 pages. 1, 1

[14] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, S. Rezapour, *Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives*, Adv. Difference Equ., **2021** (2021), 18 pages. 1

[15] D. D. Min, L. S. Liu, Y. H. Wu, *Uniqueness of positive solutions for the singular fractional differential equations involving integral boundary value conditions*, Bound. Value Probl., **2018** (2018), 18 pages. 1

[16] P. M. Rajkovć, S. D. Marinković, M. S. Stanković, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math., **1** (2017), 311–323. 2.2

[17] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad, M. K. A. Kaabar, *Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs*, Symmetry, **13** (2021), 15 pages. 1

[18] M. E. Samei, R. Ghaffari, S. W. Yao, M. K. A. Kaabar, F. Martinez, M. Inc, *Existence of Solutions for a Singular Fractional q-Differential Equations under Riemann-Liouville Integral Boundary Condition*, Symmetry, **13** (2021), 12 pages. 1

[19] M. E. Samei, L. Karimi, M. K. A. Kaabar, *To investigate a class of multi-singular pointwise defined fractional q-integro-differential equation with applications*, AIMS Math., **7** (2022), 7781–7816. 1