J. Math. Computer Sci., 29 (2023), 175–191

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

External direct products on dual UP (BCC)-algebras

Chatsuda Chanmanee[a], Ronnason Chinram[b], Rukchart Prasertpong[c], Pongpun Julatha[d], Aiyared Iampan[a,*]

[a] Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand.

[b] Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand.

[c] Division of Mathematics and Statistics, Faculty of Science and Technology, Nakhon Sawan Rajabhat University, Nakhon Sawan 60000, Thailand.

[d] Department of Mathematics, Faculty of Science and Technology, Pibulsongkram Rajabhat University, Phitsanulok 65000, Thailand.

Abstract

The concept of the direct product of finite family of B-algebras is introduced by Lingcong and Endam [J. A. V. Lingcong, J. C. Endam, Int. J. Algebra, **10** (2016), 33–40]. In this paper, we introduce the concept of the direct product of infinite family of BCC-algebras and prove that it is a dual BCC-algebras (dBCC-algebras), we call the external direct product dBCC-algebra induced by BCC-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam. We find the result of the external direct product of special subsets of BCC-algebras. Also, we introduce the concept of the weak direct product dBCC-algebras. Finally, we provide several fundamental theorems of (anti-)BCC-homomorphisms in view of the external direct product dBCC-algebras.

Keywords: BCC-algebra, dBCC-algebra, external direct product, weak direct product, BCC-homomorphism, anti-BCC-homomorphism.

2020 MSC: 03G25, 20K25.

1. Introduction and Preliminaries

Imai and Iséki defined two kinds of abstract algebras, BCK-algebras and BCI-algebras, which have been intensively studied by numerous academics. The class of BCK-algebras is known to be a proper subclass of the class of BCI-algebras [15, 16]. In 2002, Neggers and Kim [29] constructed a new algebraic structure. They took some properties from BCI and BCK-algebras be called a B-algebra. Furthermore, Kim and Kim [22] introduced a new notion, called a BG-algebra which is a generalization of B-algebra. They obtained several isomorphism theorems of BG-algebras and related properties.

Iampan [11] developed the concept of UP-algebras in 2017, and it is known that the class of KU-algebras [30] is a proper subclass of the class of UP-algebras. It have been examined by several researchers,

*Corresponding author

Email addresses: chatsuda.chanmanee@gmail.com (Chatsuda Chanmanee), ronnason.c@psu.ac.th (Ronnason Chinram), rukchart.p@nsru.ac.th (Rukchart Prasertpong), pongpun.j@psru.ac.th (Pongpun Julatha), aiyared.ia@up.ac.th (Aiyared Iampan)

doi: 10.22436/jmcs.029.02.07

Received: 2022-06-29 *Revised:* 2022-07-16 *Accepted:* 2022-07-21

for example, graphs associated with commutative UP-algebras and a graph of equivalence classes of commutative UP-algebras by Ansari et al. in 2018 [2]. In the same year Senapati et al. [35] applied the cubic set structure in UP-algebras and proved the results based on them. In 2019, Satirad et al. [32] proved every nonempty set and every nonempty totally ordered set can be a UP-algebra. In 2020, Romano and Jun [31] introduced the concept of weak implicative UP-filters in UP-algebras. In 2022, Ansari et al. [4] defined the notion of a UP-algebra valued function on a set, investigated related properties, and established the binary block codes generated by UP-algebras valued function. Following that, Koam et al. [23] proposed and researched generalized UP-valued cut functions and their many features. Using the concept of generalized UP-valued cut functions, they created n-ary block-codes for UP-algebras. In this year, Jun et al. [17] have shown that the concept of UP-algebras (see [11]) and the concept of BCC-algebras (see [24]) are the same concept. Therefore, in this article and future research, our research team will use the name BCC instead of UP in honor of Komori, who first defined it in 1984.

The concept of the direct product [37] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group, and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [25] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras and we obtained some of their properties [26]. In the same year, Endam and Teves [9] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [40] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [37] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [21] defined the direct product of GK algebra. They derived the finite form of the direct product of GK algebra and function as well. They investigated and applied the concept of the direct product of GK algebra in GK function and GK kernel and obtained interesting results.

In this paper, we introduce the concept of the direct product of infinite family of BCC-algebras and prove that it is a dBCC-algebras, we call the external direct product dBCC-algebra induced by BCC-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam [25]. Moreover, we introduce the concept of the weak direct product dBCC-algebras. Finally, we discuss several (anti-)BCC-homomorphism theorems in view of the external direct product dBCC-algebras.

The concept of BCC-algebras (see [24]) can be redefined without the condition (1.1) as follows.

Definition 1.1 ([11])**.** An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *BCC-algebra* if it satisfies the following axioms:

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0), \tag{UP-1}$$
$$(\forall x \in X)(0 * x = x), \tag{UP-2}$$
$$(\forall x \in X)(x * 0 = 0), \tag{UP-3}$$
$$(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y). \tag{UP-4}$$

Example 1.2. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	4	5	6
2	0	0	0	3	0	0	6
3	0	0	2	0	4	0	0
4	0	0	2	3	0	0	6
5	0	0	2	3	4	0	6
6	0	0	2	3	4	5	0

Then $X = (X; *, 0)$ is a BCC-algebra.

For more studies and examples of BCC-algebras, see [2, 3, 7, 11, 12, 14, 23, 32, 33, 35, 36].

Definition 1.3 ([8])**.** An algebra $X = (X; *, 0)$ of type $(2, 0)$ is called a *dual BCC-algebra* (dBCC-algebra) if it satisfies (UP-4) and the following axioms:

$$(\forall x, y, z \in X)(((z * x) * (y * x)) * (z * y) = 0), \tag{DUP-1}$$

$$(\forall x \in X)(x * 0 = x), \tag{DUP-2}$$

$$(\forall x \in X)(0 * x = 0). \tag{DUP-3}$$

The binary relation $\leqslant$ on a dBCC-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leqslant y \Leftrightarrow x * y = 0).$$

Example 1.4. Let $X = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	1
2	2	0	0	3	3	3	2
3	3	0	0	0	3	3	3
4	4	0	0	0	0	4	4
5	5	0	0	0	0	0	5
6	6	0	0	0	0	0	0

Then $X = (X; *, 0)$ is a dBCC-algebra.

Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be BCC-algebras. A map $\varphi : A \to B$ is called a *BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-BCC-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker \varphi$, is defined to be the $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker \varphi$ is a BCC-ideal of A, and $\ker \varphi = \{0_A\}$ if and only if φ is injective. A (anti-)BCC-homomorphism φ is called a (anti-) BCC-monomorphism, (anti-)BCC-epimorphism, or (anti-)BCC-isomorphism if φ is injective, surjective, or bijective, respectively.

In a BCC-algebra $X = (X; *, 0)$, the following assertions are valid (see [11, 12]).

$$(\forall x \in X)(x * x = 0), \tag{1.1}$$

$$(\forall x, y, z \in X)(x * y = 0, y * z = 0 \Rightarrow x * z = 0),$$
$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (z * x) * (z * y) = 0),$$
$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow (y * z) * (x * z) = 0),$$
$$(\forall x, y \in X)(x * (y * x) = 0),$$
$$(\forall x, y \in X)((y * x) * x = 0 \Leftrightarrow x = y * x),$$
$$(\forall x, y \in X)(x * (y * y) = 0),$$
$$(\forall u, x, y, z \in X)((x * (y * z)) * (x * ((u * y) * (u * z))) = 0),$$
$$(\forall u, x, y, z \in X)((((u * x) * (u * y)) * z) * ((x * y) * z) = 0),$$
$$(\forall x, y, z \in X)(((x * y) * z) * (y * z) = 0),$$
$$(\forall x, y, z \in X)(x * y = 0 \Rightarrow x * (z * y) = 0),$$
$$(\forall x, y, z \in X)(((x * y) * z) * (x * (y * z)) = 0),$$
$$(\forall u, x, y, z \in X)(((x * y) * z) * (y * (u * z)) = 0).$$

According to [11], the binary relation $\leqslant$ on a BCC-algebra $X = (X; *, 0)$ is defined as follows:

$$(\forall x, y \in X)(x \leqslant y \Leftrightarrow x * y = 0).$$

Definition 1.5. A BCC-algebra $X = (X; *, 0)$ is said to be

(i) *bounded* if there is an element $1 \in X$ such that $1 \leqslant x$ for all $x \in X$, that is,

$$(\forall x \in X)(1 * x = 0); \tag{Bounded}$$

(ii) *meet-commutative* [34] if it satisfies the identity

$$(\forall x, y \in X)(x \wedge y = y \wedge x), \tag{Meet-commutative}$$

where

$$(\forall x, y \in X)(x \wedge y = (y * x) * x). \tag{Meet}$$

Definition 1.6. A dBCC-algebra $X = (X; *, 0)$ is said to be

(i) *bounded* if there is an element $1 \in X$ such that $x \leqslant 1$ for all $x \in X$, that is,

$$(\forall x \in X)(x * 1 = 0); \tag{Bounded}$$

(ii) *join-commutative* if it satisfies the identity

$$(\forall x, y \in X)(x \vee y = y \vee x), \tag{Join-commutative}$$

where

$$(\forall x, y \in X)(x \vee y = x * (x * y)). \tag{Join}$$

Example 1.7. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	1
2	2	0	0	3	2
3	3	0	0	0	3
4	4	0	0	0	0

Then $X = (X; *, 0, 1)$ is a bounded dBCC-algebra.

Example 1.8. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	3	2	1
2	2	0	0	2	2
3	3	0	3	0	3
4	4	4	4	4	0

Then $X = (X; *, 0)$ is a join-commutative dBCC-algebra.

Definition 1.9 ([10, 11, 13, 18–20, 38])**.** A nonempty subset S of a BCC-algebra $X = (X; *, 0)$ is called

(i) a *BCC-subalgebra* of X if it satisfies the following condition:

$$(\forall x, y \in S)(x * y \in S); \tag{1.2}$$

(ii) a *near BCC-filter* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow x * y \in S); \tag{1.3}$$

(iii) a *BCC-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.4}$$

$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S); \tag{1.5}$$

(iv) an *implicative BCC-filter* of X if it satisfies the condition (1.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S); \tag{1.6}$$

(v) a *comparative BCC-filter* of X if it satisfies the condition (1.4) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S); \tag{1.7}$$

(vi) a *shift BCC-filter* of X if it satisfies the condition (1.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S); \tag{1.8}$$

(vii) a *BCC-ideal* of X if it satisfies the condition (1.4) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S); \tag{1.9}$$

(viii) a *strong BCC-ideal* of X if it satisfies the condition (1.4) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{1.10}$$

Definition 1.10. A nonempty subset S of a dBCC-algebra $X = (X; *, 0)$ is called

(i) a *dBCC-subalgebra* of X if it satisfies the condition (1.2);

(ii) a *near dBCC-filter* of X if it satisfies the following condition:

$$(\forall x, y \in X)(y \in S \Rightarrow y * x \in S); \tag{1.11}$$

(iii) a *dBCC-filter* of X if it satisfies the following conditions:

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.12}$$

$$(\forall x, y \in X)(y * x \in S, x \in S \Rightarrow y \in S); \tag{1.13}$$

(iv) an *implicative dBCC-filter* of X if it satisfies the condition (1.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S); \tag{1.14}$$

(v) a *comparative dBCC-filter* (CBCCF) of X if it satisfies the condition (1.12) and the following condition:

$$(\forall x, y, z \in X)(x * ((y * z) * y) \in S, x \in S \Rightarrow y \in S); \tag{1.15}$$

(vi) a *shift dBCC-filter* of X if it satisfies the condition (1.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, x \in S \Rightarrow ((z * y) * y) * z \in S); \tag{1.16}$$

(vii) a *dBCC-ideal* of X if it satisfies the condition (1.12) and the following condition:

$$(\forall x, y, z \in X)(x * (y * z) \in S, y \in S \Rightarrow x * z \in S); \tag{1.17}$$

(viii) a *strong dBCC-ideal* of X if it satisfies the condition (1.12) and the following condition:

$$(\forall x, y, z \in X)((z * y) * (z * x) \in S, y \in S \Rightarrow x \in S). \tag{1.18}$$

2. External direct product dBCC-algebras

Lingcong and Endam [25] discussed the notion of the direct product of *B*-algebras, 0-commutative *B*-algebras, and *B*-homomorphisms and obtained related properties, one of which is a direct product of two *B*-algebras, which is also a *B*-algebra. Then, they extended the concept of the direct product of *B*-algebra to finite family *B*-algebra, and some of the related properties were investigated as follows.

Definition 2.1 ([25])**.** Let $(X_i; *_i)$ be an algebra for each $i \in \{1, 2, \dots, k\}$. Define the *direct product* of algebras $X_1, X_2, \dots, X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times \dots \times X_k = \{(x_1, x_2, \dots, x_k) \mid x_i \in X_i \ \forall i = 1, 2, \dots, k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, \dots, x_k) \otimes (y_1, y_2, \dots, y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, \dots, x_k *_k y_k)$$

for all $(x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k) \in \prod_{i=1}^{k} X_i$.

Now, we extend the concept of the direct product to infinite family of BCC-algebras and provide some of its properties.

Definition 2.2. Let X_i be a nonempty set for each $i \in I$. Define the *external direct product* of sets X_i for all $i \in I$ to be the set $\prod_{i \in I} X_i$, where

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i \in I} X_i$ with a function $(x_i)_{i \in I} : I \to \bigcup_{i \in I} X_i$, where $i \mapsto x_i \in X_i$ for all $i \in I$.

Remark 2.3. Let X_i be a nonempty set and S_i a subset of X_i for all $i \in I$. Then $\prod_{i\in I} S_i$ is a nonempty subset of the external direct product $\prod_{i\in I} X_i$ if and only if S_i is a nonempty subset of X_i for all $i \in I$.

Definition 2.4. Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\boxtimes$ on the external direct product $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes)$ as follows:

$$(\forall (x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i)((x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (y_i *_i x_i)_{i\in I}).$$

We shall show that $\boxtimes$ is a binary operation on $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $y_i *_i x_i \in X_i$ for all $i \in I$. Then $(y_i *_i x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that

$$(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (y_i *_i x_i)_{i\in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x'_i)_{i\in I}, (y'_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x'_i)_{i\in I} = (y'_i)_{i\in I}$. We shall show that $(x_i)_{i\in I} \boxtimes (x'_i)_{i\in I} = (y_i)_{i\in I} \boxtimes (y'_i)_{i\in I}$. Then

$$x_i = y_i \text{ for all } i \in I \text{ and } x'_i = y'_i \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x'_i *_i x_i = y'_i *_i y_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \boxtimes (x'_i)_{i\in I} = (x'_i *_i x_i)_{i\in I} = (y'_i *_i y_i)_{i\in I} = (y_i)_{i\in I} \boxtimes (y'_i)_{i\in I}.$$

Hence, $\boxtimes$ is a binary operation on $\prod_{i\in I} X_i$.

Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(f_{x_i}(j) = \begin{cases} x_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right). \tag{2.1}$$

Then $f_{x_i} \in \prod_{i\in I} X_i$.

Lemma 2.5. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \boxtimes f_{y_i} = f_{y_i *_i x_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i, & \text{if } j = i, \\ 0_j *_j 0_j, & \text{otherwise,} \end{cases}\right).$$

By (1.1), we have

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right).$$

By (2.1), we have $f_{x_i} \boxtimes f_{y_i} = f_{y_i *_i x_i}$. □

The following theorem shows that the direct product of BCC-algebras in term of infinite family of BCC-algebras is also.

Theorem 2.6. $X_i = (X_i; *_i, 0_i)$ *is a BCC-algebra for all* $i \in I$ *if and only if* $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ *is a dBCC-algebra, where the binary operation* $\boxtimes$ *is defined in Definition 2.4.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$.

(i) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (UP-1), we have $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$ for all $i \in I$. Thus

$$\begin{aligned}(((z_i)_{i\in I} \boxtimes (x_i)_{i\in I}) \boxtimes ((y_i)_{i\in I} \boxtimes (x_i)_{i\in I})) \boxtimes ((z_i)_{i\in I} \boxtimes (y_i)_{i\in I})\\ &= ((x_i *_i z_i)_{i\in I} \boxtimes (x_i *_i y_i)_{i\in I}) \boxtimes (y_i *_i z_i)_{i\in I}\\ &= ((x_i *_i y_i) *_i (x_i *_i z_i))_{i\in I} \boxtimes (y_i *_i z_i)_{i\in I}\\ &= ((y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)))_{i\in I} = (0_i)_{i\in I}.\end{aligned}$$

(ii) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (UP-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(x_i)_{i\in I} \boxtimes (0_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(iii) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (UP-3), we have $x_i *_i 0_i = 0_i$ for all $i \in I$. Thus

$$(0_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i 0_i)_{i\in I} = (0_i)_{i\in I}.$$

UP-4: Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (0_i)_{i\in I}$. Then $(y_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$ and $(x_i *_i y_i)_{i\in I} = (0_i)_{i\in I}$, so $y_i *_i x_i = 0_i$ and $x_i *_i y_i = 0_i$ for all $i \in I$. Since X_i satisfies (UP-4), we have $x_i = y_i$ for all $i \in I$. Therefore, $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4. Let $i \in I$.

UP-1: Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies (i), we have $((f_{z_i} \boxtimes f_{x_i}) \boxtimes (f_{y_i} \boxtimes f_{x_i})) \boxtimes (f_{z_i} \boxtimes f_{y_i}) = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left(((f_{z_i} \boxtimes f_{x_i}) \boxtimes (f_{y_i} \boxtimes f_{x_i})) \boxtimes (f_{z_i} \boxtimes f_{y_i}))(j) = \begin{cases}(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)), & \text{if } j = i,\\ (0_j *_j 0_j) *_j ((0_j *_j 0_j) *_j (0_j *_j 0_j)), & \text{otherwise,}\end{cases}\right),$$

this implies that $(y_i *_i z_i) *_i ((x_i *_i y_i) *_i (x_i *_i z_i)) = 0_i$.

UP-2: Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies (ii), we have $f_{x_i} \boxtimes (0_i)_{i\in I} = f_{x_i}$. Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes (0_i)_{i\in I})(j) = \begin{cases}0_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

this implies that $0_i *_i x_i = x_i$.

UP-3: Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ satisfies ((iii)), we have $(0_i)_{i\in I} \boxtimes f_{x_i} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left(((0_i)_{i\in I} \boxtimes f_{x_i})(j) = \begin{cases}x_i *_i 0_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

this implies that $x_i *_i 0_i = 0_i$.

UP-4: Let $x_i, y_i \in X_i$ be such that $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases}y_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \boxtimes f_{x_i})(j) = \begin{cases} x_i *_i y_i, & \text{if } j = i, \\ 0_j *_j 0_j, & \text{otherwise,} \end{cases}\right).$$

By assumption and (1.1), we have

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} 0_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right) \quad \text{and} \quad (\forall j \in I)\left((f_{y_i} \boxtimes f_{x_i})(j) = \begin{cases} 0_i, & \text{if } j = i, \\ 0_j, & \text{otherwise,} \end{cases}\right).$$

Thus $f_{x_i} \boxtimes f_{y_i} = (0_i)_{i\in I}$ and $f_{y_i} \boxtimes f_{x_i} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ satisfies (UP-4), we have $f_{x_i} = f_{y_i}$. Therefore, $x_i = y_i$.

Hence, $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$. □

We call the dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ in Theorem 2.6 the external direct product dBCC-algebra induced by a BCC-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Theorem 2.7. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a bounded BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; , (0_i)_{i\in I})$ is a bounded BCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.6, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4. We are left to prove that X_i is bounded for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is bounded.

Assume that X_i is bounded for all $i \in I$. Then there exists $1_i \in X_i$ be such that $1_i \leqslant x_i$ for all $x_i \in X_i$. That is, $1_i *_i x_i = 0_i$ for all $i \in I$. Now, $(1_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Thus

$$(x_i)_{i\in I} \boxtimes (1_i)_{i\in I} = (1_i *_i x_i)_{i\in I} = (0_i)_{i\in I}.$$

That is, $(x_i)_{i\in I} \leqslant (1_i)_{i\in I}$. Hence, $\prod_{i\in I} X_i$ is bounded.

Conversely, assume that $\prod_{i\in I} X_i$ is bounded. Then there exists $(1_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $(x_i)_{i\in I} \leqslant (1_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. That is, $(x_i)_{i\in I} \boxtimes (1_i)_{i\in I} = (0_i)_{i\in I}$ for all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $i \in I$. Now, $1_i \in X_i$. Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since $\prod_{i\in I} X_i$ is bounded, we have $f_{x_i} \boxtimes (1_i)_{i\in I} = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left((f_{x_i} \boxtimes (1_i)_{i\in I})(j) = \begin{cases} 1_i *_i x_i, & \text{if } j = i, \\ 0_j *_j 0_j, & \text{otherwise,} \end{cases}\right),$$

this implies that $1_i *_i x_i = 0_i$. That is, $1_i \leqslant x_i$. Hence, X_i is bounded for all $i \in I$. □

Theorem 2.8. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then X_i is a meet-commutative BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a join-commutative BCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4.*

Proof. By Theorem 2.6, we have $X_i = (X_i; *_i, 0_i)$ is a BCC-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$ is a dBCC-algebra, where the binary operation $\boxtimes$ is defined in Definition 2.4. We are left to prove that X_i is meet-commutative for all $i \in I$ if and only if $\prod_{i\in I} X_i$ is join-commutative.

Assume that X_i is meet-commutative for all $i \in I$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i is meet-commutative, we have $x_i \wedge y_i = y_i \wedge x_i$ for all $i \in I$. That is, $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$ for all $i \in I$. Thus

$$\begin{aligned}(x_i)_{i\in I} \vee (y_i)_{i\in I} &= (x_i)_{i\in I} \boxtimes ((x_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \\ &= (x_i)_{i\in I} \boxtimes (y_i *_i x_i)_{i\in I} \\ &= ((y_i *_i x_i) *_i x_i)_{i\in I}\end{aligned}$$

$$
\begin{aligned}
&= ((x_i *_i y_i) *_i y_i)_{i\in I} \\
&= (y_i)_{i\in I} \boxtimes (x_i *_i y_i)_{i\in I} \\
&= (y_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes (x_i)_{i\in I}) \\
&= (y_i)_{i\in I} \vee (x_i)_{i\in I}.
\end{aligned}
$$

Hence, $\prod_{i\in I} X_i$ is join-commutative.

Conversely, assume that $\prod_{i\in I} X_i$ is join-commutative. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Since $\prod_{i\in I} X_i$ is join-commutative, we have $f_{x_i} \vee f_{y_i} = f_{y_i} \vee f_{x_i}$. That is, $f_{x_i} \boxtimes (f_{x_i} \boxtimes f_{y_i}) = f_{y_i} \boxtimes (f_{y_i} \boxtimes f_{x_i})$. By Lemma 2.5, we have $f_{(y_i *_i x_i) *_i x_i} = f_{(x_i *_i y_i) *_i y_i}$. By (2.1), we have $(y_i *_i x_i) *_i x_i = (x_i *_i y_i) *_i y_i$. That is, $x_i \wedge y_i = y_i \wedge x_i$. Hence, X_i is meet-commutative for all $i \in I$. □

Next, we introduce the concept of the weak direct product of infinite family of dBCC-algebras and obtain some of its properties as follows.

Definition 2.9. Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Define the *weak direct product dBCC-algebra* induced by X_i for all $i \in I$ to be the structure $\prod^w_{i\in I} X_i = (\prod^w_{i\in I} X_i; \boxtimes)$, where

$$\prod^w_{i\in I} X_i = \left\{ (x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite} \right\}.$$

Then $(0_i)_{i\in I} \in \prod^w_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 2.10. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra for all $i \in I$. Then $\prod^w_{i\in I} X_i$ is a dBCC-subalgebra of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^w_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^w_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} 0_j *_j x_j, & \text{if } j \in I_1 - I_2, \\ y_j *_j x_j, & \text{if } j \in I_1 \cap I_2, \\ y_j *_j 0_j, & \text{if } j \in I_2 - I_1, \\ 0_j *_j 0_j, & \text{otherwise}, \end{cases} \right).$$

By (UP-2) and (UP-3), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j) = \begin{cases} x_j, & \text{if } j \in I_1 - I_2, \\ y_j *_j x_j, & \text{if } j \in I_1 \cap I_2, \\ 0_j, & \text{if } j \in I_2 - I_1, \\ 0_j, & \text{otherwise}, \end{cases} \right).$$

This implies that the number of such $((x_i)_{i\in I} \boxtimes (y_i)_{i\in I})(j)$ is not more than $|I_1|$, that is, it is finite. Thus $(x_i)_{i\in I} \boxtimes (y_i)_{i\in I} \in \prod^w_{i\in I} X_i$. Hence, $\prod^w_{i\in I} X_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 2.11. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-subalgebra of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (1.2), we have $x_i *_i y_i \in S_i$ for all $i \in I$, so $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By (1.2) and Lemma 2.5, we have $f_{x_i *_i y_i} = f_{y_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a BCC-subalgebra of X_i for all $i \in I$. □

Theorem 2.12. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a near BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a near dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a near BCC-filter of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.3, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $y_i \in S_i$ for all $i \in I$, it follows from (1.3) that $x_i *_i y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a near dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a near dBCC-filter of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.3, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By (1.11) and by Lemma 2.5, we have $f_{x_i *_i y_i} = f_{y_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i y_i \in S_i$. Hence, S_i is a near BCC-filter of X_i for all $i \in I$. □

Theorem 2.13. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (1.5) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $f_{y_i} \boxtimes f_{x_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.13), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $y_i \in S_i$. Hence, S_i is a BCC-filter of X_i for all $i \in I$. □

Theorem 2.14. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an implicative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an implicative dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an implicative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (1.6) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an implicative dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an implicative dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \boxtimes f_{x_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.14) and Lemma 2.5, we have $f_{x_i *_i z_i} = f_{z_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is an implicative BCC-filter of X_i for all $i \in I$. □

Theorem 2.15. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a comparative BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a comparative dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a comparative BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((y_i)_{i\in I} \boxtimes ((z_i)_{i\in I} \boxtimes (y_i)_{i\in I})) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i ((y_i *_i z_i) *_i y_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$, it follows from (1.7) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a comparative dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a comparative dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i ((y_i *_i z_i) *_i y_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{y_i} \boxtimes (f_{z_i} \boxtimes f_{y_i})) \boxtimes f_{x_i} = f_{x_i *_i ((y_i *_i z_i) *_i y_i)} \in \prod_{i\in I} S_i$. By (1.15), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.1), we have $y_i \in S_i$. Hence, S_i is a comparative BCC-filter of X_i for all $i \in I$. □

Theorem 2.16. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a shift BCC-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a shift dBCC-filter of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a shift BCC-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$, it follows from (1.8) that $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes ((y_i)_{i\in I} \boxtimes (z_i)_{i\in I})) = (((z_i *_i y_i) *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a shift dBCC-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a shift dBCC-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (1.16) and Lemma 2.5, we have $f_{((z_i *_i y_i) *_i y_i) *_i z_i} = f_{z_i} \boxtimes (f_{y_i} \boxtimes (f_{y_i} \boxtimes f_{z_i})) \in \prod_{i\in I} S_i$. By (2.1), we have $((z_i *_i y_i) *_i y_i) *_i z_i \in S_i$. Hence, S_i is a shift BCC-filter of X_i for all $i \in I$. □

Theorem 2.17. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a dBCC-ideal of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \boxtimes (y_i)_{i\in I}) \boxtimes (x_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$, it follows from (1.9) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(z_i)_{i\in I} \boxtimes (x_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a dBCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a dBCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{z_i} \boxtimes f_{y_i}) \boxtimes f_{x_i} = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$. By (1.17) and Lemma 2.5, we have $f_{x_i *_i z_i} = f_{z_i} \boxtimes f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i *_i z_i \in S_i$. Hence, S_i is a BCC-ideal of X_i for all $i \in I$. □

Theorem 2.18. *Let $X_i = (X_i; *_i, 0_i)$ be a BCC-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong BCC-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong dBCC-ideal of the external direct product dBCC-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \boxtimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a strong BCC-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((x_i)_{i\in I} \boxtimes (z_i)_{i\in I}) \boxtimes ((y_i)_{i\in I} \boxtimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((z_i *_i y_i) *_i (z_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and $y_i \in S_i$, it follows from (1.10) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong dBCC-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a strong dBCC-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(z_i *_i y_i) *_i (z_i *_i x_i) \in S_i$ and

$y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.1). By Lemma 2.5, we have $(f_{x_i} \boxtimes f_{z_i}) \boxtimes (f_{y_i} \boxtimes f_{z_i}) = f_{(z_i *_i y_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$. By (1.18), we have $f_{x_i} \in \prod_{i\in I} S_i$. By (2.1), we have $x_i \in S_i$. Hence, S_i is a strong BCC-ideal of X_i for all $i \in I$. □

Moreover, we discuss several BCC-homomorphism theorems in view of the external direct product of dBCC-algebras.

Definition 2.19 ([6]). Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}).$$

Then $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function (see [6]).

Theorem 2.20 ([6]). *Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$.*

(i) *ψ_i is injective for all $i \in I$ if and only if ψ is injective which is defined in Definition 2.19.*

(ii) *ψ_i is surjective for all $i \in I$ if and only if ψ is surjective.*

(iii) *ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.*

Theorem 2.21. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is a BCC-homomorphism for all $i \in I$ if and only if ψ is a dBCC-homomorphism which is defined in Definition 2.19;*

(ii) *ψ_i is a BCC-monomorphism for all $i \in I$ if and only if ψ is a dBCC-monomorphism;*

(iii) *ψ_i is a BCC-epimorphism for all $i \in I$ if and only if ψ is a dBCC-epimorphism;*

(iv) *ψ_i is a BCC-isomorphism for all $i \in I$ if and only if ψ is a dBCC-isomorphism;*

(v) *$\ker\psi = \prod_{i\in I} \ker\psi_i$ and $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.*

Proof.

(i) Assume that ψ_i is a BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) = \psi(x'_i *_i x_i)_{i\in I} &= (\psi_i(x'_i *_i x_i))_{i\in I}\\ &= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I}\\ &= (\psi_i(x_i))_{i\in I} \boxtimes (\psi_i(x'_i))_{i\in I} = \psi(x_i)_{i\in I} \boxtimes \psi(x'_i)_{i\in I}.\end{aligned}$$

Hence, ψ is a dBCC-homomorphism.

Conversely, assume that ψ is a dBCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (2.1). Since ψ is a dBCC-homomorphism, we have $\psi(f_{x_i} \boxtimes f_{y_i}) = \psi(f_{x_i}) \boxtimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i), & \text{if } j = i,\\ \psi_j(0_j *_j 0_j), & \text{otherwise,}\end{cases}\right). \tag{2.2}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i), & \text{if } j = i, \\ \psi_j(0_j), & \text{otherwise,} \end{cases}\right) \quad \text{and} \quad (\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i), & \text{if } j = i, \\ \psi_j(0_j), & \text{otherwise,} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \boxtimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i), & \text{if } j = i, \\ \psi_j(0_j) \circ_j \psi_j(0_j), & \text{otherwise,} \end{cases}\right). \tag{2.3}$$

By (2.2) and (2.3), we have $\psi_i(y_i *_i x_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is a BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.20 (i).

(iii) It is straightforward from (i) and Theorem 2.20 (ii).

(iv) It is straightforward from (i) and Theorem 2.20 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned} (x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I} \\ &\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I \\ &\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I \\ &\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i. \end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned} (y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I} \\ &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I} \\ &\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I \\ &\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i). \end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

Finally, we discuss several anti-BCC-homomorphism theorems in view of the external direct product of dBCC-algebras.

Theorem 2.22. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be BCC-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is an anti-BCC-homomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-homomorphism which is defined in Definition 2.19;*

(ii) *ψ_i is an anti-BCC-monomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-monomorphism;*

(iii) *ψ_i is an anti-BCC-epimorphism for all $i \in I$ if and only if ψ is an anti-dBCC-epimorphism;*

(iv) *ψ_i is an anti-BCC-isomorphism for all $i \in I$ if and only if ψ is an anti-dBCC-isomorphism.*

Proof.

(i) Assume that ψ_i is an anti-BCC-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}\psi((x_i)_{i\in I} \boxtimes (x'_i)_{i\in I}) = \psi(x'_i *_i x_i)_{i\in I} &= (\psi_i(x'_i *_i x_i))_{i\in I}\\ &= (\psi_i(x_i) *_i \psi_i(x'_i))_{i\in I}\\ &= (\psi_i(x'_i))_{i\in I} \boxtimes (\psi_i(x_i))_{i\in I} = \psi(x'_i)_{i\in I} \boxtimes \psi(x_i)_{i\in I}.\end{aligned}$$

Hence, ψ is an anti-dBCC-homomorphism.

Conversely, assume that ψ is an anti-dBCC-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.1). Since ψ is an anti-dBCC-homomorphism, we have $\psi(f_{x_i} \boxtimes f_{y_i}) = \psi(f_{y_i}) \boxtimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} y_i *_i x_i, & \text{if } j = i,\\ 0_j *_j 0_j, & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \boxtimes f_{y_i})(j) = \begin{cases} \psi_i(y_i *_i x_i), & \text{if } j = i,\\ \psi_j(0_j *_j 0_j), & \text{otherwise,}\end{cases}\right). \tag{2.4}$$

Since

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right) \quad \text{and} \quad (\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i), & \text{if } j = i,\\ \psi_j(0_j), & \text{otherwise,}\end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{y_i}) \boxtimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i), & \text{if } j = i,\\ \psi_j(0_j) \circ_j \psi_j(0_j), & \text{otherwise,}\end{cases}\right). \tag{2.5}$$

By (2.4) and (2.5), we have $\psi_i(y_i *_i x_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is an anti-BCC-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.20 (i).

(iii) It is straightforward from (i) and Theorem 2.20 (ii).

(iv) It is straightforward from (i) and Theorem 2.20 (iii). □

3. Conclusions and Future Work

In this paper, we have introduced the concept of the direct product of infinite family of BCC-algebras and prove that it is a dBCC-algebras, we call the external direct product dBCC-algebra induced by BCC-algebras, which is a general concept of the direct product in the sense of Lingcong and Endam [25]. We proved that the external direct product of BCC-algebras is dBCC-algebras, the external direct product of a bounded BCC-algebra is a bounded dBCC-algebra, and the external direct product of a meet-commutative BCC-algebra is a join-commutative dBCC-algebra. Also, we have introduced the concept of the weak direct product dBCC-algebras. We proved that the weak direct product of BCC-algebras is dBCC-subalgebras and the external direct product of BCC-subalgebras (resp., near BCC-filters, BCC-filters, comparative BCC-filters, shift BCC-filters, implicative BCC-filters, BCC-ideals, strong BCC-ideals) is a dBCC-subalgebra (resp., near dBCC-filter, dBCC-filter, comparative dBCC-filter, shift dBCC-filter, implicative dBCC-filter, dBCC-ideal, strong dBCC-ideal) of the external direct product dBCC-algebras. Finally, we have provided several fundamental theorems of (anti-)BCC-homomorphisms in view of the external direct product dBCC-algebras.

Based on the concept of the external direct product dBCC-algebras in this article, we can apply it to the study of the external direct product in other algebraic systems. Researching the external and weak direct products that we will study in the future will be the internal direct products dBCC-algebras.

The research topics of interest by our research team being studied in the external direct product dBCC-algebras are as follows:

(1) to study fuzzy set theory (with respect to a triangular norm) based on the concept of Somjanta et al. [38] and Thongarsa et al. [5, 39];

(2) to study bipolar fuzzy set theory based on the concept of Muhiuddin [27];

(3) to study interval-valued fuzzy set theory based on the concept of Muhiuddin et al. [28];

(4) to study interval-valued intuitionistic fuzzy set theory based on the concept of Senapati et al. [36].

Acknowledgments

This work was supported by the revenue budget in 2022, School of Science, University of Phayao, Thailand.

References

[1] G. A. Abebe, *On the theory of BRK-algebras*, Doctoral dissertation, Department of Mathematics, Addis Ababa University, (2018). 1
[2] M. A. Ansari, A. Haidar, A. N. A. Koam, *On a graph associated to UP-algebras*, Math. Comput. Appl., **23** (2018), 12 pages. 1, 1
[3] M. A. Ansari, A. N. A. Koam, A. Haider, *Rough set theory applied to UP-algebras*, Ital. J. Pure Appl. Math., **42** (2019), 388–402. 1
[4] M. A. Ansari, A. N. A. Koam, A. Haidar, *On binary block codes associated to UP-algebras*, Ital. J. Pure Appl. Math., **47** (2022), 205–220. 1
[5] P. Burandate, S. Thongarsa, A. Iampan, *Fuzzy sets in UP-algebras with respect to a triangular norm*, Konuralp J. Math., **7** (2019), 410–432. (1)
[6] C. Chanmanee, R. Chinram, R. Prasertpong, P. Julatha, A. Iampan, *Direct product of infinite family of B-algebras*, Eur. J. Pure Appl. Math., **15** (2022), 999-1014. 2.19, 2.20
[7] R. Chinram, A. Iampan, *Codewords generated by UP-valued functions*, AIMS Math., **6** (2021), 4771–4785. 1
[8] W. A. Dudek, X. H. Zhang, *On ideals and congruences in BCC-algebras*, Czechoslovak Math. J., **48** (1998), 21–29. 1.3
[9] J. C. Endam, R. C. Teves, *Direct product of BF-algebras*, Int. J. Algebra, **10** (2016), 125–132. 1
[10] T. Guntasow, S. Sajak, A. Jomkham, A. Iampan, *Fuzzy translations of a fuzzy set in UP-algebras*, J. Indones. Math. Soc., **23** (2017), 1–19. 1.9
[11] A. Iampan, *A new branch of the logical algebra: UP-algebras*, J. Algebra Relat. Topics, **5** (2017), 35–54. 1, 1.1, 1, 1, 1, 1.9
[12] A. Iampan, *Introducing fully UP-semigroups*, Discuss. Math. Gen. Algebra Appl., **38** (2018), 297–306. 1, 1
[13] A. Iampan, *Multipliers and near UP-filters of UP-algebras*, J. Discrete Math. Sci. Cryptogr., **24** (2021), 667–680. 1.9
[14] A. Iampan, M. Songsaeng, G. Muhiuddin, *Fuzzy duplex UP-algebras*, Eur. J. Pure Appl. Math., **13** (2020), 459–471. 1
[15] Y. Imai, K. Iséki, *On axiom systems of propositional calculi. XIV*, Proc. Japan Acad., **42** (1966), 19–22. 1
[16] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad., **42** (1966), 26–29. 1
[17] Y. B. Jun, B. Brundha, N. Rajesh, R. K. Bandaru, *$(3,2)$-Fuzzy UP-subalgebras and $(3,2)$-fuzzy UP-filters*, J. Mahani Math. Res. Cent., **11** (2022), 1–14. 1
[18] Y. B. Jun, A. Iampan, *Comparative and allied UP-filters*, Lobachevskii J. Math., **40** (2019), 60–66. 1.9
[19] Y. B. Jun, A. Iampan, *Implicative UP-filters*, Afr. Mat., **30** (2019), 1093–1101.
[20] Y. B. Jun, A. Iampan, *Shift UP-filters and decompositions of UP-filters in UP-algebras*, Missouri J. Math. Sci., **31** (2019), 36–45. 1.9
[21] J. Kavitha, R. Gowri, *Direct product of GK algebra*, Indian J. Technol., **14** (2018), 2802–2805. 1
[22] C. B. Kim, H. S. Kim, *On BG-algebras*, Demonstratio Math., **41** (2008), 497–505. 1
[23] A. N. A. Koam, A. Haider, M. A. Ansari, *Generalized cut functions and n-ary block codes on UP-algebras*, J. Math., **2022** (2022), 5 pages. 1, 1
[24] Y. Komori, *The class of BCC-algebras is not a variety*, Math. Japon., **29** (1984), 391–394. 1
[25] J. A. V. Lingcong, J. C. Endam, *Direct product of B-algebras*, Int. J. Algebra, **10** (2016), 33–40. 1, 2, 2.1, 3

[26] J. A. V. Lingcong, J. C. Endam, *Mappings of the direct product of B-algebras*, Int. J. Algebra, **10** (2016), 133–140. 1
[27] G. Muhiuddin, *Bipolar fuzzy* KU-*subalgebras/ideals of* KU-*algebras*, Ann. Fuzzy Math. Inform., **8** (2014), 409–418. (2)
[28] G. Muhiuddin, D. Al-Kadi, A. Mahboob, *More general form of interval-valued fuzzy ideals of BCK/BCI-algebras*, Secur. Commun. Netw., **2021** (2021), 10 pages. (3)
[29] J. Neggers, H. S. Kim, *On* B-*algebras*, Mat. Vesnik, **54** (2002), 21–29. 1
[30] C. Prabpayak, U. Leerawat, *On ideals and congruences in KU-algebras*, Sci. Magna, **5** (2009), 54–57. 1
[31] D. A. Romano, Y. B. Jun, *Weak implicative UP-filters of UP-algebras*, Open J. Math. Sci., **4** (2020), 442–447. 1
[32] A. Satirad, P. Mosrijai, A. Iampan, *Formulas for finding UP-algebras*, Int. J. Math. Comput. Sci., **14** (2019), 403–409. 1, 1
[33] A. Satirad, P. Mosrijai, A. Iampan, *Generalized power UP-algebras*, Int. J. Math. Comput. Sci., **14** (2019), 17–25. 1
[34] K. Sawika, R. Intasan, A. Kaewwasri, A. Iampan, *Derivations of UP-algebras*, Korean J. Math., **24** (2016), 345–367. (ii)
[35] T. Senapati, Y. B. Jun, K. P. Shum, *Cubic set structure applied in UP-algebras*, Discrete Math. Algorithms Appl., **10** (2018), 23 pages. 1, 1
[36] T. Senapati, G. Muhiuddin, K. P. Shum, *Representation of UP-algebras in interval-valued intuitionistic fuzzy environment*, Ital. J. Pure Appl. Math., **38** (2017), 497–517. 1, (4)
[37] A. Setiani, S. Gemawati, L. Deswita, *Direct product of BP-algebra*, Int. J. Math. Trends Tech., **66** (2020), 63–66. 1
[38] J. Somjanta, N. Thuekaew, P. Kumpeangkeaw, A. Iampan, *Fuzzy sets in UP-algebras*, Ann. Fuzzy Math. Inform., **12** (2016), 739–756. 1.9, (1)
[39] S. Thongarsa, P. Burandate, A. Iampan, *Some operations of fuzzy sets in UP-algebras with respect to a triangular norm*, Ann. Commun. Math., **2** (2019), 1–10. (1)
[40] S. Widianto, S. Gemawati, Kartini, *Direct product in BG-algebras*, Int. J. Algebra, **13** (2019), 239–247. 1