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Conformable Gehring inequalities and conformable higher integrability

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Abstract

In this paper, we prove some reverse conformable inequalities with weights and employ them to prove some conformable inequalities of Gehring type. Moreover, we prove some interpolation theorems which are powerful tools in the study of operators in function spaces. Our results develop a technique based on the applications of a refinement of conformable inequalities.

Keywords: Conformable Gehring's inequality, conformable Hölder's inequality, reverse inequality.

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1. Introduction

We fix an interval $I_0 \subset \mathbb{R}_+ = [0, \infty)$, and consider the subinterval I of I_0 of the form $[0, s]$, for $0 < s < \infty$ and donate by $|I|$ the Lebesgue measure of I. The nonnegative weight ω is said to belong to the Muckenhoupt class $A_p(\mathcal{C})$ on the interval I_0 for $p > 1$ and $\mathcal{C} > 1$ (independent of p) if the inequality

$$\frac{1}{|I|}\int_I \omega(x)dx \leqslant \mathcal{C}\left(\frac{1}{|I|}\int_I \omega^{\frac{1}{1-p}}(x)dx\right)^{1-p}, \tag{1.1}$$

holds for every subinterval $I \subset I_0$. For $p > 1$, we define the A_p-norm of the weight ω by

$$[A_p(\omega)] := \sup_{I\subset I_0}\left(\frac{1}{|I|}\int_I \omega(x)dx\right)\left(\frac{1}{|I|}\int_I \omega^{\frac{-1}{p-1}}(x)dx\right)^{p-1}.$$

The weight ω is said to belong to the Muckenhoupt class $A_1(\mathcal{C})$ on the interval I_0, if the inequality

$$\frac{1}{|I|}\int_I \omega(x)dx \leqslant \mathcal{C}\omega(x), \text{ for } \mathcal{C} > 1, \tag{1.2}$$

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holds for every subinterval $I \subset I_0$, and we define the A_1-norm by

$$[A_1(\omega)] := \sup_{I \subset I_0} \frac{1}{|I|} \left(\frac{1}{\inf \omega} \int_I \omega(x) dx \right).$$

In [14], Muckenhoupt proved that if ω is a nonincreasing weight satisfying condition (1.2), then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that

$$\frac{1}{|I|} \int_I \omega^p(x) dx \leqslant \frac{\mathcal{C}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|I|} \int_I \omega(x) dx \right)^p.$$

The authors in [3] improved the Muckenhoupt result by excluding the property of monotonicity on the weight ω by using a rearrangement ω^* of the function ω over the interval I and established the best constant. In particular, they proved that if ω is a nonincreasing weight satisfying condition (1.2) with $\mathcal{C} > 1$, then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that

$$\frac{1}{|I|} \int_I \omega^p(x) dx \leqslant \frac{\mathcal{C}^{1-p}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|I|} \int_I \omega(x) dx \right)^p.$$

Further in [14], Muckenhoupt proved the following result: if $1 < p < \infty$ and ω satisfies the A_p-condition (1.1) on the interval I, with constant $\mathcal{C}$, then there exist constants q and $\mathcal{C}_1$ depending on p and $\mathcal{C}$ such that $1 < q < p$ and ω satisfies the A_q-condition

$$\left(\frac{1}{|I|} \int_I \omega(x) dx \right) \subset \left(\frac{1}{|I|} \int_I \omega^{-\frac{1}{q-1}}(x) dx \right)^{q-1} \leqslant \mathcal{C}_1,$$

for every subinterval $I \subset I_0$. On other words, Muckenhoupt's result for *self-improving* property states that: if $\nu \in A_p(\mathcal{C})$, then there exists a constant $\epsilon > 0$ and a positive constant $\mathcal{C}_1$ such that $\nu \in A_{p-\epsilon}(\mathcal{C}_1)$, and then $A_p(\mathcal{C}) \subset A_{p-\epsilon}(\mathcal{C}_1)$. Despite of a variety of ideas related to weighted inequalities appeared with the birth of singular integrals, it was only in the 1970s that a better understanding of the subject was obtained and the full characterization of the weights ω for which the Hardy-Littlewood maximal operator

$$\mathcal{M}\omega(x) := \sup_{x \in I} \frac{1}{|I|} \int_I \omega(y) dy,$$

is bounded on $L^p_w(\mathbb{R})$ by means of the so-called A_p-*condition* was achieved by Muckenhoupt and published in 1972 (see[14]). Muckenhoupt's result became a landmark in the theory of weighted inequalities because most of the previously known results for classical operators had been obtained for special classes of weights (like power weights) and has been extended to cover several operators like Hardy operator, Hilbert operator, Calderón-Zygmund singular integral operators, fractional integral operators, etc. The weight ω is said to belong to the Gehring class $G_q(\mathcal{K})$, $1 < q < \infty$ for the interval I_0, if there exists a constant $\mathcal{K} > 1$ such that the inequality

$$\left(\frac{1}{|I|} \int_I \omega^q(x) dx \right)^{\frac{1}{q}} \leqslant \mathcal{K} \left(\frac{1}{|I|} \int_I \omega(x) dx \right), \tag{1.3}$$

holds for every subinterval $I \subset I_0$ and we define the G_q-norm by

$$[G_q(\omega)] := \sup_{I \subset I_0} \left[\left(\frac{1}{|I|} \int_I \omega^q(x) dx \right)^{\frac{1}{q}} \left(\frac{1}{|I|} \int_I \omega(x) dx \right)^{-1} \right]^{\frac{q}{q-1}}.$$

In [8, 9], Gehring proved that if (1.3) holds, then there exist $p > q$ and a positive constant $\mathcal{K}_1$ such that

$$\frac{1}{|I|} \int_I \omega^p(x) dx \leqslant \mathcal{K}_1 \left(\frac{1}{|I|} \int_I \omega(x) dx \right)^p \quad \text{for every } I \subset I_0.$$

On other words, Gehring's result for *self-improving* property states that: if $\omega \in G_q(\mathcal{K})$, then there exists $\epsilon > 0$ and a positive constant $\mathcal{K}_1$ such that $\omega \in G_{q+\epsilon}(\mathcal{K}_1)$, and then

$$G_q(\mathcal{K}) \subset G_{q+\epsilon}(\mathcal{K}_1).$$

The proof of the Gehring inequality based on the use of the Calderón-Zygmund decomposition. This and the scale structure of L^p-spaces. In [8], the author extended the Gehring's inequality by means of connecting it to the real method of interpolation by considering maximal operators, and via rearrangements reinterpreted the underlying estimates through the use of K-functionals. This technique allowed to quantify in a precise way, via reiteration, how the Calderón-Zygmund decompositions have to be reparameterized in order to characterize different L^p-spaces. Various integral inequalities and reverse integral inequalities (cf. [8, 9]) and their many variants and extensions are important in qualitative analysis of differential equations (see [2, 4, 5, 12]) and partial differential equations (see [11, 13]), in the study of weighted norm inequalities for classical operators of harmonic analysis, as well as in functional analysis. These inequalities also appear in different fields of analysis such as quasiconformal mappings, weighted Sobolev embedding theorems, and regularity theory of variational problems (see [6, 7, 10]).

Our aim in this paper, is to prove some reverse inequalities with weights and employ them to prove some conformable inequalities of Gehring types and some interpolation theorems. Moreover, our results develop a technique based on the applications of a refinement of conformable inequalities. The paper is organized as follows. Section 2 is devoted to preliminaries on conformable calculus. In Section 3, we state and prove some basic lemmas which will used to prove our main results. In Section 4, we state and prove the main results in our paper.

2. Preliminaries on conformable calculus

In this section, we present some preliminaries and definitions on conformable calculus. Throughout, we assume that $\alpha \in (0,1]$ and the weight f is a nonnegative locally α-integrable defined on $I_0 \subset \mathbb{R}_+$ and p is a positive real number. In addition, in our proofs, we will use the convention $0 \cdot \infty = 0$ and $0/0 = 0$.

Definition 2.1 ([1])**.** Let $\alpha \in (0,1]$. Given a function $f : [0,\infty) \to \mathbb{R}$. Then conformable derivative is defined as follows

$$D^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for $t > 0$.

Now, we present some useful properties of α-derivative as the following.

Theorem 2.2 ([1])**.** *Let* $\alpha \in (0,1]$, *and* f, g *be* α*-differentiable at a point* $t > 0$. *Then*

1. $D^{\alpha}(af(t) + bg(t)) = aD^{\alpha}f(t) + bD^{\alpha}g(t)$;
2. $D^{\alpha}(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$;
3. $D^{\alpha}(f(t)g(t)) = f(t)D^{\alpha}g(t) + g(t)D^{\alpha}f(t)$;
4. $D^{\alpha}(\frac{f(t)}{g(t)}) = \frac{g(t)D^{\alpha}f(t) - f(t)D^{\alpha}g(t)}{g^2(t)}$;
5. $D^{\alpha}(k) = 0$, *where* k *is a constant;*
6. *if* f *is differentiable, then* $D^{\alpha}f(t) = t^{1-\alpha}\frac{df(t)}{dt}$;
7. *if* f *and* g *are differentiable functions, then*

$$D^{\alpha}(f \circ g)(t) = t^{1-\alpha}g'(t)f'(g(t)) = D^{\alpha}g(t)f'(g(t)).$$

Definition 2.3. Let $\alpha \in (0,1]$. A function $f : [0,t] \to \mathbb{R}$ is conformable integrable on $[0,t]$ if the integral

$$I^{\alpha}(f)(t) = I(t^{\alpha-1}f)(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}}dx,$$

exists.

Theorem 2.4 ([1]). *Let* $a, b, c \in \mathbb{R}$. *Then*

$$\int_a^b D^\alpha f(t)g(t)d^\alpha t = f(t)g(t)\,|_a^b - \int_a^b f\,(t)D^\alpha g(t)d^\alpha t.$$

Lemma 2.5. *Let* $f : [0, \infty) \to \mathbb{R}$ *be a continuous function and* $0 < \alpha \leqslant 1$. *Then for all* $t > 0$ *we have*

1. $D^\alpha I^\alpha f(t) = f(t)$;
2. $I^\alpha D^\alpha f(t) = f(t) - f(0)$.

The conformable Hölder inequality is given by

$$\int_0^t |f(s)g(s)|\, d^\alpha s \leqslant \left(\int_0^t |f(s)|^p\, d^\alpha s\right)^{\frac{1}{p}} \left(\int_0^t |g(s)|^q\, d^\alpha s\right)^{\frac{1}{q}},$$

where $f,\ g \in C([a, b], \mathbb{R})$, $p > 1$ and $1/p + 1/q$. We say that f satisfies a reverse Hölder inequality if for some constants $p > q$ the following holds

$$\left(\int_0^t |f(s)|^p\, d^\alpha s\right)^{\frac{1}{p}} \leqslant \mathcal{C} \left(\int_0^t |f(s)|^q\, d^\alpha s\right)^{\frac{1}{q}}.$$

3. Auxiliary results

In this section, we prove the main results. We will introduce some definitions and notations that will be needed in the paper.

Definition 3.1. Let $f : I \longrightarrow \mathbb{R}^+$ be a nonnegative and nondecreasing. We define the conformable operator $\mathcal{M}^\alpha f : I \longrightarrow \mathbb{R}^+$ by

$$\mathcal{M}^\alpha f(t) := \frac{1}{t^\alpha} \int_0^t f(s)d^\alpha s, \text{ for all } t > 0. \tag{3.1}$$

Now we prove some properties of the operator $\mathcal{M}^\alpha$ that will be needed in the proofs later. From the definition of $\mathcal{M}^\alpha$, we see that if f is nonincreasing, then

$$\mathcal{M}^\alpha f(t) = \frac{1}{t^\alpha} \int_0^t f(s)d^\alpha s \geqslant \frac{1}{t^\alpha} \int_0^t f(t)d^\alpha s = \frac{1}{t^\alpha} \int_0^t s^{\alpha-1} ds f(t) = \frac{f(t)}{\alpha}.$$

From this, the following lemma gives some properties of the operator $\mathcal{M}^\alpha f$.

Lemma 3.2. *Let* $\mathcal{M}^\alpha f$ *be defined as in* (3.1). *Then we have the following properties:*

(a) *if* f *is nonincreasing, then* $\mathcal{M}^\alpha f(t) \geqslant \frac{1}{\alpha} f(t)$;
(b) *if* f *is nondecreasing, then* $\mathcal{M}^\alpha f(t) \leqslant \frac{1}{\alpha} f(t)$.

Now, we will define $\mathcal{M}^\alpha[\mathcal{M}^\alpha f]^p$ by,

$$\mathcal{M}^\alpha[\mathcal{M}^\alpha f]^p = \frac{1}{t^\alpha} \int_0^t \left(\frac{1}{s^\alpha} \int_0^s f(x)d^\alpha x\right)^p d^\alpha s, \text{ for } t > 0.$$

Theorem 3.3. *Let* $f : I \longrightarrow \mathbb{R}^+$ *be a nonnegative and nondecreasing. If* $p > 1$, *then*

$$\mathcal{M}^\alpha[\mathcal{M}^\alpha f]^p \leqslant \left(\frac{p}{\alpha(1-p)}\right)^p \mathcal{M}^\alpha f^p(t), \textit{ for } t > 0. \tag{3.2}$$

Proof. Let $\mathcal{M}^{\alpha}f = F$ and $t > 0$. Moreover, since $t^{\alpha}F(t) = \int_0^t f(s)d^{\alpha}s$, the product rule (see Theorem 2.2), we see that

$$t^{\alpha}D^{\alpha}F(t) + \alpha F(t) = f(t). \tag{3.3}$$

Letting $u(t) = t^{\alpha}$, $v(t) = F^p(t)$ and integrating by parts (see Theorem 2.4), we see that

$$\alpha\int_0^t F^p(s)d^{\alpha}s = t^{\alpha}F^p(t) - \int_0^t s^{\alpha}D^{\alpha}F^p(s)d^{\alpha}s. \tag{3.4}$$

By using the chain rule (see Theorem 2.2), we see that

$$D^{\alpha}F^p(s) = pD^{\alpha}F(s)F^{p-1}(s). \tag{3.5}$$

From (3.4) and (3.5), we see that

$$\alpha\int_0^t F^p(s)d^{\alpha}s = t^{\alpha}F^p(t) - p\int_0^t s^{\alpha}D^{\alpha}F(s)F^{p-1}(s)d^{\alpha}s. \tag{3.6}$$

From (3.3) and (3.6), we obtain

$$\begin{aligned}\alpha\int_0^t F^p(s)d^{\alpha}s &= t^{\alpha}F^p(t) - p\int_0^t \left[f(s) - \alpha F(s)\right]F^{p-1}(s)d^{\alpha}s \\ &= t^{\alpha}F^p(t) - p\int_0^t f(s)F^{p-1}(s)d^{\alpha}s + \alpha p\int_0^t F^p(s)d^{\alpha}s.\end{aligned} \tag{3.7}$$

Note

$$\lim_{s\to 0^+} u(s)v(s) = \lim_{s\to 0^+} s^{\alpha}\left(\frac{1}{s^{\alpha}}\int_0^s f(x)d^{\alpha}x\right)^p = \lim_{s\to 0^+} s^{\alpha}\left(\frac{1}{s^{\alpha}}\int_0^s f(x)x^{\alpha-1}dx\right)^p = \lim_{s\to 0^+} s^{\alpha}\frac{f^p(s)}{\alpha^p} = 0.$$

From (3.7), we obtain

$$\alpha(1-p)\int_0^t F^p(s)d^{\alpha}s = t^{\alpha}F^p(t) + p\int_0^t f(s)F^{p-1}(s)d^{\alpha}s.$$

Applying Hölder's inequality with exponents p and $p/(p-1)$, we have

$$\alpha(1-p)\int_0^t F^p(s)d^{\alpha}s \leqslant p\left(\int_0^t f^p(s)d^{\alpha}s\right)^{1/p}\left(\int_0^t F^p(s)d^{\alpha}s\right)^{(p-1)/p}.$$

From this we get the desired inequality (3.2) after using the definition of $\mathcal{M}^{\alpha}$. The proof is complete. □

Lemma 3.4. *Let* $x(t) = t^{\alpha}$. *If* $\gamma > 1$, *then*

$$D^{\alpha}x^{1-\gamma}(t) = \frac{\alpha(1-\gamma)}{t^{\alpha\gamma}}. \tag{3.8}$$

Proof. By using the chain rule (see Theorem 2.2), we see that

$$D^{\alpha}x^{1-\gamma}(t) = (1-\gamma)x^{-\gamma}(t)t^{1-\alpha}x'(t) = \alpha(1-\gamma)t^{-\alpha\gamma},$$

which is (3.8). □

4. Main results

The first theorem will be used later in the proof of conformable Gehring's inequality.

Theorem 4.1. *If* $p > 1$ *and* $f : I \longrightarrow \mathbb{R}^+$ *is a nonnegative and nonincreasing function, then for any* $q \in (0, p)$, *we have*

$$\frac{1}{\alpha^{\frac{p}{q}}}\mathcal{M}^\alpha f^p(t) \leqslant \frac{q}{\alpha p}[\mathcal{M}^\alpha f^q(t)]^{p/q} + \frac{(p-q)}{p}\mathcal{M}^\alpha[\mathcal{M}^\alpha f^q(t)]^{p/q}, \textit{ for } t > 0.$$

Proof. Let $t > 0$ and let

$$\gamma = \frac{p}{q} \text{ and } \mathcal{F} = \int_0^t f^q(s)d^\alpha s.$$

By using Lemma 3.4, we have

$$\begin{aligned}\frac{(p-q)}{p}\mathcal{M}^\alpha[\mathcal{M}^\alpha f^q(t)]^{p/q} &= \frac{(\gamma-1)}{\gamma t^\alpha}\int_0^t \left(\frac{\mathcal{F}(s)}{s^\alpha}\right)^\gamma d^\alpha s \\ &= \frac{-1}{\gamma t^\alpha}\int_0^t \mathcal{F}^\gamma(s)\frac{(1-\gamma)}{s^{\alpha\gamma}}d^\alpha s = \frac{-1}{\alpha\gamma t^\alpha}\int_0^t \mathcal{F}^\gamma(s)D^\alpha x^{1-\gamma}(s)d^\alpha s.\end{aligned}$$

Putting $u(s) = x^{1-\gamma}(s), \nu(s) = \mathcal{F}^\gamma(s)$ and integrating by parts, we obtain that

$$\begin{aligned}\frac{(p-q)}{p}\mathcal{M}^\alpha[\mathcal{M}^\alpha f^q(t)]^{p/q} &= \lim_{s\to 0}\frac{x^{1-\gamma}(s)\mathcal{F}^\gamma(s)}{\alpha\gamma t^\alpha} - \frac{x^{1-\gamma}(t)\mathcal{F}^\gamma(t)}{\alpha\gamma t^\alpha} + \frac{1}{\alpha\gamma t^\alpha}\int_0^t x^{1-\gamma}(s)D^\alpha\mathcal{F}^\gamma(s)d^\alpha s \\ &= \frac{1}{\alpha\gamma t^\alpha}\int_0^t s^{\alpha(1-\gamma)}D^\alpha\mathcal{F}^\gamma(s)d^\alpha s - \frac{1}{\alpha\gamma}\left(\frac{\mathcal{F}(t)}{t^\alpha}\right)^\gamma.\end{aligned}$$

By using Lemma 3.2, since $\mathcal{M}^\alpha f^q(s) \geqslant 1/(\alpha)f^q(s)$ for $s > 0$, we see that

$$\begin{aligned}\frac{(p-q)}{p}\mathcal{M}^\alpha[\mathcal{M}^\alpha f^q(t)]^{p/q} &= \frac{1}{\alpha t^\alpha}\int_0^t f^q(s)[\mathcal{M}^\alpha f^q(s)]^{\gamma-1}d^\alpha s - \frac{1}{\alpha\gamma}[\mathcal{M}^\alpha f^q(t)]^\gamma \\ &\geqslant \frac{1}{\alpha^\gamma t^\alpha}\int_0^t f^q(s)[f^q(s)]^{\gamma-1}d^\alpha s - \frac{1}{\alpha\gamma}[\mathcal{M}^\alpha f^q(t)]^\gamma \\ &= \frac{1}{\alpha^\gamma t^\alpha}\int_0^t [f^q(s)]^\gamma d^\alpha s - \frac{1}{\alpha\gamma}[\mathcal{M}^\alpha f^q(t)]^\gamma \\ &= \frac{1}{\alpha^{\frac{p}{q}}}\mathcal{M}^\alpha f^p(t) - \frac{q}{\alpha p}[\mathcal{M}^\alpha f^q(t)]^{p/q}.\end{aligned}$$

The proof is complete. □

Now, we state and prove first conformable version of the Gehring inequality for monotone functions.

Theorem 4.2. *Let* $p > 1$ *and* $f : I \longrightarrow \mathbb{R}^+$ *is a nonnegative and nonincreasing function. If*

$$\mathcal{M}^\alpha f^q(t) \leqslant \lambda[\mathcal{M}^\alpha f(t)]^q, \textit{ for some } \lambda > 0, \tag{4.1}$$

then for $p > q$, *we have*

$$\mathcal{M}^\alpha f^p(t) \leqslant A[\mathcal{M}^\alpha f(t)]^p, \textit{ for } t > 0, \tag{4.2}$$

where

$$A = \frac{q\lambda^{p/q}}{\alpha^{1-\frac{p}{q}}p - \alpha(p-q)\lambda^{p/q}\left(\frac{p}{\alpha(1-p)}\right)^p} > 0.$$

Proof. From Theorem 4.1, we have that

$$\frac{1}{\alpha^{\frac{p}{q}}}\mathcal{M}^{\alpha}f^{p}(t) \leqslant \frac{q}{\alpha p}[\mathcal{M}^{\alpha}f^{q}(t)]^{p/q} + \frac{(p-q)}{p}\mathcal{M}^{\alpha}[\mathcal{M}^{\alpha}f^{q}(t)]^{p/q}.$$

By using (4.1), we obtain

$$\begin{aligned}\frac{1}{\alpha^{\frac{p}{q}}}\mathcal{M}^{\alpha}f^{p}(t) &\leqslant \frac{q}{\alpha p}[\lambda[\mathcal{M}^{\alpha}f(t)]^{q}]^{p/q} + \frac{(p-q)}{p}\mathcal{M}^{\alpha}[\lambda[\mathcal{M}^{\alpha}f(t)]^{q}]^{p/q}\\ &= \frac{q}{\alpha p}\lambda^{p/q}[\mathcal{M}^{\alpha}f(t)]^{p} + \lambda^{p/q}\frac{(p-q)}{p}\mathcal{M}^{\alpha}[[\mathcal{M}^{\alpha}f(t)]^{p}].\end{aligned}$$

By applying Theorem 3.3, since $\mathcal{M}^{\alpha}[\mathcal{M}^{\alpha}f]^{p} \leqslant (p/\alpha(1-p))^{p}\,\mathcal{M}^{\alpha}f^{p}(t)$, we obtain

$$\frac{1}{\alpha^{\frac{p}{q}}}\mathcal{M}^{\alpha}f^{p}(t) \leqslant \frac{q}{\alpha p}\lambda^{p/q}[\mathcal{M}^{\alpha}f(t)]^{p} + \lambda^{p/q}\frac{(p-q)}{p}\left(\frac{p}{\alpha(1-p)}\right)^{p}\mathcal{M}^{\alpha}f^{p}(t).$$

So

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant A[\mathcal{M}^{\alpha}f(t)]^{p},$$

which is the desired result. □

Theorem 4.3. *If* $p > 1$, $f : I \longrightarrow \mathbb{R}^{+}$ *is a nonnegative and nonincreasing function and* (4.2) *holds, then for* $0 < r < 1$, *we have*

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant \bar{A}[\mathcal{M}^{\alpha}f(t)]^{p/r}, \text{ for } t > 0,$$

where $\bar{A} = A^{1/\theta}$ *with* $\theta = (1-\frac{1}{p})/(\frac{1}{r}-\frac{1}{p})$.

Proof. Note that $\theta \in (0,1)$ and $(1-\theta)/p + (\theta/r) = 1$. By applying Hölder's inequality with exponents $1/\theta$ and $p/(1-\theta)$, and (4.2), we have

$$\begin{aligned}\left(\frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s\right)^{1/p} &\leqslant \frac{A^{1/p}}{t^{\alpha}}\int_0^t f(s)d^{\alpha}s\\ &= \frac{A^{1/p}}{t^{\alpha}}\int_0^t f^{1-\theta}(s)f^{\theta}(s)d^{\alpha}s\\ &\leqslant A^{1/p}\left(\frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s\right)^{(1-\theta)/p}\left(\frac{1}{t^{\alpha}}\int_0^t f^{r}(s)d^{\alpha}s\right)^{\theta/r}\\ &= A^{1/p}[\mathcal{M}^{\alpha}f^{p}(t)]^{(1-\theta)/p} \times [\mathcal{M}^{\alpha}f^{r}(t)]^{\theta/r}.\end{aligned}$$

By substituting, we find

$$\left(\frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s\right)^{\theta/p} \leqslant A^{1/p}\left(\frac{1}{t^{\alpha}}\int_0^t f^{r}(s)d^{\alpha}s\right)^{\theta/r}.$$

The proof is complete. □

Theorem 4.4. *If* $0 < p_0 < p_1 < \infty, 0 < \theta < 1$ *and* f *is a nonnegative and nonincreasing function, then*

(1) *if* $p = (1-\theta)p_0 + \theta p_1$, *then*

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant [\mathcal{M}^{\alpha}f^{p_0}(t)]^{1-\theta}[\mathcal{M}^{p_1}f(t)]^{\theta}, \text{ for } t > 0;$$

(2) *if* $p = \frac{1}{\frac{1-\theta}{p_0}+\frac{\theta}{p_1}}$, *then*

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant [\mathcal{M}^{\alpha}f^{p_0}(t)]^{(1-\theta)p/p_0}[\mathcal{M}^{p_1}f(t)]^{\theta p/p_1}, \text{ for } t > 0.$$

Proof.
(1). By applying Hölder's inequality with exponents $1/\theta$ and $1/(1-\theta)$, we have

$$\begin{aligned}\mathcal{M}^{\alpha}f^{p}(t) &= \frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s = \frac{1}{t^{\alpha}}\int_0^t f^{(1-\theta)p_0}(s)f^{\theta p_1}(s)d^{\alpha}s \left(\frac{1}{t^{\alpha}}\int_0^t f^{p_0}(s)d^{\alpha}s\right)^{1-\theta}\left(\frac{1}{t^{\alpha}}\int_0^t f^{p_1}(s)d^{\alpha}s\right)^{\theta} \\ &= [\mathcal{M}^{\alpha}f^{p_0}(t)]^{1-\theta}[\mathcal{M}^{\alpha}f^{p_1}(t)]^{\theta},\end{aligned}$$

which is (1).
(2). By applying Hölder's inequality with exponents $1/\gamma$ and $1/(1-\gamma)$, where $\gamma = \theta p/p_1, 1-\gamma = (1-\theta)p/p_0$, we have

$$\begin{aligned}\mathcal{M}^{\alpha}f^{p}(t) &= \frac{1}{t^{\alpha}}\int_0^t f^{(1-\theta)p}(s)f^{\theta p}(s)d^{\alpha}s \\ &\leqslant \left(\frac{1}{t^{\alpha}}\int_0^t f^{(1-\theta)p/(1-\gamma)}(s)d^{\alpha}s\right)^{1-\gamma}\left(\frac{1}{t^{\alpha}}\int_0^t f^{\theta p/\gamma}(s)d^{\alpha}s\right)^{\gamma} \\ &= [\mathcal{M}^{\alpha}f^{p_0}(t)]^{(1-\theta)p/p_0}[\mathcal{M}^{p_1}f(t)]^{\theta p/p_1},\end{aligned}$$

which shows (2). The proof is complete. □

In the following, we give a new proof of conformable Gehring's inequality.

Theorem 4.5. *If* $p > 1$, $f : I \longrightarrow \mathbb{R}^{+}$ *is a nonnegative and nonincreasing function and*

$$\mathcal{M}^{\alpha}f(t) \leqslant \lambda f(t) \text{ for some } \lambda > 1, \tag{4.3}$$

then for $p \in [1, \lambda/(\lambda-1)]$, *where* $k = \alpha\lambda$, *we have*

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant A[\mathcal{M}^{\alpha}f(t)]^{p} \text{ for } t > 0, \text{ where } A = \frac{\alpha^{p-1}k}{k-p(k-1)} > 0.$$

Proof. Let $F(t) = \int_0^t f(s)d^{\alpha}s$. By using property (5) in Theorem 2.2, we obtain

$$D^{\alpha}\log t^{\alpha} = t^{1-\alpha}\frac{d}{dt}\log t^{\alpha} = \frac{\alpha}{t^{\alpha}}. \tag{4.4}$$

Now, since $t^{\alpha}\mathcal{M}^{\alpha}f(t) = F(t)$, we get by using (4.3) that

$$\frac{\alpha}{t^{\alpha}} = \frac{\alpha\mathcal{M}^{\alpha}f(t)}{F(t)} \leqslant \alpha\lambda\frac{f(t)}{F(t)} = k\frac{f(t)}{F(t)}.$$

This and (4.4) give us that

$$D^{\alpha}\log t^{\alpha} \leqslant k\frac{D^{\alpha}(F(t))}{F(t)}.$$

By integrating the last inequality from s to t, we get that

$$\int_s^t D^{\alpha}(\log x^{\alpha})d^{\alpha}x \leqslant k\int_s^t \frac{D^{\alpha}(F(x))}{F(x)}d^{\alpha}x \leqslant k\int_s^t \frac{x^{1-\alpha}\frac{d}{dx}(F(x))}{F(x)}x^{\alpha-1}dx.$$

That is,

$$\log\left(\frac{t^{\alpha}}{s^{\alpha}}\right)^{1/k} \leqslant \log\left(\frac{F(t)}{F(s)}\right). \tag{4.5}$$

Now, since f is nonincreasing, we have from Lemma 3.2 and (4.5) that

$$\frac{1}{\alpha}f(s) \leqslant \mathcal{M}^{\alpha}f(s) = \frac{F(s)}{s^{\alpha}} \leqslant \frac{1}{s^{\alpha}}\left(\frac{s^{\alpha}}{t^{\alpha}}\right)^{1/k}F(t),$$

then

$$\frac{1}{\alpha^{p}}f^{p}(s) \leqslant \left(\frac{1}{s^{\alpha}}\right)^{p}\left(\frac{s^{\alpha}}{t^{\alpha}}\right)^{p/k}F^{p}(t).$$

Integrating from 0 to t, and putting $\gamma = p(1-1/k) \in (0,1)$, we obtain

$$\frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s \leqslant \frac{\alpha^{p}F^{p}(t)}{t^{\alpha(1+p/k)}}\int_0^t \frac{1}{s^{\alpha p(1-1/k)}}d^{\alpha}s = \frac{\alpha^{p}F^{p}(t)}{t^{\alpha(1+p/k)}}\int_0^t \frac{1}{s^{\alpha\gamma}}d^{\alpha}s.$$

Letting $x(t) = t^{\alpha}$, from Lemma (3.4), we see that

$$D^{\alpha}x^{1-\gamma}(t) = \frac{\alpha(1-\gamma)}{t^{\alpha\gamma}}. \tag{4.6}$$

Since $\gamma > 1$, and by using (4.6), then we get that

$$\begin{aligned}\frac{1}{t^{\alpha}}\int_0^t f^{p}(s)d^{\alpha}s &\leqslant \frac{\alpha^{p}F^{p}(t)}{\alpha(1-\gamma)t^{\alpha(1+p/k)}}\int_0^t D^{\alpha}\left(x^{1-\gamma}(t)\right)d^{\alpha}s\\ &= \frac{\alpha^{p-1}F^{p}(t)}{(1-\gamma)t^{\alpha(1+p/k)}}\int_0^t D^{\alpha}\left(x^{1-\gamma}(t)\right)d^{\alpha}s\\ &= \frac{\alpha^{p-1}F^{p}(t)t^{\alpha(1-\gamma)}}{(1-\gamma)t^{\alpha(1+p/k)}} = \frac{\alpha^{p-1}}{(1-\gamma)}\left(\frac{F(t)}{t^{\alpha}}\right)^{p},\end{aligned}$$

which is the desired result. □

5. Conformable higher integrability

For $t > 0$, f is a nonnegative and nonincreasing and $q > 1$, we have

$$\mathcal{M}^{\alpha}f^{q}(t) = \frac{1}{t^{\alpha}}\int_0^t f^{q}(s)d^{\alpha}s = \frac{1}{t^{\alpha}}\int_0^t f^{q-1}(s)f(s)d^{\alpha}s \geqslant \frac{f^{q-1}(t)}{t^{\alpha}}\int_0^t f(s)d^{\alpha}s = f^{q-1}(t)\mathcal{M}^{\alpha}f(t). \tag{5.1}$$

Now, consider the class of nonnegative and nonincreasing functions f that satisfy the reverse of (5.1). Then

$$\mathcal{M}^{\alpha}f^{q}(t) \leqslant \lambda f^{q-1}(t)\mathcal{M}^{\alpha}f(t) \text{ for some } \lambda > 1. \tag{5.2}$$

Theorem 5.1. *If $q > 1$, f is a nonnegative and nonincreasing function, (5.2) holds, and $\lambda_q > 1-q$, then for $p \in [q, q\lambda_q/(\lambda_q-1))$, we have*

$$\mathcal{M}^{\alpha}f^{p}(t) \leqslant \mathcal{K}[\mathcal{M}^{\alpha}f^{q}(t)]^{p/q} \text{ for } t > 0, \tag{5.3}$$

where

$$\mathcal{K} = \frac{\alpha^{\frac{p}{q}-1}\lambda_q^{1+p/q}}{\lambda_q - \frac{p}{q}(\lambda_q - 1)}, \text{ with } \lambda_q = \frac{q\lambda}{\alpha(1-q)}.$$

Proof. Let $\mathcal{M}^{\alpha}f^{q} = F$. By applying Hölder's inequality with exponents q and $q/(q-1)$, and (5.2), we obtain

$$\frac{1}{t^{\alpha}}\int_0^t F(s)d^{\alpha}s \leqslant \frac{\lambda}{t^{\alpha}}\int_0^t f^{q-1}(s)\left(\frac{1}{s^{\alpha}}\int_0^s f(\tau)d^{\alpha}\tau\right)d^{\alpha}s$$

$$\leqslant \lambda \left(\frac{1}{t^\alpha} \int_0^t f^q(s) d^\alpha s \right)^{(q-1)/q} \left(\frac{1}{t^\alpha} \int_0^t \left(\frac{1}{s^\alpha} \int_0^s f^q(\tau) d^\alpha \tau \right)^q d^\alpha s \right)^{1/q}.$$

By using the definition of $\mathcal{M}^\alpha f$, we obtain

$$\mathcal{M}^\alpha F(t) = \frac{1}{t^\alpha} \int_0^t F(s) d^\alpha s \leqslant \lambda [\mathcal{M}^\alpha f^q(t)]^{(q-1)/q} [\mathcal{M}^\alpha [\mathcal{M}^\alpha f(t)]^q]^{1/q}. \tag{5.4}$$

From Theorem 3.3, we have

$$[\mathcal{M}^\alpha [\mathcal{M}^\alpha f(t)]^q]^{1/q} \leqslant \frac{q}{\alpha(1-q)} [\mathcal{M}^\alpha f^q(t)]^{1/q}. \tag{5.5}$$

From (5.4) and (5.5), we have that

$$\begin{aligned} \mathcal{M}^\alpha F(t) = \frac{1}{t^\alpha} \int_0^t F(s) d^\alpha s &\leqslant \frac{q\lambda}{\alpha(1-q)} [\mathcal{M}^\alpha f^q(t)]^{(q-1)/q} [\mathcal{M}^\alpha f^q(t)]^{1/q} \\ &= \frac{q\lambda}{\alpha(1-q)} \mathcal{M}^\alpha f^q(t) = \lambda_q \mathcal{M}^\alpha f^q(t) = \lambda_q F(t). \end{aligned} \tag{5.6}$$

Note $\lambda_q > 1$ since $\lambda_q > 1-q$. Since F is a nonnegative and nonincreasing (see Lemma 3.2), then we have

$$\mathcal{M}^\alpha F^r(t) \leqslant A [\mathcal{M}^\alpha f(t)]^r, \tag{5.7}$$

with

$$A = \frac{\alpha^{r-1} \lambda_q}{\lambda_q - r(\lambda_q - 1)} \text{ and } r = \frac{p}{q} \in [1, \frac{\lambda_q}{\lambda_q - 1}).$$

Note

$$F(t) = \frac{1}{t^\alpha} \int_0^t f^q(s) d^\alpha s \geqslant \frac{1}{\alpha} f^q(t).$$

From (5.7) and (5.6), we have that

$$\begin{aligned} \mathcal{M}^\alpha f^p(t) = \frac{1}{t^\alpha} \int_0^t (f^q(s))^r d^\alpha s &\leqslant \frac{1}{t^\alpha} \int_0^t F^r(s) d^\alpha s \\ &= \mathcal{M}^\alpha f^r(t) \leqslant A [\mathcal{M}^\alpha f(t)]^r \leqslant A \lambda_q^r [F(t)]^r = \mathcal{K} [F(t)]^r = \mathcal{K} [\mathcal{M}^\alpha f^q(t)]^{p/q}, \end{aligned}$$

which proves (5.3). □

6. Conclusion

In this paper, we proved some reverse conformable inequalities with weights and employed them to prove some conformable inequalities of Gehring type. Also, we proved some interpolation theorems which are powerful tools in the study of operators in function spaces. We aim to generalize the results of the weighted classes and use the results to prove the boundedness of operators in conformable version.

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