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Properties of Muckenhoupt and Gehring classes via conformable calculus

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Abstract

In this paper, we study the relationship between the Muckenhoupt class $\mathcal{A}_1^{\alpha}(\mathcal{C})$ and the Gehring class $\mathcal{G}_q^{\alpha}(\mathcal{K})$ via conformable calculus. We also establish the constants of the classes for the power law functions.

Keywords: Conformable Muckenhoupt class, conformable Gehring class, Hölder's inequality, reverse Hölder's inequality.

2020 MSC: 40D05, 40D25, 42C10 43A55, 46A35, 46B15.

1. Introduction

We fix an interval $I_0 \subset \mathbb{R}_+ = [0, \infty)$, and consider the subinterval I of I_0 of the form $[0, s]$, for $0 < s < \infty$ and donate by $|I|$ the Lebesgue measure of I. The nonnegative weight ν is said to belong to the Muckenhoupt class $A_p(\mathcal{C})$ on the interval I_0 for $p > 1$ and $\mathcal{C} > 1$ (independent of p) if the inequality

$$\frac{1}{|I|}\int_I \nu(x)dx \leqslant \mathcal{C}\left(\frac{1}{|I|}\int_I \nu^{\frac{1}{1-p}}(x)dx\right)^{1-p}, \tag{1.1}$$

holds for every subinterval $I \subset I_0$. For $p > 1$, we define the A_p-norm of the weight ν by

$$[A_p(\nu)] := \sup_{I \subset I_0}\left(\frac{1}{|I|}\int_I \nu(x)dx\right)\left(\frac{1}{|I|}\int_I \nu^{\frac{-1}{p-1}}(x)dx\right)^{p-1}.$$

The weight ν is said to belong to the Muckenhoupt class $A_1(\mathcal{C})$ on the interval I_0, if the inequality

$$\frac{1}{|I|}\int_I \nu(x)dx \leqslant \mathcal{C}\nu(x), \text{ for } \mathcal{C} > 1, \tag{1.2}$$

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holds for every subinterval $I \subset I_0$, and we define the A_1-norm by

$$[A_1(\nu)] := \sup_{I \subset I_0} \frac{1}{|I|} \left(\frac{1}{\inf \nu} \int_I \nu(x) dx \right).$$

In [20], Muckenhoupt proved that if ν is a nonincreasing weight satisfying condition (1.2), then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that

$$\frac{1}{|I|} \int_I \nu^p(x) dx \leqslant \frac{\mathcal{C}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|I|} \int_I \nu(x) dx \right)^p.$$

The authors in [7] improved the Muckenhoupt result by excluding the property of monotonicity on the weight ν by using a rearrangement ν^* of the function ν over the interval I and established the best constant. In particular, they proved that if ν is a nonincreasing weight satisfying condition (1.2) with $\mathcal{C} > 1$, then there exists $p \in [1, \mathcal{C}/(\mathcal{C}-1)]$ such that

$$\frac{1}{|I|} \int_I \nu^p(x) dx \leqslant \frac{\mathcal{C}^{1-p}}{\mathcal{C} - p(\mathcal{C}-1)} \left(\frac{1}{|I|} \int_I \nu(x) dx \right)^p.$$

Further in [20], Muckenhoupt proved the following result. If $1 < p < \infty$ and ν satisfies the A_p-condition (1.1) on the interval I, with constant $\mathcal{C}$, then there exist constants q and $\mathcal{C}_1$ depending on p and $\mathcal{C}$ such that $1 < q < p$ and ν satisfies the A_q-condition

$$\left(\frac{1}{|I|} \int_I \nu(x) dx \right) \left(\frac{1}{|I|} \int_I \nu^{-\frac{1}{q-1}}(x) dx \right)^{q-1} \leqslant \mathcal{C}_1,$$

for every subinterval $I \subset I_0$. On other words, Muckenhoupt's result for *self-improving* property states that: if $\nu \in A_p(\mathcal{C})$, then there exists a constant $\epsilon > 0$ and a positive constant $\mathcal{C}_1$ such that $\nu \in A_{p-\epsilon}(\mathcal{C}_1)$, and then $A_p(\mathcal{C}) \subset A_{p-\epsilon}(\mathcal{C}_1)$. Despite of a variety of ideas related to weighted inequalities appeared with the birth of singular integrals, it was only in the 1970s that a better understanding of the subject was obtained and the full characterization of the weights ν for which the Hardy-Littlewood maximal operator

$$\mathcal{M}\nu(x) := \sup_{x \in I} \frac{1}{|I|} \int_I \nu(y) dy,$$

is bounded on $L^p_w(\mathbb{R})$ by means of the so-called A_p-*condition* was achieved by Muckenhoupt and published in 1972 (see[20]). Muckenhoupt's result became a landmark in the theory of weighted inequalities because most of the previously known results for classical operators had been obtained for special classes of weights (like power weights) and has been extended to cover several operators like Hardy operator, Hilbert operator, Calderón-Zygmund singular integral operators, fractional integral operators, etc.

The weight ν is said to belong to the Gehring class $G_q(\mathcal{K})$, $1 < q < \infty$ for the interval I_0, if there exists a constant $\mathcal{K} > 1$ such that the inequality

$$\left(\frac{1}{|I|} \int_I \nu^q(x) dx \right)^{\frac{1}{q}} \leqslant \mathcal{K} \left(\frac{1}{|I|} \int_I \nu(x) dx \right), \tag{1.3}$$

holds for every subinterval $I \subset I_0$, and we define the G_q-norm by

$$[G_q(\nu)] := \sup_{I \subset I_0} \left[\left(\frac{1}{|I|} \int_I \nu^q(x) dx \right)^{\frac{1}{q}} \left(\frac{1}{|I|} \int_I \nu(x) dx \right)^{-1} \right]^{\frac{q}{q-1}}.$$

In [9] Gehring proved that if (1.3) holds, then there exist $p > q$ and a positive constant $\mathcal{K}_1$ such that

$$\frac{1}{|I|} \int_I \nu^p(x) dx \leqslant \mathcal{K}_1 \left(\frac{1}{|I|} \int_I \nu(x) dx \right)^p, \text{ for every } I \subset I_0.$$

On other words Gehring's result for *self-improving* property states that: if $\nu \in G_q(\mathcal{K})$, then there exists $\epsilon > 0$ and a positive constant $\mathcal{K}_1$ such that $\nu \in G_{q+\epsilon}(\mathcal{K}_1)$, and then

$$G_q(\mathcal{K}) \subset G_{q+\epsilon}(\mathcal{K}_1).$$

For power-low functions, Malaksiano in [18] proved that if $I = [0, 1]$, $p > 1$ and $\gamma > -1/p$, then

$$(G_p(x^\gamma))^{(p-1)/p} = \frac{1+\gamma}{(1+p\gamma)^{1/p}}. \tag{1.4}$$

Moreover, if $0 < \gamma < \beta$ and $p > 1$, then $G_p(x^\gamma) < G_p(x^\beta)$. Further in [19], Malaksiano proved that if $I = [0, 1]$, $q > 1$, and $\gamma \in (-1, q-1)$, then

$$(A_q(x^\gamma)) = \frac{(q-1)^{(q-1)}}{(\gamma+1)(q-1-\gamma)^{q-1}}. \tag{1.5}$$

Moreover, if $0 < \gamma < \beta < q-1$ and $q > 1$, then $A_q(x^{-\gamma}) < A_q(x^{-\beta})$.

In the last decade, discrete analogues in harmonic analysis became a very active and attractive field of research. For example, the research on regularity and boundedness of discrete analogues of the corresponding L^p operators, as well as the higher summability theorems analogues for higher integrability theorems have been studied by numerous authors (see, e.g., [6, 8, 16, 17, 24, 28] and references therein). In the articles [23, 26] the authors studied the structure and basic properties of the weighted discrete Gehring classes, as well as the relationship between the weighted discrete Gehring and Muckenhoupt classes. In recent years, by utilizing the conformable calculus, many authors proved several results related to some integral inequalities like Chebyshev type inequality [3], Hardy type inequalities [25], Hermite-Hadamard type inequalities [2, 12, 13], and Iyengar type inequalities [27]. For more details of conformable calculus we refer the reader to the papers [10, 14, 15, 21, 22, 29] and the references cited therein.

Following this trend and to develop the studies in this directions, we study the structure of conformable Muckenhoupt class $\mathcal{A}_p^\alpha(\mathcal{C})$ and conformable Gehring class $\mathcal{G}_q^\alpha(\mathcal{K})$. In particular, we prove that if $\nu \in \mathcal{A}_q^\alpha(\mathcal{C})$, such that $1 < q < \infty$, then $\nu \in \mathcal{A}_1^\alpha(\mathcal{C}_1)$. For the relation between conformable Muckenhoupt class and conformable Gehring class, we prove that if $\nu \in \mathcal{A}_1^\alpha(\mathcal{C})$ then $\nu \in \mathcal{G}_p^\alpha(\mathcal{K})$. Also, we prove the exact values of the Muckenhoupt norm $\mathcal{A}_p^\alpha(t^\gamma)$ and the Gehring norm $\mathcal{G}_p^\alpha(t^\gamma)$ for power-low property on conformable calculus. We believe that the results of this paper will act as fundamental infrastructure for all topics dealing with conformable Muckenhoupt and Gehring classes. The paper is organized as follows. Section 2 is devoted to present some preliminaries on conformable calculus which will be involved throughout the remaining part of the paper. In Section 3, first we state and prove some basic lemmas which will used to prove our main results. Next, we prove the relationship between the Muckenhoupt class $\mathcal{A}_1^\alpha(\mathcal{C})$ and the Gehring class $\mathcal{G}_p^\alpha(\mathcal{K})$ via conformable calculus. Particularly, we prove that if the weight ν belongs to the Muckenhoupt class $\mathcal{A}_1^\alpha(\mathcal{C})$, then it belongs to the Gehring class $\mathcal{G}_p^\alpha(\mathcal{K})$ for some p. We also prove the analogues of (1.4) and (1.5) via conformable calculus and show that the results depend on the constant α. The results proved by employing some inequalities designed and proved for this purpose.

2. Preliminaries

In this section, we present some preliminaries and definitions on conformable calculus. Throughout, we assume that $\alpha \in (0, 1]$ and the weight ν is a nonnegative locally α-integrable defined on $I_0 \subset \mathbb{R}_+$ and p is a positive real number. In addition, in our proofs, we will use the convention $0 \cdot \infty = 0$ and $0/0 = 0$.

Definition 2.1 ([1])**.** The conformable derivative of the function $f : [0, \infty) \to \mathbb{R}$ is defined as follows

$$D^\alpha f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for $t > 0$.

Theorem 2.2 ([1]). *Let* $\alpha \in (0,1]$ *and* f, g *be* α*-differentiable at a point* $t > 0$. *Then*

(i) $D^{\alpha}(af(t)+bg(t)) = aD^{\alpha}f(t)+bD^{\alpha}g(t)$;
(ii) $D^{\alpha}(t^{p}) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$;
(iii) $D^{\alpha}(f(t)g(t)) = f(t)D^{\alpha}g(t)+g(t)D^{\alpha}f(t)$;
(iv) $D^{\alpha}(\frac{f(t)}{g(t)}) = \frac{g(t)D^{\alpha}f(t)-f(t)D^{\alpha}g(t)}{g^2(t)}$;
(v) $D^{\alpha}(k) = 0$, *where* k *is a constant;*
(vi) *if* f *is differentiable, then* $D^{\alpha}f(t) = t^{1-\alpha}\frac{df(t)}{dt}$.
(vii) $D^{\alpha}(f \circ g) = t^{1-\alpha}g'(t)f'(g(t)) = D^{\alpha}g(t)f'(g(t))$.

Definition 2.3. A function $f : [0,\infty) \to \mathbb{R}$ is conformable integrable on $[0,t]$ if the integral

$$I^{\alpha}(f)(t) = I(t^{\alpha-1}f)(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}}dx,$$

exists.

Theorem 2.4 ([1]). *Let* $a, b, c \in \mathbb{R}$. *Then*

$$\int_a^b D^{\alpha}f(t)g(t)d^{\alpha}t = f(t)g(t)\mid_a^b - \int_a^b f(t)D^{\alpha}g(t)d^{\alpha}t.$$

Lemma 2.5. *Let* $f : [0,\infty) \to \mathbb{R}$ *be continuous. Then for all* $t > 0$, *we have*

(i) $D^{\alpha}I^{\alpha}f(t) = f(t)$;
(ii) $I^{\alpha}D^{\alpha}f(t) = f(t) - f(0)$.

The conformable Hölder inequality is given by

$$\int_0^t |f(s)g(s)|\, d^{\alpha}s \leqslant \left(\int_0^t |f(s)|^p\, d^{\alpha}s\right)^{\frac{1}{p}} \left(\int_0^t |g(s)|^q\, d^{\alpha}s\right)^{\frac{1}{q}},$$

where $f, g \in C([a,b],\mathbb{R})$, $p > 1$ and $1/p + 1/q$. We say that f satisfies a reverse Hölder inequality for some constants $p > q$ if the following holds,

$$\left(\int_0^t |f(s)|^p\, d^{\alpha}s\right)^{\frac{1}{p}} \leqslant \mathcal{C}\left(\int_0^t |f(s)|^q\, d^{\alpha}s\right)^{\frac{1}{q}}.$$

Let $g \in C([0,t],(c,d))$ and $F \in C([c,d],\mathbb{R})$ is convex. The conformable Jensen inequality (see [5]) is given by

$$F\left(\frac{\alpha I^{\alpha}g(t)}{t^{\alpha}}\right) \leqslant \frac{\alpha}{t^{\alpha}}I^{\alpha}F(g(t)).$$

Now, we present the definitions of Muckenhoupt and Gehring weights via conformable calculus. The nonnegative weight ν is said to belong to the Muckenhoupt class $\mathcal{A}_p^{\alpha}(\mathcal{C})$ on the interval I_0 for $p > 1$ and $\mathcal{C} > 1$ (independent of p), if

$$\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s \leqslant \mathcal{C}\left(\frac{1}{t^{\alpha}}\int_0^t \nu^{\frac{1}{1-p}}(s)d^{\alpha}s\right)^{1-p},$$

for every $t \in I_0$. For $p > 1$, we define the $\mathcal{A}_p^{\alpha}$-norm of the weight ν by

$$[\mathcal{A}_p^{\alpha}(\nu)] := \sup_{t\in I_0}\left(\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s\right)\left(\frac{1}{t^{\alpha}}\int_0^t \nu^{\frac{-1}{p-1}}(s)d^{\alpha}s\right)^{p-1}.$$

The weight ν is said to belong to the Muckenhoupt class $\mathcal{A}_1^{\alpha}(\mathcal{C})$ if

$$\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s \leqslant \mathcal{C}\nu(s), \text{ for } \mathcal{C}>1,$$

for every $t \in I_0$. The weight ν is said to belong to the Gehring class $\mathcal{G}_q^{\alpha}(\mathcal{K})$ on the interval I_0 for $q>1$ and $\mathcal{K}>1$ (independent of q) if

$$\left(\frac{1}{t^{\alpha}}\int_0^t \nu^q(s)d^{\alpha}s\right)^{\frac{1}{q}} \leqslant \mathcal{K}\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s,$$

for every $t \in I_0$. We define the $\mathcal{G}_q^{\alpha}$-norm is by

$$[\mathcal{G}_q^{\alpha}(\nu)] := \sup_{t\in I_0}\left[\left(\frac{1}{t^{\alpha}}\int_0^t \nu^q(s)d^{\alpha}s\right)^{\frac{1}{q}}\left(\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s\right)^{-1}\right]^{\frac{q}{q-1}}.$$

We mention here that when $\alpha=1$ the definitions will be reduced to the classical definitions of the Muckenhoupt and Gehring weights.

3. Main results

For any function $\nu : I_0 \longrightarrow \mathbb{R}^+$, we define the α-operator $\mathcal{M}^{\alpha}\nu : I_0 \longrightarrow \mathbb{R}^+$ by

$$\mathcal{M}^{\alpha}\nu(t) := \frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s, \text{ for all } I_0 \subseteq \mathbb{R}^+. \tag{3.1}$$

Now we give some properties of the operator $\mathcal{M}^{\alpha}$ that will be needed in the proofs later. From the definition of $\mathcal{M}^{\alpha}$, we see that if ν is nonincreasing, then

$$\mathcal{M}^{\alpha}\nu(t) = \frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s \geqslant \frac{1}{t^{\alpha}}\int_0^t \nu(t)d^{\alpha}s = \frac{1}{t^{\alpha}}\int_0^t s^{\alpha-1}ds\nu(t) = \frac{\nu(t)}{\alpha}.$$

The following lemma gives some properties of the operator $\mathcal{M}^{\alpha}\nu$.

Lemma 3.1. *Let $\mathcal{M}^{\alpha}\nu$ be defined as in (3.1). Then we have the following properties:*

(i) *if ν is nonincreasing, then $\mathcal{M}^{\alpha}\nu(t) \geqslant \frac{1}{\alpha}\nu(t)$;*
(ii) *if ν is nondecreasing, then $\mathcal{M}^{\alpha}\nu(t) \leqslant \frac{1}{\alpha}\nu(t)$.*

Lemma 3.2. *If $\nu \in \mathcal{A}_p^{\alpha}(\mathcal{C})$, and $p>1$, then*

$$\mathcal{M}^{\alpha}\nu \leqslant \frac{\mathcal{C}}{\alpha^{1-p}}\exp\left(\alpha\mathcal{M}^{\alpha}\log\nu\right).$$

Proof. Since $\nu \in \mathcal{A}_p^{\alpha}(c)$, for $p>1$, then we have

$$\frac{1}{t^{\alpha}}\int_0^t \nu(s)d^{\alpha}s \leqslant \mathcal{C}\left(\frac{1}{t^{\alpha}}\int_0^t \nu^{\frac{1}{1-p}}(s)d^{\alpha}s\right)^{1-p}. \tag{3.2}$$

By applying conformable Jensen inequality for the convex function $F(x)=\exp(x)$ and g replaced by

$$\frac{\alpha}{1-p}\log\nu(s),$$

we have

$$\exp\left(\frac{\alpha}{1-p}\left(\frac{1}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right)\right) \leqslant \frac{\alpha}{t^{\alpha}}\int_0^t \exp\left(\frac{1}{1-p}\log v(s)\right)d^{\alpha}s$$
$$= \frac{\alpha}{t^{\alpha}}\int_0^t \left(\exp\left(\log v^{\frac{1}{1-p}}(s)\right)\right)d^{\alpha}s = \frac{\alpha}{t^{\alpha}}\int_0^t v^{\frac{1}{1-p}}(s)d^{\alpha}s. \tag{3.3}$$

The left hand side of (3.3) can be written as follows:

$$\exp\left(\frac{\alpha}{1-p}\left(\frac{1}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right)\right) = \left(\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right)\right)^{\frac{1}{1-p}}. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\left(\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right)\right)^{\frac{1}{1-p}} \leqslant \frac{\alpha}{t^{\alpha}}\int_0^t v^{\frac{1}{1-p}}(s)d^{\alpha}s,$$

and then

$$\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right) \geqslant \left(\frac{\alpha}{t^{\alpha}}\int_0^t v^{\frac{1}{1-p}}(s)d^{\alpha}s\right)^{1-p}. \tag{3.5}$$

From (3.2) and (3.5), we obtain

$$\frac{1}{t^{\alpha}}\int_0^t v(s)d^{\alpha}s \leqslant \frac{\mathcal{C}}{\alpha^{1-p}}\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right),$$

which is the desired inequality. □

Lemma 3.3. *Let $1 < q < \infty$ and v be a nonincreasing weight. If $v \in \mathcal{A}_q^{\alpha}(\mathcal{C})$, then $v \in \mathcal{A}_1^{\alpha}(\mathcal{C}_1)$.*

Proof. To prove this lemma, we shall prove that if

$$\left(\frac{1}{t^{\alpha}}\int_0^t v(s)d^{\alpha}s\right)\left(\frac{1}{t^{\alpha}}\int_0^t v^{\frac{-1}{q-1}}(s)d^{\alpha}s\right)^{q-1} \leqslant \mathcal{C}, \text{ for some } \mathcal{C} > 1,$$

then

$$\frac{1}{t^{\alpha}}\int_0^t v(s)d^{\alpha}s \leqslant \mathcal{C}_1 v(t). \tag{3.6}$$

By using (3.6) and employing Lemma 3.2, we get that

$$\frac{1}{t^{\alpha}}\int_0^t v(s)d^{\alpha}s \leqslant \frac{\mathcal{C}}{\alpha^{1-p}}\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right).$$

By applying Lemma 3.1 for the nondecreasing weight $\log v(s)$, we get

$$\frac{1}{t^{\alpha}}\int_0^t v(s)d^{\alpha}s \leqslant \frac{\mathcal{C}}{\alpha^{1-p}}\exp\left(\frac{\alpha}{t^{\alpha}}\int_0^t \log v(s)d^{\alpha}s\right) \leqslant \frac{\mathcal{C}}{\alpha^{1-p}}\exp\left(\log v(t)\right) = \mathcal{C}_1 v(t),$$

where $\mathcal{C}_1 = \mathcal{C}/\alpha^{1-p}$. The proof is complete. □

Lemma 3.4. *Let $1 < p < \infty$ and v be a nonnegative weight. Then $v \in \mathcal{A}_p^{\alpha}$ if and only if $v^{1-p'} \in \mathcal{A}_{p'}^{\alpha}$, with $[\mathcal{A}_{p'}^{\alpha}(v^{1-p'})] = [\mathcal{A}_p^{\alpha}(v)]^{p'-1}$, where p' is the conjugate of p.*

Proof. From the definition of the class $\mathcal{A}_p^\alpha$ and since $1-p' = 1/(1-p) < 0$, we have for $c > 1$, that

$$\begin{aligned} v \in \mathcal{A}_p^\alpha &\iff \left(\frac{1}{t^\alpha}\int_0^t v(s)d^\alpha s\right) \leqslant \mathcal{C}\left(\frac{1}{t^\alpha}\int_0^t v^{\frac{1}{1-p}}(s)d^\alpha s\right)^{1-p} \\ &\iff \left(\frac{1}{t^\alpha}\int_0^t v(s)d^\alpha s\right)^{\frac{1}{1-p}} \geqslant \mathcal{C}^{\frac{1}{1-p}}\frac{1}{t^\alpha}\int_0^t v^{\frac{1}{1-p}}(s)d^\alpha s \\ &\iff \frac{1}{t^\alpha}\int_0^t v^{1-p'}(s)d^\alpha s \leqslant \mathcal{C}^{p'-1}\left(\frac{1}{t^\alpha}\int_0^t (v^{1-p'}(s))^{\frac{1}{1-p'}}d^\alpha s\right)^{1-p'} \\ &\iff v^{1-p'} \in \mathcal{A}_{p'}^\alpha, \end{aligned}$$

with $[\mathcal{A}_{p'}^\alpha(v^{1-p'})] = [\mathcal{A}_p^\alpha(v)]^{p'-1}$. The proof is complete. □

The following Lemma will be used to prove our main results.

Lemma 3.5. *Let v be a nonincreasing weight. If $p > 1$, then*

$$\frac{1}{t^\alpha}\int_0^t \left[v(s)\left(\mathcal{M}^\alpha v(s)\right)^{p-1} - \frac{p-\alpha}{p}\left(\mathcal{M}^\alpha v(s)\right)^p\right]d^\alpha s \leqslant \frac{1}{p}\left(\mathcal{M}^\alpha v(t)\right)^p,$$

for all $t \in I_0$.

Proof. Let $\mathcal{M}^\alpha v(t) = V(t)$ and $t \in I_0$. Moreover, since $t^\alpha V(t) = \int_0^t v(s)d^\alpha s$, the product rule (see Theorem 2.2) yields

$$t^\alpha D^\alpha V(t) + \alpha V(t) = v(t).$$

Put $f(t) = t^\alpha$ and $g(t) = V^p(t)$, and by integrating by parts (see Theorem 2.4), we have

$$\alpha\int_0^t V^p(s)d^\alpha s = t^\alpha V^p(t) - \int_0^t s^\alpha D^\alpha V^p(s)d^\alpha s. \tag{3.7}$$

Note that

$$\lim_{s\to 0^+} s^\alpha V^p(s) = \lim_{s\to 0^+} s^\alpha\left(\frac{1}{s^\alpha}\int_0^s v(x)d^\alpha x\right)^p = 0.$$

By using the chain rule (see Theorem 2.2), we see that

$$D^\alpha V^p(s) = pV^{p-1}(s)D^\alpha V(s). \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\begin{aligned} \alpha\int_0^t V^p(s)d^\alpha s &= t^\alpha V^p(t) - p\int_0^t \left[v(s) - \alpha V(s)\right]V^{p-1}(s)d^\alpha s \\ &= t^\alpha V^p(t) - p\int_0^t v(s)V^{p-1}(s)d^\alpha s + \alpha p\int_0^t V^p(s)d^\alpha s. \end{aligned} \tag{3.9}$$

From (3.9) and (3.7), we obtain

$$\alpha\int_0^t V^p(s)d^\alpha s = t^\alpha V^p(t) - p\int_0^t v(s)V^{p-1}(s)d^\alpha s + \alpha p\int_0^t V^p(s)d^\alpha s.$$

Since $\alpha p < p$, we see that

$$\alpha\int_0^t V^p(s)d^\alpha s \leqslant t^\alpha V^p(t) - p\int_0^t v(s)V^{p-1}(s)d^\alpha s + p\int_0^t V^p(s)d^\alpha s,$$

i.e.,

$$\frac{1}{t^\alpha}\int_0^t \left[v(s)V^{p-1}(s) - \frac{p-\alpha}{p}V^p(s)\right] d^\alpha s \leqslant \frac{V^p(t)}{p}.$$

The proof is complete. □

Theorem 3.6. *Let v be a nonnegative and nonincreasing weight. If $\mathcal{M}^\alpha v(t) \leqslant \mathcal{C}v(t)$ for some $\mathcal{C} > 1$, then for $p \in [1, \alpha\mathcal{C}/(\mathcal{C}-1))$, we have that*

$$\mathcal{M}^\alpha(v(t))^p \leqslant A[\mathcal{M}^\alpha v(t)]^p, \text{ for all } t \in I_0,$$

where A is given by

$$A := \frac{\mathcal{C}^{1-p}}{\alpha\mathcal{C} - p(\mathcal{C}-1)}.$$

Proof. From the definition of $\mathcal{M}^\alpha v(t)$ and Lemma 3.5 with $p = p > 1$, we see that

$$\frac{1}{t^\alpha}\int_0^t \left[v(s)\left(\mathcal{M}^\alpha v(s)\right)^{p-1} - \frac{p-\alpha}{p}\left(\mathcal{M}^\alpha v(s)\right)^p\right] d^\alpha s \leqslant \frac{1}{p}\left(\mathcal{M}^\alpha v(t)\right)^p. \tag{3.10}$$

Define the function

$$\Omega(\beta) = \gamma\beta^{p-1} - \frac{p-\alpha}{p}\beta^p, \text{ for all } \gamma > 0 \text{ and } \beta \geqslant \gamma, \tag{3.11}$$

and

$$D^\alpha\Omega(\beta) = \gamma(p-1)\beta^{p-1-\alpha} - (p-\alpha)\beta^{p-\alpha} \leqslant (p-1)\beta^{p-\alpha} - (p-\alpha)\beta^{p-\alpha} = (\alpha-1)\beta^{p-\alpha} < 0.$$

That is $\Omega(\beta)$ is decreasing for $\beta \geqslant \gamma$. From Lemma 3.1, we see that

$$\mathcal{M}^\alpha v(s) \geqslant \frac{1}{\alpha}v(s).$$

Now, by taking that $\gamma = v(s)$, $\beta = \mathcal{M}^\alpha v(s)$, and $\theta = \mathcal{C}v(t)$, we see that $\gamma \leqslant \beta \leqslant \theta$, and then we have

$$\Omega(\gamma) \geqslant \Omega(\beta) \geqslant \Omega(\theta) \text{ for } \gamma \leqslant \beta \leqslant \theta.$$

This implies, by using (3.11), that

$$\begin{aligned} v(s)\left(\mathcal{M}^\alpha v(s)\right)^{p-1} - \frac{p-\alpha}{p}\left(\mathcal{M}^\alpha v(s)\right)^p &\geqslant v(s)\left(\mathcal{C}v(s)\right)^{p-1} - \frac{p-\alpha}{p}\left(\mathcal{C}v(s)\right)^p \\ &= \mathcal{C}^{p-1}\left(v(s)\right)^p - \frac{p-\alpha}{p}\mathcal{C}^p\left(v(s)\right)^p \\ &= \mathcal{C}^{p-1}\left[1 - \frac{p-\alpha}{p}\mathcal{C}\right]\left(v(s)\right)^p. \end{aligned} \tag{3.12}$$

By combining (3.10) and (3.12), we get that

$$\mathcal{C}^{p-1}\left[\frac{p-(p-\alpha)\mathcal{C}}{p}\right]\frac{1}{t^\alpha}\int_0^t \left(v(s)\right)^p d^\alpha s \leqslant \frac{1}{p}\left(\mathcal{M}^\alpha v(t)\right)^p,$$

which implies that

$$\frac{1}{t^\alpha}\int_0^t \left(v(s)\right)^p d^\alpha s \leqslant \frac{\mathcal{C}^{1-p}}{\alpha\mathcal{C}-p(\mathcal{C}-1)}\left(\mathcal{M}^\alpha v(t)\right)^p.$$

The proof is complete. □

In the following we will prove the property of the parameter of Muckenhoupt and Gehring classes for power-low on conformable calculus.

Lemma 3.7.

(i) *If* $p > 1$ *and* $-\alpha < \gamma < p - \alpha$, *then the norm* $\mathcal{A}_p^\alpha(t^\gamma) = \Psi(p, \gamma, \alpha)$, *where*

$$\Psi(p, \alpha, \gamma) = \frac{1}{(\alpha + \gamma)}\left(\frac{p-1}{\alpha(p-1) - \gamma}\right)^{p-1}.$$

(ii) *If* $0 < \gamma < \beta$, *then* $\mathcal{A}_p^\alpha(t^{-\gamma}) < \mathcal{A}_p^\alpha(t^{-\beta})$.

Proof. From the definition of the norm of $\mathcal{A}_p^\alpha(v)$, we have

$$\mathcal{A}_p^\alpha(t^\gamma) := \sup_{t \in I_0} \left(\frac{1}{t^\alpha}\int_0^t s^\gamma d^\alpha s\right)\left(\frac{1}{t^\alpha}\int_0^t s^{\frac{\gamma}{1-p}} d^\alpha s\right)^{p-1}.$$

Now, we determine the integration in the right-hand side. We start by $\int_0^t s^\gamma d^\alpha s$, then

$$\frac{1}{t^\alpha}\int_0^t s^\gamma d^\alpha s = \frac{1}{t^\alpha}\int_0^t s^\gamma s^{\alpha-1} ds = \frac{1}{t^\alpha}\frac{t^{\alpha+\gamma}}{(\alpha+\gamma)}. \tag{3.13}$$

Also, we have that

$$\left(\frac{1}{t^\alpha}\int_0^t s^{\frac{\gamma}{1-p}} d^\alpha s\right)^{p-1} = \left(\frac{1}{t^\alpha}\int_0^t s^{\frac{\gamma}{1-p}} s^{\alpha-1} ds\right)^{p-1} = \left(\frac{p-1}{\alpha(p-1)-\gamma}\right)^{p-1}\frac{1}{t^{\alpha(p-1)}}\left(t^{\alpha+\frac{\gamma}{1-p}}\right)^{p-1}. \tag{3.14}$$

By combining (3.13) and (3.14), we see that

$$\mathcal{A}_p^\alpha(t^\gamma) = \frac{1}{(\alpha+\gamma)}\left(\frac{p-1}{\alpha(p-1)-\gamma}\right)^{p-1}\sup_{t\in I_0}\frac{t^{\alpha+\gamma}}{t^{\alpha p}}\left(t^{\alpha+\frac{\gamma}{1-p}}\right)^{p-1}. \tag{3.15}$$

From (3.15), we see that

$$\mathcal{A}_p^\alpha(t^\gamma) = \Psi(p,\alpha,\gamma)\sup_{t\in I_0}\frac{t^{\alpha+\gamma}}{t^{\alpha p}}\left(t^{\alpha+\frac{\gamma}{1-p}}\right)^{p-1}.$$

We define

$$\zeta(t,p,\alpha,\gamma) = t^{\alpha+\gamma}t^{-\alpha p}\left(t^{\alpha+\frac{\gamma}{1-p}}\right)^{p-1}, \text{ for } t>1,\ p>1,\ -\alpha<\gamma<p-1.$$

Now, we see that

$$\sup_{t>1}\zeta(t,p,\alpha,\gamma) = \sup_{t>1} t^{\alpha+\gamma}t^{-\alpha p}\left(t^{\alpha+\frac{\gamma}{1-p}}\right)^{p-1} = \sup_{t>1} t^0 = 1,$$

for all fixed $p > 1$ and $-\alpha < \gamma < p-1$. This gives us that $\mathcal{A}_p^\alpha(t^\gamma) = \Psi(p,\alpha,\gamma)$, which proves statement (i). By noting that

$$F(x) = \frac{1}{(\alpha+x)}\left(\frac{p-1}{\alpha(p-1)-x}\right)^{p-1}$$

is a decreasing function for $x > 0$, we have that $F(-\gamma) < F(-\beta)$ if $0 < \gamma < \beta$. The proof is complete. □

Lemma 3.8.

(i) *If* $p > 1$ *and* $\gamma > -1/p$, *then the norm* $\left(\mathcal{G}_p^\alpha(t^\gamma)\right)^{\frac{p-1}{p}} = \Psi(p,\gamma,\alpha)$, *where*

$$\Phi(p,\alpha,\gamma) = \frac{(\alpha+\gamma)}{(\alpha+p\gamma)^{1/p}}.$$

(ii) *If* $0<\gamma<\beta$, *then* $\left(\mathcal{G}_p^\alpha(t^\gamma)\right)^{\frac{p-1}{p}} < \left(\mathcal{G}_p^\alpha(t^{-\beta})\right)^{\frac{p-1}{p}}$.

Proof. From the definition of the norm of $\mathcal{G}_p^\alpha(\nu)$, we have

$$(\mathcal{G}_p^\alpha(t^\gamma))^{\frac{p-1}{p}} := \sup_{t\in I_0}\left(\frac{1}{t^\alpha}\int_0^t s^\gamma d^\alpha s\right)^{-1}\left(\frac{1}{t^\alpha}\int_0^t s^{p\gamma} d^\alpha s\right)^{\frac{1}{p}}.$$

Now, we determine the integration in the right-hand side. We start by $\int_0^t s^\gamma d^\alpha s$, then

$$\frac{1}{t^\alpha}\int_0^t s^\gamma d^\alpha s = \frac{1}{t^\alpha}\int_0^t s^\gamma s^{\alpha-1} ds = \frac{1}{t^\alpha}\frac{t^{\alpha+\gamma}}{(\alpha+\gamma)}. \tag{3.16}$$

Also, we have that

$$\left(\frac{1}{t^\alpha}\int_0^t s^{p\gamma} d^\alpha s\right)^{\frac{1}{p}} = \frac{1}{(\alpha+p\gamma)^{1/p}}\frac{1}{(t^\alpha)^{1/p}}\left(t^{\alpha+p\gamma}\right)^{1/p}. \tag{3.17}$$

By combining (3.16) and (3.17), we see that

$$(\mathcal{G}_p^\alpha(t^\gamma))^{\frac{p-1}{p}} = \frac{(\alpha+\gamma)}{(\alpha+p\gamma)^{1/p}}\sup_{t\in I_0}\left(t^{\alpha+p\gamma}\right)^{1/p}(t^\alpha)^{1-\frac{1}{p}}\left(t^{\alpha+\gamma}\right)^{-1}. \tag{3.18}$$

From (3.18), we see that

$$(\mathcal{G}_p^\alpha(t^\gamma))^{\frac{p-1}{p}} = \Phi(p,\alpha,\gamma)\sup_{t\in I_0}\left(t^{\alpha+p\gamma}\right)^{1/p}(t^\alpha)^{1-\frac{1}{p}}\left(t^{\alpha+\gamma}\right)^{-1}.$$

We define

$$\zeta(t,p,\alpha,\gamma) = \left(t^{\alpha+p\gamma}\right)^{1/p}(t^\alpha)^{1-\frac{1}{p}}\left(t^{\alpha+\gamma}\right)^{-1}, \text{ for } t>1, p>1, \alpha\in(0,1) \text{ and } \gamma>-1/p.$$

Now, we see that

$$\sup_{t>1}\zeta(t,p,\alpha,\gamma) = \sup_{t>1}\left(t^{\alpha+p\gamma}\right)^{1/p}(t^\alpha)^{1-\frac{1}{p}}\left(t^{\alpha+\gamma}\right)^{-1} = \sup_{t>1} t^0 = 1,$$

for all fixed $p>1$ and $\gamma>-1/p$. This gives us that $(\mathcal{G}_p^\alpha(t^\gamma))^{\frac{p-1}{p}} = \Phi(p,\alpha,\gamma)$, which proves statement (i). By noting that

$$F(x) = \frac{(\alpha+x)}{(\alpha+px)^{1/p}}$$

is a decreasing function for $x>0$, we have that $F(\gamma)<F(\beta)$ if $0<\gamma<\beta$. The proof is complete. □

Conclusion 3.9. In this paper, we study the structure of the Muckenhoupt class $\mathcal{A}_p^\alpha(\mathcal{C})$ and the Gehring class $\mathcal{G}_q^\alpha(\mathcal{K})$ via conformable calculus. Also, we study the relationships between the two classes and prove that if $\nu\in\mathcal{A}_1^\alpha(\mathcal{C})$, then $\nu\in\mathcal{G}_p^\alpha(\mathcal{K})$. We generalize the property of the parameter of Muckenhoupt and Gehring classes for power-low via conformable calculus. We aim to generalize the results of the weighted classes and use the results to prove the boundedness of operators in conformable version.

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