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Variational iteration method: A tools for solving partial differential equations

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Abstract

In this paper, He's variational iteration method (VIM) has been used to obtain solution nonlinear gas dynamics equation and Stefan equation. This method is based on Lagrange multipliers for identification of optimal values of parameters in a functional. Using this method creates a sequence which tends to the exact solution of the problem.

Keywords

Variational iteration method; nonlinear gas dynamics equation; Stefan equation; Partial differential equation.

1 Introduction

Analytical methods commonly used to solve nonlinear equations are very restricted and numerical techniques involving discretization of the variables on the other hand gives rise to rounding off errors.
Recently introduced variational iteration method by He [5,6,7,8], which gives rapidly convergent successive approximations of the exact solution if such a

solution exists, has proved successful in deriving analytical solutions of linear and nonlinear differential equations. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory. He [6,7] has applied this method for obtaining analytical solutions of autonomous ordinary differential equation, nonlinear partial differential equations with variable coefficients and integro-differential equations. The variational iteration method was successfully applied to Seventh order Sawada-Kotera equations [11], to Schruodinger-KdV, generalized KdV and shallow water equations [1], to linear Helmholtz partial differential equation [12]. Linear and nonlinear wave equations, KdV, K(2,2), Burgers, and cubic Boussinesq equations have been solved by Wazwaz [14,15] using the variational iteration method. In the present paper we employ VIM method for solving following equations. Consider the homogeneous gas dynamics equation

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}=u(1-u), \qquad 0\le \mathrm{x}\le 1\ ,\mathrm{t}\ge 0 \tag{1}$$

with initial condition

$$u(x,0)=g(x). \tag{2}$$

Also we employ VIM method for solving following equations. Consider the Stefan equation

$$\frac{\partial u}{\partial t}=\frac{\partial u^2}{\partial x^2}, \qquad 0\le \mathrm{x}\le \mathrm{s}(t)\ ,\mathrm{t}\ge 0 \tag{3}$$

with initial condition

$$u(x,0)=-1\ , \quad \mathrm{t}\ge 0$$

$$u(s(x),t)=0, \quad \mathrm{t}\ge 0$$

$$\frac{\partial u}{\partial x}=\frac{\partial s(x)}{\partial t}.$$

Further we compare the result with given solutions using ADM [2,3, 4]. The paper has been organized as follows. Section 2, gives a brief review of VIM. Section 3,5 consists of main results of the paper, in which variational iteration method of nonlinear gas dynamics equation and nonlinear Stefan equation have been developed. In Section 4,6 illustrative examples are given.In Section 6,illustrative examples are given. Conclusions are presented in Section 7.

2 He's variational iteration method

According to the variational iteration method, we consider the following differential equation:

$$Lu+Nu=g(x,t), \tag{4}$$

Where L is a linear operator, N a non-linear operator and *g(t)* is the source inhomogeneous term. According to the variational interation method, we can construct a Correction functional as follow

$$u_{n+1}(x,t)=u_n(x,t)+\int_o^t \lambda(\xi)\{Lu_n(\xi)+N\tilde{u}_n(\xi)-g(\xi)\}d\xi, \quad \mathrm{n}\ge 0, \tag{5}$$

Where λ is a general Lagrangian multiplier, which can be identified optimally via the variational theory, the second term on the right is called the correction and $\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$.
So, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), n \geq 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0 .Consequently, the solution

$$\mathbf{u(x,t)} = \lim_{\mathbf{n}\to\infty} \mathbf{u_n(x,t)}. \qquad \mathbf{(6)}$$

3 Applying VIM for nonlinear gas dynamics equation

In this section, we shall introduce a reliable algorithm based on VIM to handle singular initial value problem in a realistic and efficient way considering gas dynamics equation as a model problem.
Its correction variational functional in t-direction to obtain the solution of gas dynamics equation (1) by variational iteration method can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}(u_n) - u_n + (N(\tilde{u}_n))\right] d\xi, \ \mathrm{n} \geq 0, \qquad (7)$$

where $N\tilde{u}_n(x,\xi) = \frac{1}{2}\tilde{u}_{nx}^2(x,\xi) + \tilde{u}^2(x,\xi)$. taking variation with respect to the independent variable u_n noticing that $\delta N\tilde{u}_n(x,\xi) = 0$.

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^t \lambda(\xi)\left[\frac{\partial}{\partial\xi}(u_n) - u_n + (N(\tilde{u}_n))\right] d\xi$$

$$= \delta u_n(x,t) + \lambda\delta u_n\big|_{\xi=t} - \int_0^t \lambda'(\xi)\delta u_n + \lambda(\xi)\delta u_n d\xi = 0, \qquad (8)$$

This yields the stationary conditions

$$1 + \lambda(\xi) = 0, \quad \lambda(\xi) + \lambda'(\xi)\big|_{\xi=t} = 0. \qquad (9)$$

This in turn gives $\lambda(\xi) = -\mathrm{e}^{\mathrm{t}-\xi}$. Substituting this value of the Lagrange multiplier into the functional (13)gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial\xi}(u_n) - u_n + (N(\tilde{u}_n))\right] d\xi, \quad \mathrm{n} \geq 0 \qquad (10)$$

thus, we can obtain approximation solutions for u(x,t), considering boundary conditions and insert $u\ (x,t) \cong u_n(x,t)$.

4 Illustrative Example

To demonstrate the effectiveness of the method we consider here Eqs. [3] with given initial condition.

Example 4.1 Consider the nonlinear gas dynamics equations (1) with the initial condition

$u(x,0) \cong e^{-x}$. (11)

Substituting(11) into Eq.(10) we obtain the following successive approximations

$u_0(x,t) \cong e^{-x}$,

$$u_1(x,t) = u_0(x,t) - \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial \xi}(u_0) - u_0 + \frac{1}{2}\tilde{u}_{0x}^2(x,\xi) + \tilde{u}_0^2(x,\xi)\right]d\xi,$$

$$= e^{t-x},$$

$$u_2(x,t) = u_1(x,t) - \int_0^t e^{t-\xi}\left[\frac{\partial}{\partial \xi}(u_1) - u_1 + \frac{1}{2}\tilde{u}_{1x}^2(x,\xi) + \tilde{u}_1^2(x,\xi)\right]d\xi,$$

$$= e^{t-x},$$

Finally, $u_n(x,t) = e^{t-x}$, then we can write, $u(x,t) = e^{t-x}$.

we can see that calculated solution will be the following exact solution:

$$u(x,t) = e^{t-x} \quad (12)$$

.

Remark 1: Nonlinear gas dynamics equation has been solved by ADM by Evans and Bulut [3]. they obtained exact solution after some iteration but we obtain exact solution after 2 iteration.

5 Applying VIM for Stefan problem

In this section, we apply VIM to stefan prablem(Eq.(3)). Its correction variational functional in t-direction to obtain the solution of stefan prablem (3) by variational iteration method can be expressed as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(\xi,t)\left[\frac{\partial^2}{\partial \xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi, \ n \geq 0, \quad (13)$$

where $N\tilde{u}_n(\xi,t) = -\tilde{u}_{nt}(\xi,t)$. taking variation with respect to the independent variable u_n noticing that $\delta N\tilde{u}_n(\xi,t) = 0$

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\int_0^x \lambda(\xi,t)\left[\frac{\partial^2}{\partial \xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi \quad (14)$$

$$= \delta u_n(x,t) + \lambda(\xi,t)\delta u_{nx}\big|_{\xi=x}$$

$$- \lambda'(\xi,t)\delta u_n\big|_{\xi=x} + \int_0^x \lambda''(\xi,t)\delta u_n \, d\xi = 0 \quad (15)$$

This yields the stationary conditions

$$1 - \lambda'(\xi,t)_{\xi=x} = 0, \quad \lambda(\xi,t)_{\xi=x} = 0, \quad \lambda''(\xi,t)_{\xi=x} = 0. \quad (16)$$

This in turn gives $\lambda(\xi,t)_{\xi=x} = \xi - x$. Substituting this value of the Lagrange multiplier into the functional[13] gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (\xi - x)\left[\frac{\partial^2}{\partial \xi^2}(u_n) + (N(\tilde{u}_n))\right]d\xi, \quad (17)$$

thus, we can obtain approximation or exact solution for u(x,t).

6 Illustrative Example

To demonstrate the effectiveness of the method we consider here Eqs. [3] with given initial condition.

Example 6.1 Consider Stefan problem (3) with the I.C.

$$u_0(x,t) = -x + e^{-t}. \qquad (18)$$

Substituting [18] into [17] we obtain the following successive approximations

$$u_1(x,t) = -x + e^t + \frac{1}{2}x^2e^t,$$

$$u_2(x,t) = -x + e^t + \frac{1}{2}x^2e^t + \frac{1}{24}x^4e^t,$$

$$u_3(x,t) = -x + e^t + \frac{1}{2}x^2e^t + \frac{1}{24}x^4e^t + \frac{1}{720}x^6e^t,$$

finally, $u_n(x,t) = -x + e^t \cosh x$, we have $u(x,t) = -x + e^t \cosh x$ which is exact solution.

7 Conclusion

Variational iteration method is a powerful tool which is capable of handling linear/nonlinear partial differential equations. The method have been successfully applied to nonlinear gas dynamics equation and Stefan equation.

Also, comparisons were made between He's variational iteration method and Adomian decomposition method (ADM) for nonlinear gas dynamics equation and Stefan equation.

The VIM reduces the volume of calculations without requiring to compute the Adomian polynomials. He's variational iteration method facilitates the computational work and gives the solution rapidly if compared with Adomian method.

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