

Allee effect in a Ricker type predator-prey model



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Abstract

The stability of the predator-prey model subject to the Allee effect is an interesting topic in recent times. The impact of a weak Allee effect on the stability of a discrete-time predator-prey model is investigated in this paper. Equilibrium analysis, stability analysis, and bifurcation theory are used to examine the mathematical properties of the proposed model. By using the Allee parameter as the bifurcation parameter, we provide sufficient conditions for the flip bifurcation. Numerical simulations are used to demonstrate our analytical conclusions.

Keywords: Allee effect, predator-prey system, chaos.

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1. Introduction

Theoretical ecologists and mathematical biologists have become more interested in mathematical modelling of population dynamics in recent years [10, 13, 16, 27]. Continuous and discrete models are two types of mathematical models often used to model population dynamics. Continuous-time models defined by differential equations and discrete-time models represented by difference equations. Discrete-time population models have gotten a lot of attention in recent years. The following are the reasons for this. When populations have non-overlapping generations or the number of populations is minimal, discrete-time models are more appropriate than continuous-time models. Second, discrete-time models provide more accurate numerical simulation results. Furthermore, discretization is used to produce numerical simulations of continuous-time models. Finally, discrete-time models feature complicated dynamical behaviors; single-species discrete-time models, for example, have bifurcations, chaos, and more complex dynamical behaviors [12, 14, 20, 28, 29]. The Lotka-Volterra model of predator-prey interactions is the first and most basic. Lotka [25] and Volterra [30] developed the model independently. The Lotka-Volterra model assumes that a predator's prey consumption rate is directly proportional to the abundance of prey. This indicates that predator feeding is only limited by the availability of prey. While this may be plausible at low prey densities, it is impractical at large prey densities, when predators are constrained

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by time and digestion. Researchers have proposed a number of improvements to the model, including Holling type functional responses and density-dependent prey growth. The Allee effect can be used to alter the Lotka-Volterra model in another way. It is generally recognized that introducing the Allee effect into the system makes modelling the prey-predator interaction more realistic. Allee first described the Allee effect in 1931 [1]. It describes a positive correlation between any measure of species fitness and population numbers. The difficulty in finding mates, inbreeding depression, social dysfunction in small populations, predator avoidance, and food exploitation are the main drivers of the Allee effect [4, 8, 15]. Many natural species have shown evidence of the Allee effect, including plants, insects, marine invertebrates, birds and mammals [6]. According to recent research, the Allee effect has significant dynamical impacts on population model stability analyses. The Allee effect causes either instability or stabilization in the system. A positive fixed point's local stability can be altered from stable to unstable or vice versa. The Allee effect can cause complex dynamics in predator-prey systems by changing the dynamics of the system in unexpected ways. However, few publications investigate the dynamical behaviors of predator-prey models with the Allee effect, such as bifurcations and chaos occurrences for discrete-time models. Celik and Duman examined a discrete-time predator-prey model with the Allee effect on the prey population and discovered that it stabilizes the population [4]. In a discrete-time predator-prey model, Wang et al. investigated the Allee effect on both populations [31]. Chen et al. investigated a discrete-time predator-prey model with the Allee effect and discovered that as the Allee parameter is increased, the model may transition from complex unstable states to stable ones [5].

In [18], the author has modified the density-dependent prey growth to follow Ricker model. Ricker introduced his model in the context of stock and recruitment in fisheries in 1954 [26]. In this paper, we consider that the density of prey follows the Ricker model. In addition, we suppose that the predator eats the prey in accordance with the Holling type-I functional response. Moreover, we will subject weak Allee effect to that prey growth function as follows:

$$x_{n+1} = x_n + rx_n e^{1-x_n} (1 - e^{-\epsilon x_n}) - \alpha x_n y_n, \quad y_{n+1} = y_n + \alpha x_n y_n - \delta y_n, \quad (1.1)$$

where x_n and y_n are the densities of prey and predator populations; r , ϵ , α and δ are positive parameters. The following is the interpretation of the the components of the model (1.1):

- $rx_n e^{1-x_n}$ represents the rate of the increase of the prey population in the absence of predator, and $1 - e^{-\epsilon x_n}$ is the term for mate-finding Allee effect.
- The term $\alpha x_n y_n$ stands for the rate of decrease due to predation. It also stands for predator density variation as a function of prey population.
- The drop in predator population due to natural death is represented by δy_n .

The main goal of this research is to investigate the impact of subject Allee effect to Ricker model. The structure of this paper is as follows. The existence and local stability of equilibria in model (1.1) are discussed in Section 2. In Section 3, we examine flip bifurcation and Neimark-Sacker bifurcation for model (1.1) by using ϵ as a bifurcation parameter. We give numerical simulations in Section 4, which not only demonstrate our theoretical results, but also show sophisticated dynamical behaviors such as the cascade of period-doubling bifurcation in periods 2, 4, and 8, as well as quasi-periodic orbits and chaotic sets. The discussion is presented in Section 5.

2. Fixed points: existence and stability

In this section, we investigate the existence and stability of equilibrium points of the model (1.1) in $\mathbb{R}_+^2$. We begin by discussing the existence of model (1.1) equilibria. Clearly, $E_0 = (0, 0)$ is a model (1.1) equilibria. The other equilibria of model (1.1) satisfy

$$r e^{1-x^*} (1 - e^{-\epsilon x^*}) - \alpha y^* = 0, \quad \alpha x^* - \delta = 0, \quad (2.1)$$

by solving the system (2.1) for x^* and y^* , we obtain

$$E_1 = \left(\frac{\delta}{\alpha}, \frac{re^{\frac{\alpha-\delta}{\alpha}} \left(1 - e^{-\frac{\epsilon\delta}{\alpha}} \right)}{\alpha} \right),$$

which always exists for positive parameter values. The biological meaning of these two fixed points is as follows. The fixed point E_0 refers to a situation in which there is no prey and no predators. The fixed point E_1 denotes the coexistence of a fixed nonzero number of predators and prey.

After finding the equilibrium points, we need to investigate their stability. In order to do that, we have to find the variation matrix. $J(x, y)$ is the Jacobian matrix of the model (1.1) at the fixed point (x, y) , which is given by

$$J(x, y) = \begin{pmatrix} r(((\epsilon + 1)x - 1)e^{-\epsilon x} - x + 1)e^{1-x} - \alpha y + 1 & -\alpha x \\ \alpha y & \alpha x - \delta + 1 \end{pmatrix},$$

the Jacobian matrix has the following characteristic equation:

$$\rho^2 - p(x, y)\rho + q(x, y) = 0, \quad (2.2)$$

where

$$\begin{aligned} p(x, y) &= r(((\epsilon + 1)x - 1)e^{-\epsilon x} - x + 1)e^{1-x} - \alpha y + 2 + \alpha x - \delta, \\ q(x, y) &= r(((\epsilon + 1)x - 1)e^{-\epsilon x} - x + 1)(\alpha x - \delta + 1)e^{1-x} + \alpha x + (\delta - 1)(\alpha y - 1). \end{aligned}$$

Proposition 2.1. *The equilibrium point $E_0(0, 0)$ of model (1.1) is a non-hyperbolic equilibrium point.*

Proof. It is clear that the eigenvalues of $J(E_1)$ are $\rho_1 = 1$ and $\rho_2 = 1 - \delta$, as a result, the proof is completed. $\square$

The following proposition demonstrates the local dynamics of the equilibrium point E_1 .

Proposition 2.2. *If the following requirements are met, the equilibrium point E_1 of model (1.1) is locally asymptotically stable:*

- (i) $r\delta \left((\alpha - 2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha - 2\epsilon - 2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} \right) + 4\alpha > 0;$
- (ii) $(\alpha - \epsilon - 1)e^{-\frac{\epsilon\delta}{\alpha}} > \alpha - 1.$

Proof. Evaluating the characteristic equation (2.2) of the Jacobian matrix J of the linearized system of model (1.1) about E_1 we get:

$$p(x, y) = \frac{r\delta(\epsilon + 1)e^{\frac{\alpha+(-\epsilon-1)\delta}{\alpha}} - e^{\frac{\alpha-\delta}{\alpha}}\delta r + 2\alpha}{\alpha}, \quad q(x, y) = \frac{-\delta \left((\alpha - \epsilon - 1)e^{-\frac{\epsilon\delta}{\alpha}} + 1 - \alpha \right) re^{\frac{\alpha-\delta}{\alpha}} + \alpha}{\alpha},$$

the positive fixed point E_1 is locally asymptotically stable, using [11, Theorem 4.4 p.200], if

$$|p(x, y)| < 1 + q(x, y) < 2,$$

the criterion $1 - p(x, y) + q(x, y) > 0$ is satisfied when

$$r\delta \left(e^{\frac{\alpha-\delta}{\alpha}} - e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} \right) > 0,$$

that easily seen to be equivalent to

$$e^{\frac{\alpha-\delta}{\alpha}} > e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}},$$

clearly, this condition is always satisfied. Next, the criterion $1 + p(x, y) + q(x, y) > 0$ is

$$\frac{r\delta\left((\alpha-2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha-2\epsilon-2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}}\right) + 4\alpha}{\alpha} > 0$$

which is the same as

$$r\delta\left((\alpha-2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha-2\epsilon-2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}}\right) + 4\alpha > 0.$$

The criterion $1 - q(x, y) > 0$ translates as

$$\frac{\delta\left((\alpha-\epsilon-1)e^{-\frac{\epsilon\delta}{\alpha}} + 1 - \alpha\right)re^{\frac{\alpha-\delta}{\alpha}}}{\alpha} > 0,$$

which is positive when

$$(\alpha - \epsilon - 1)e^{-\frac{\epsilon\delta}{\alpha}} > \alpha - 1.$$

□

The case of the nonhyperbolic fixed point is more complicated. There are various possibilities based on eigenvalues of 1 or -1. When one eigenvalue is on the unit circle and the other is inside the unit circle, center manifold theory is commonly used to establish the stability of this fixed point [11, 22, 32].

Proposition 2.3. E_1 loses stability:

(i) via a period doubling bifurcation if the conditions listed below are met:

$$(a) \quad r = \frac{4\alpha}{\delta\left((\alpha-2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha-2\epsilon-2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}}\right)};$$

$$(b) \quad r\delta(\epsilon+1)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} - r\delta e^{\frac{\alpha-\delta}{\alpha}} + 2\alpha < 0;$$

(ii) via a Neimark-Sacker bifurcation if the conditions listed below are met:

$$(a) \quad \left| \frac{r\delta(\epsilon+1)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} - r\delta e^{\frac{\alpha-\delta}{\alpha}} + 2\alpha}{\alpha} - 1 \right| < \frac{-\delta\left((\alpha-\epsilon-1)e^{-\frac{\epsilon\delta}{\alpha}} + 1 - \alpha\right)re^{\frac{\alpha-\delta}{\alpha}} + \alpha}{\alpha};$$

$$(b) \quad \delta = -\frac{\ln\left(\frac{\alpha-1}{\alpha-(\epsilon+1)}\right)\alpha}{\epsilon}.$$

Proof. By using [11, Theorem 4.5 p.203], the period doubling bifurcation conditions are $1 + p(x, y) + q(x, y) = 0$ and $q(x, y) > 0$. The condition $1 + p(x, y) + q(x, y) = 0$ is satisfied, using Proposition (2.2), when

$$\frac{r\delta\left((\alpha-2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha-2\epsilon-2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}}\right) + 4\alpha}{\alpha} = 0.$$

Solving for r , we get

$$r = \frac{4\alpha}{\delta\left((\alpha-2)e^{\frac{\alpha-\delta}{\alpha}} - (\alpha-2\epsilon-2)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}}\right)},$$

the other condition is met when

$$\frac{r\delta(\epsilon+1)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} - r\delta e^{\frac{\alpha-\delta}{\alpha}} + 2\alpha}{\alpha} < 0,$$

that easily seen to be equivalent to

$$r\delta(\epsilon+1)e^{\frac{\alpha-(\epsilon+1)\delta}{\alpha}} - r\delta e^{\frac{\alpha-\delta}{\alpha}} + 2\alpha < 0.$$

Part (ii) can be proved in a similar way. □

We may conclude from the above considerations that the Allee parameter ϵ influences the stability of the system's inner equilibrium points.

3. Analysis of bifurcation

3.1. Neimark-Sacker bifurcation about E_1

Consider the parameter ϵ in a vicinity of ϵ^* , i.e., $\epsilon = \epsilon^* + \mu$ in which $\mu \ll 1$, then the discrete model (1.1) becomes

$$x_{n+1} = x_n + rx_n e^{1-x_n} (1 - e^{-(\epsilon^* + \mu)x_n}) - \alpha x_n y_n, \quad y_{n+1} = y_n + \alpha x_n y_n - \delta y_n, \quad (3.1)$$

the characteristic equation of $J(E'_1)$ about $E'_1 = \left(\frac{\delta}{\alpha}, \frac{re^{\frac{\alpha-\delta}{\alpha}} \left(1 - e^{-\frac{(\epsilon^* + \mu)\delta}{\alpha}} \right)}{\alpha} \right)$ of the model (3.1) is

$$\rho^2 - p(\mu)\rho + q(\mu) = 0,$$

where

$$p(\mu) = \frac{r\delta((\epsilon^* + \mu) + 1)e^{\frac{\alpha - ((\epsilon^* + \mu) + 1)\delta}{\alpha}} - e^{\frac{\alpha - \delta}{\alpha}} \delta r + 2\alpha}{\alpha}, \quad q(\mu) = \frac{-\delta \left((\alpha - \epsilon - 1)e^{-\frac{(\epsilon^* + \mu)\delta}{\alpha}} + 1 - \alpha \right) re^{\frac{\alpha - \delta}{\alpha}} + \alpha}{\alpha},$$

the roots of characteristic equation of $J(E'_1)$ are

$$\begin{aligned} \rho_{1,2} &= \frac{p(\mu) \pm \sqrt{4q(\mu) - p^2(\mu)}}{2}, \\ &= \frac{r\delta((\epsilon^* + \mu) + 1)e^{\frac{\alpha - ((\epsilon^* + \mu) + 1)\delta}{\alpha}} - e^{\frac{\alpha - \delta}{\alpha}} \delta r + 2\alpha}{2\alpha} \\ &\quad \pm \frac{1}{2} \sqrt{-\frac{\left(r\delta(\mu + \epsilon + 1)^2 e^{\frac{(-2\mu - 2\epsilon - 2)\delta + 2\alpha}{\alpha}} - 2r\delta(\mu + \epsilon + 1)e^{\frac{(-\mu - \epsilon - 2)\delta + 2\alpha}{\alpha}} + e^{\frac{2\alpha - 2\delta}{\alpha}} \delta r - 4e^{\frac{\alpha - \delta}{\alpha}} \alpha^2 + 4e^{\frac{\alpha + (-\epsilon - \mu - 1)\delta}{\alpha}} \alpha^2 \right) r\delta}{\alpha^2}}, \\ |\rho_{1,2}| &= \sqrt{q(\mu)} = \sqrt{\frac{r\delta((\epsilon^* + \mu) + 1)e^{\frac{\alpha - ((\epsilon^* + \mu) + 1)\delta}{\alpha}} - e^{\frac{\alpha - \delta}{\alpha}} \delta r + 2\alpha}{\alpha}}, \end{aligned}$$

and

$$\left. \frac{d|\rho_{1,2}|}{d\mu} \right|_{\mu=0} = \frac{(\alpha + (-1 + \alpha - \epsilon)\delta)e^{\frac{\alpha + (-\epsilon - 1)\delta}{\alpha}} \delta r}{2\alpha^{3/2} \sqrt{-r\delta(-1 + \alpha - \epsilon)e^{\frac{\alpha + (-\epsilon - 1)\delta}{\alpha}} + \delta re^{\frac{\alpha - \delta}{\alpha}} (\alpha - 1) + \alpha}} > 0,$$

additionally, we required that when $\mu = 0$, $\rho_{1,2}^m \neq 1$, $m = 1, 2, 3, 4$, which corresponds to $p(0) \neq -2, 0, 1, 2$. This can be shown by calculation. If $u_n = x_n - x^*$, $v_n = y_n - y^*$, then the fixed point E_1 of model (1.1) is transformed into $O(0, 0)$. After manipulation, one obtains

$$\begin{aligned} u_{n+1} &= u_n + r(u_n + x^*) e^{1-x^* - u_n} (1 - e^{-(\epsilon^* + \mu)(u_n + x^*)}) - \alpha(u_n + x^*)(v_n + y^*), \\ v_{n+1} &= v_n + \alpha(u_n + x^*)(v_n + y^*) - \delta(v_n + y^*), \end{aligned} \quad (3.2)$$

where $x^* = \frac{\delta}{\alpha}$, $y^* = \frac{re^{\frac{\alpha - \delta}{\alpha}} \left(1 - e^{-\frac{(\epsilon^* + \mu)\delta}{\alpha}} \right)}{\alpha}$. The normal form of model (3.2) is investigated further when $\mu = 0$. Up to fourth order, (3.2) is expanded about $(u_n, v_n) = (0, 0)$. We obtain from the Taylor series

$$\begin{aligned} u_{n+1} &= \omega_{11}u_n + \omega_{12}v_n + \omega_{13}u_n^2 + \omega_{14}u_nv_n + \omega_{15}u_n^3 + O(|u_n|^4), \\ v_{n+1} &= \omega_{21}u_n + \omega_{22}v_n + \omega_{23}u_nv_n, \end{aligned} \quad \Bigg\}$$

where

$$\begin{aligned} \omega_{11} &= r \left(((\epsilon^* + 1)x^* - 1)e^{-\epsilon^*x^*} + 1 - x^* \right) e^{1-x^*} - \alpha y^* + 1, \quad \omega_{12} = -\alpha x^*, \\ \omega_{13} &= -\frac{1}{2} \left[re^{1-x^*} \left(((\epsilon^* + 1)x^* - 2)(\epsilon^* + 1)e^{-\epsilon^*x^*} - x^* + 2 \right) \right], \quad \omega_{14} = -\alpha, \\ \omega_{15} &= \frac{1}{6} \left[re^{1-x^*} \left((\epsilon^* + 1)^2 (\epsilon^* x^* + x^* - 3)e^{-\epsilon^*x^*} - x^* + 3 \right) \right], \\ \omega_{21} &= \alpha y^*, \quad \omega_{22} = \alpha x^* - \delta + 1, \quad \omega_{23} = \alpha. \end{aligned}$$

Now, let

$$\eta = \frac{r\delta((\epsilon^* + \mu) + 1)e^{\frac{\alpha - ((\epsilon^* + \mu) + 1)\delta}{\alpha}} - e^{\frac{\alpha - \delta}{\alpha}} \delta r + 2\alpha}{2\alpha},$$

$$\zeta = \frac{1}{2} \sqrt{-\frac{\left(r\delta(\mu + \epsilon + 1)2e^{\frac{(-2\mu - 2\epsilon - 2)\delta + 2\alpha}{\alpha}} - 2r\delta(\mu + \epsilon + 1)e^{\frac{(-\mu - \epsilon - 2)\delta + 2\alpha}{\alpha}} + e^{\frac{2\alpha - 2\delta}{\alpha}} \delta r - 4e^{\frac{\alpha - \delta}{\alpha}} \alpha^2 + 4e^{\frac{\alpha + (-\epsilon - \mu - 1)\delta}{\alpha}} \alpha^2\right)r\delta}{\alpha^2}},$$

in addition to, the invertible matrix T defined by

$$T = \begin{pmatrix} \omega_{12} & 0 \\ \eta - \omega_{11} & -\zeta \end{pmatrix}.$$

Using the next translation:

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} \omega_{12} & 0 \\ \eta - \omega_{11} & -\zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

(3.2) gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \eta & -\zeta \\ \zeta & \eta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \Phi(X_n, Y_n) \\ \Psi(X_n, Y_n) \end{pmatrix}, \quad (3.3)$$

where

$$\Phi(X_n, Y_n) = \Pi_{11}X_n^2 + \Pi_{12}X_nY_n + \Pi_{13}X_n^3 + O(|X_n|^4),$$

$$\Psi(X_n, Y_n) = \Pi_{21}X_n^2 + \Pi_{22}X_nY_n + \Pi_{23}X_n^3 + O(|X_n|^4),$$

and

$$\Pi_{11} = \omega_{12}\omega_{13} + \omega_{14}(\eta - \omega_{11}),$$

$$\Pi_{12} = -\zeta\omega_{14},$$

$$\Pi_{13} = \omega_{15}\omega_{12}^2,$$

$$\Pi_{21} = \frac{\eta - \omega_{11}}{\zeta} [\omega_{12}(\omega_{13} - \omega_{23}) + \omega_{14}(\eta - \omega_{11})],$$

$$\Pi_{22} = \omega_{12}\omega_{23} - \omega_{14}(\eta - \omega_{11}),$$

$$\Pi_{23} = \frac{1}{\zeta} ((\eta - \omega_{11})\omega_{15}\omega_{12}^2).$$

In addition,

$$\Phi_{X_n X_n}|_{(0,0)} = 2\Pi_{11}, \quad \Phi_{X_n Y_n}|_{(0,0)} = \Pi_{12}, \quad \Phi_{Y_n Y_n}|_{(0,0)} = 0,$$

$$\Phi_{X_n X_n X_n}|_{(0,0)} = 6\Pi_{13}, \quad \Phi_{X_n X_n Y_n}|_{(0,0)} = \Phi_{X_n Y_n Y_n}|_{(0,0)} = \Phi_{Y_n Y_n Y_n}|_{(0,0)} = 0,$$

and

$$\Psi_{X_n X_n}|_{(0,0)} = 2\Pi_{21}, \quad \Psi_{X_n Y_n}|_{(0,0)} = \Pi_{22}, \quad \Psi_{Y_n Y_n}|_{(0,0)} = 0,$$

$$\Psi_{X_n X_n X_n}|_{(0,0)} = 6\Pi_{23}, \quad \Psi_{X_n X_n Y_n}|_{(0,0)} = \Psi_{X_n Y_n Y_n}|_{(0,0)} = \Psi_{Y_n Y_n Y_n}|_{(0,0)} = 0.$$

In order for (3.3) to undergo a Neimark-Sacker bifurcation, it is mandatory that the following discriminatory quantity, i.e., $\xi \neq 0$ (see [11, 22, 24]),

$$\xi = -\operatorname{Re} \left[\frac{(1 - 2\bar{\rho})\bar{\rho}^2}{1 - \rho} \tau_{11}\tau_{20} \right] - \frac{1}{2} \|\tau_{11}\|^2 - \|\tau_{02}\|^2 + \operatorname{Re}(\bar{\rho}\tau_{21}),$$

where

$$\tau_{02} = \frac{1}{8} [\Phi_{X_n X_n} - \Phi_{Y_n Y_n} + 2\Psi_{X_n Y_n} + \iota(\Psi_{X_n X_n} - \Psi_{Y_n Y_n} + 2\Phi_{X_n Y_n})] \Big|_{(0,0)},$$

$$\tau_{11} = \frac{1}{4} [\Phi_{X_n X_n} + \Phi_{Y_n Y_n} + \iota(\Psi_{X_n X_n} + \Psi_{Y_n Y_n})] \Big|_{(0,0)},$$

$$\tau_{20} = \frac{1}{8} [\Phi_{X_n X_n} - \Phi_{Y_n Y_n} + 2\Psi_{X_n Y_n} + \iota(\Psi_{X_n X_n} - \Psi_{Y_n Y_n} - 2\Phi_{X_n Y_n})] \Big|_{(0,0)},$$

$$\tau_{21} = \frac{1}{16} [\Phi_{X_n X_n X_n} + \Phi_{X_n Y_n Y_n} + \Psi_{X_n X_n Y_n} + \Psi_{Y_n Y_n Y_n} + \iota(\Psi_{X_n X_n X_n} + \Psi_{X_n Y_n Y_n} - \Phi_{X_n X_n Y_n} - \Phi_{Y_n Y_n Y_n})] \Big|_{(0,0)},$$

after calculating, we get

$$\tau_{02} = \frac{1}{4} [\Pi_{11} + \Pi_{22} + \iota(\Pi_{21} + \Pi_{12})], \quad \tau_{11} = \frac{1}{2} [\Pi_{11} + \iota\Pi_{21}],$$

$$\tau_{20} = \frac{1}{4} [\Pi_{11} + \Pi_{22} + \iota(\Pi_{21} - \Pi_{12})], \quad \tau_{21} = \frac{3}{8} [\Pi_{13} + \iota(\Pi_{23})].$$

Based on this study and the Neimark-Sacker bifurcation theorem (explained in [7, 17, 19, 22, 23]), we obtain the following Proposition.

Proposition 3.1. *If $\xi \neq 0$, then the discrete model (1.1) experiences a Neimark-Sacker bifurcation at E_1 as the parameters satisfy (ii) from Proposition (2.3). Moreover, an attracting (resp. repelling) closed curve bifurcates from E_2 if $\xi < 0$ (resp. $\xi > 0$).*

Remark 3.2. Based on the bifurcation theory presented in [23], if the discriminatory quantity ξ is negative, the bifurcation is called a super-critical Neimark-Sacker bifurcation.

3.2. Period-doubling bifurcation about E_1

The flip bifurcation of a system (1.1) is discussed in this section. According to [17], implementing the center manifold theorem, which allows us to focus our attention on the center manifold, is an excellent strategy for bifurcation analysis. Because the center manifold theorem is dependent on the normal form of the map, we must first utilize co-ordinate transformations to simplify the system's analytic expressions in normal form before applying center manifold theory. In order to achieve flip bifurcation, the linearized system's Jacobian matrix must have one eigenvalue $\rho_1 = -1$ and the modulus of another eigenvalue $|\rho_2| \neq 1$. To examine the flip bifurcation of the equilibrium point E_1 of system, we use the Allee parameter ϵ as a bifurcation parameter. The equilibrium point is neither stable nor unstable at the bifurcation point, but when the flip bifurcation occurs, the equilibrium point loses its stability and the system transitions to a new behavior with period-2. We take ϵ^* to be a new dependent variable, and we obtain

$$x_{n+1} = x_n + rx_n e^{1-x_n} (1 - e^{-(\epsilon+\epsilon^*)x_n}) - \alpha x_n y_n, \quad y_{n+1} = y_n + \alpha x_n y_n - \delta y_n. \quad (3.4)$$

Let $u_n = x_n - x^*, v_n = y_n - y^*$, then, model (3.4)'s equilibrium E_1 transforms to $O(0,0)$. Calculating yields

$$u_{n+1} = \widehat{\omega}_{11}u_n + \widehat{\omega}_{12}v_n + \widehat{\omega}_{13}u_n^2 + \widehat{\omega}_{14}u_nv_n + \widehat{\omega}_{15}u_n\epsilon^* + \widehat{\omega}_{16}u_n^3 + \widehat{\omega}_{17}u_n(\epsilon^*)^2 + \widehat{\omega}_{18}u_n^2\epsilon^* + O(|u_n|, |\epsilon^*|)^4,$$

$$v_{n+1} = \widehat{\omega}_{21}u_n + \widehat{\omega}_{22}v_n + \widehat{\omega}_{23}u_nv_n, \quad (3.5)$$

where

$$\widehat{\omega}_{11} = r \left(((\epsilon+1)x^* - 1)e^{-\epsilon x^*} + 1 - x^* \right) e^{1-x^*} - \alpha y^* + 1, \quad \widehat{\omega}_{12} = -\alpha x^*,$$

$$\widehat{\omega}_{13} = -\frac{re^{1-x^*} (((\epsilon+1)x^* - 2)(\epsilon+1)e^{-\epsilon x^*} - x^* + 2)}{2}, \quad \widehat{\omega}_{14} = -\alpha,$$

$$\widehat{\omega}_{15} = -rx^*((\epsilon+1)x^* - 2)e^{1-(\epsilon+1)x^*}, \quad \widehat{\omega}_{16} = \frac{re^{1-x^*} ((\epsilon+1)^2((\epsilon+1)x^* - 3)e^{-\epsilon x^*} - x^* + 3)}{6},$$

$$\widehat{\omega}_{17} = \frac{r(x^*)^2((\epsilon+1)x^* - 3)e^{1-(\epsilon+1)x^*}}{2}, \quad \widehat{\omega}_{18} = \frac{r(2 + (\epsilon+1)^2(x^*)^2 - 4(\epsilon+1)x^*)e^{1-(\epsilon+1)x^*}}{2},$$

$$\widehat{\omega}_{21} = \alpha y^*, \quad \widehat{\omega}_{22} = \alpha x^* - \delta + 1, \quad \widehat{\omega}_{23} = \alpha.$$

Now, construct an invertible matrix T

$$T = \begin{pmatrix} \widehat{\omega}_{12} & \widehat{\omega}_{12} \\ -1 - \widehat{\omega}_{11} & \rho_2 - \widehat{\omega}_{11} \end{pmatrix},$$

and use the translation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} \widehat{\omega}_{12} & \widehat{\omega}_{12} \\ -1 - \widehat{\omega}_{11} & \rho_2 - \widehat{\omega}_{11} \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

(3.5) gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \rho_2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \widehat{\Phi}(u_n, v_n, \epsilon^*) \\ \widehat{\Psi}(u_n, v_n, \epsilon^*) \end{pmatrix}, \quad (3.6)$$

where

$$\begin{aligned} \widehat{\Phi}(u_n, v_n, \epsilon^*) &= \frac{\widehat{\omega}_{13}(\rho_2 - \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^2 + \frac{\widehat{\omega}_{14}(\rho_2 - \widehat{\omega}_{11}) - \widehat{\omega}_{12}\widehat{\omega}_{23}}{\widehat{\omega}_{12}(1 + \rho_2)} u_n v_n + \frac{\widehat{\omega}_{15}(\rho_2 - \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n \epsilon^* \\ &\quad + \frac{\widehat{\omega}_{18}(\rho_2 - \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^2 \epsilon^* + \frac{\widehat{\omega}_{16}(\rho_2 - \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^3 + \frac{\widehat{\omega}_{17}(\rho_2 - \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n (\epsilon^*)^2 + O(|u_n|, |v_n|, |\epsilon^*|)^4, \\ \widehat{\Psi}(u_n, v_n, \epsilon^*) &= \frac{\widehat{\omega}_{13}(1 + \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^2 + \frac{\widehat{\omega}_{14}(1 + \widehat{\omega}_{11}) + \widehat{\omega}_{12}\widehat{\omega}_{23}}{\widehat{\omega}_{12}(1 + \rho_2)} u_n v_n + \frac{\widehat{\omega}_{15}(1 + \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n \epsilon^* \\ &\quad + \frac{\widehat{\omega}_{18}(1 + \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^2 \epsilon^* + \frac{\widehat{\omega}_{16}(1 + \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n^3 + \frac{\widehat{\omega}_{17}(1 + \widehat{\omega}_{11})}{\widehat{\omega}_{12}(1 + \rho_2)} u_n (\epsilon^*)^2 + O(|u_n|, |v_n|, |\epsilon^*|)^4, \\ u_n^2 &= \widehat{\omega}_{12}^2 (X_n^2 + 2X_n Y_n + Y_n^2), \\ u_n v_n &= -\widehat{\omega}_{12}(1 + \widehat{\omega}_{11}) X_n^2 + (\widehat{\omega}_{12}(\rho_2 - \widehat{\omega}_{11}) - \widehat{\omega}_{12}(1 + \widehat{\omega}_{11})) X_n Y_n + \widehat{\omega}_{12}(\rho_2 - \widehat{\omega}_{11}) Y_n^2, \\ u_n \epsilon^* &= \widehat{\omega}_{12} X_n \epsilon^* + \widehat{\omega}_{12} Y_n \epsilon^*, \\ u_n^2 \epsilon^* &= \widehat{\omega}_{12}^2 (X_n^2 \epsilon^* + 2X_n Y_n \epsilon^* + Y_n^2 \epsilon^*). \end{aligned}$$

Following this, we identify the center manifold $W^c(0,0)$ of (3.6) at $(0,0)$ in a small neighborhood of ϵ^* [3, 22, 23, 33]. According to the center manifold theorem, a center manifold $W^c(0,0)$ can be expressed as follows:

$$W^c(0,0) = \left\{ (X_n, Y_n) : Y_n = c_0 \epsilon^* + c_1 X_n^2 + c_2 X_n \epsilon^* + c_3 (\epsilon^*)^2 + O(|X_n|, |\epsilon^*|)^3 \right\},$$

where $O(|X_n| + |\epsilon^*|)^3$ is a function with order at least three in their variables (X_n, ϵ^*) , and

$$c_0 = 0, \quad c_1 = \frac{(1 + \widehat{\omega}_{11})(\widehat{\omega}_{11} + 1)\widehat{\omega}_{14} - (\widehat{\omega}_{13} - \widehat{\omega}_{23})\widehat{\omega}_{12}}{\rho_2^2 - 1}, \quad c_2 = -\frac{\widehat{\omega}_{15}(1 + \widehat{\omega}_{11})}{(1 + \rho_2)^2}, \quad c_3 = 0.$$

Consequently, the map (3.6) is restricted to $W^c(0,0)$ as follows:

$$f(X_n) = -X_n + h_1 X_n^2 + h_2 X_n \epsilon^* + h_3 X_n^2 \epsilon^* + h_4 X_n (\epsilon^*)^2 + h_5 X_n^3 + O(|X_n|, |\epsilon^*|)^4, \quad (3.7)$$

where

$$\begin{aligned} h_1 &= \frac{1}{1 + \rho_2} \left[\widehat{\omega}_{11}^2 \widehat{\omega}_{14} + ((\widehat{\omega}_{23} - \widehat{\omega}_{13})\widehat{\omega}_{12} - \widehat{\omega}_{14}(\rho_2 - 1)) \widehat{\omega}_{11} + (\rho_2 \widehat{\omega}_{13} + \widehat{\omega}_{23}) \widehat{\omega}_{12} - \rho_2 \widehat{\omega}_{14} \right], \\ h_2 &= \frac{1}{1 + \rho_2} [\widehat{\omega}_{15}(\rho_2 - \widehat{\omega}_{11})], \\ h_3 &= \frac{1}{(\rho_2 - 1)(\rho_2 + 1)^3} \left[\rho_2^4 \widehat{\omega}_{12} \widehat{\omega}_{18} - (\widehat{\omega}_{14} \widehat{\omega}_{15}(1 + \widehat{\omega}_{11}) + \widehat{\omega}_{12} \widehat{\omega}_{18}(\widehat{\omega}_{11} - 1)) \rho_2^3 + 4(1 + \widehat{\omega}_{11}) \right. \\ &\quad \left. + \left(\left(\frac{\widehat{\omega}_{23}}{2} - \frac{3\widehat{\omega}_{13}}{4} \right) \widehat{\omega}_{12} + \widehat{\omega}_{14} \left(\widehat{\omega}_{11} + \frac{3}{4} \right) \right) \widehat{\omega}_{15} - \frac{\widehat{\omega}_{12} \widehat{\omega}_{18}}{4} \right] \rho_2^2 + \left(-3(-(\widehat{\omega}_{13} - \widehat{\omega}_{23})(\widehat{\omega}_{11} + \frac{1}{3})) \widehat{\omega}_{12} \right. \end{aligned}$$

$$\begin{aligned}
& + \widehat{\omega}_{11}\widehat{\omega}_{14}(\widehat{\omega}_{11} + \frac{3}{4}))(1 + \widehat{\omega}_{11})\widehat{\omega}_{15} + \widehat{\omega}_{12}\widehat{\omega}_{18}(\widehat{\omega}_{11} - 1))\rho_2 + (1 + \widehat{\omega}_{11})((\widehat{\omega}_{23} - \widehat{\omega}_{13})\widehat{\omega}_{11} \\
& + \widehat{\omega}_{23})\widehat{\omega}_{12} + \widehat{\omega}_{11}^2\widehat{\omega}_{14})\widehat{\omega}_{15} + \widehat{\omega}_{11}\widehat{\omega}_{12}\widehat{\omega}_{18}], \\
h_4 = & \frac{(\rho_2 - \widehat{\omega}_{11})(\widehat{\omega}_{17}(\rho_2 + 1)^2 - \widehat{\omega}_{15}^2(1 + \widehat{\omega}_{11}))}{(\rho_2 + 1)^3}, \\
h_5 = & \frac{1}{(\rho_2 + 1)^2(\rho_2 - 1)} \left[(2(\widehat{\omega}_{13} - \widehat{\omega}_{23})^2\widehat{\omega}_{11}^2 + (-\rho_2^2\widehat{\omega}_{16} + (3\widehat{\omega}_{13}\widehat{\omega}_{23} - 2\widehat{\omega}_{13}^2 - \widehat{\omega}_{23}^2)\rho_2 + 2\widehat{\omega}_{13}^2 \right. \\
& - 5\widehat{\omega}_{23}\widehat{\omega}_{13} + 3\widehat{\omega}_{23}^2 + \widehat{\omega}_{16})\widehat{\omega}_{11} + \rho_2^3\widehat{\omega}_{16} + (3\widehat{\omega}_{13}\widehat{\omega}_{23} - 2\widehat{\omega}_{13}^2 - \widehat{\omega}_{23}^2 - \widehat{\omega}_{16})\rho_2 - \widehat{\omega}_{23}(\widehat{\omega}_{13} \\
& - \widehat{\omega}_{23}))\widehat{\omega}_{12}^2 - (4(\widehat{\omega}_{13} - \widehat{\omega}_{23})\widehat{\omega}_{11}^2 + ((4\widehat{\omega}_{23} - 5\widehat{\omega}_{13})\rho_2 + 3\widehat{\omega}_{13} - 4\widehat{\omega}_{23})\widehat{\omega}_{11} + (\widehat{\omega}_{13} - \widehat{\omega}_{23})\rho_2^2 \\
& \left. + (2\widehat{\omega}_{23} - 3\widehat{\omega}_{13})\rho_2 - \widehat{\omega}_{23})(1 + \widehat{\omega}_{11})\widehat{\omega}_{12}\widehat{\omega}_{14} + \widehat{\omega}_{14}^2(1 + \widehat{\omega}_{11})^2(\rho_2 - \widehat{\omega}_{11})(\rho_2 - 2\widehat{\omega}_{11} - 1) \right].
\end{aligned}$$

The following discriminating quantities must be nonzero for the map (3.7) to experience a period-doubling bifurcation:

$$\omega_1 = \left(\frac{\partial^2 f}{\partial X_n \partial \epsilon^*} + \frac{1}{2} \frac{\partial f}{\partial \epsilon^*} \frac{\partial^2 f}{\partial X_n^2} \right) \Big|_{(0,0)}, \quad \omega_2 = \left(\frac{1}{6} \frac{\partial^3 f}{\partial X_n^3} + \left(\frac{1}{2} \frac{\partial^2 f}{\partial X_n^2} \right)^2 \right) \Big|_{(0,0)}.$$

After calculating we obtain

$$\omega_1 = h_2 + \frac{1}{2}h_3 \quad \text{and} \quad \omega_2 = h_5 + h_1^2.$$

From the above analysis in [21] and theorem in [7, 17, 19, 22, 23], we have next Proposition.

Proposition 3.3. *If $\omega_2 \neq 0$, map (3.4) experiences a period-doubling bifurcation about the unique positive equilibrium E_1 , when ϵ^* varies in a small vicinity of $O(0,0)$. Moreover, if $\omega_2 > 0$ (resp. $\omega_2 < 0$), then the period-2 points that bifurcate from E_1 are stable (resp. unstable).*

4. Numerical simulation

In this section, we give some numerical simulations for model (1.1) to support our theoretical results. The bifurcation parameters are explored in the following two cases.

Case 1: Varying ϵ in range $0.07 \leq \epsilon \leq 0.25$ and fixing the other parameters at $r = 12, \alpha = 0.5, \delta = 0.9$. When $0.115082 \leq \epsilon < 0.25$ model (1.1) has a unique stable equilibrium point E_1 , see Figure 1 (a)-(b). E_1 loses its stability via a Neimark-Sacker bifurcation when $\epsilon = 0.115082$ with

$$\frac{(\alpha + (-1 + \alpha - \epsilon)\delta)e^{\frac{\alpha + (-\epsilon - 1)\delta}{\alpha}}\delta r}{2\alpha^{3/2}\sqrt{-r\delta(-1 + \alpha - \epsilon)e^{\frac{\alpha + (-\epsilon - 1)\delta}{\alpha}} + \delta re^{\frac{\alpha - \delta}{\alpha}}(\alpha - 1) + \alpha}} = -0.4226760790 < 0.$$

Moreover, the eigenvalues of J_{E_1} about E_1 are

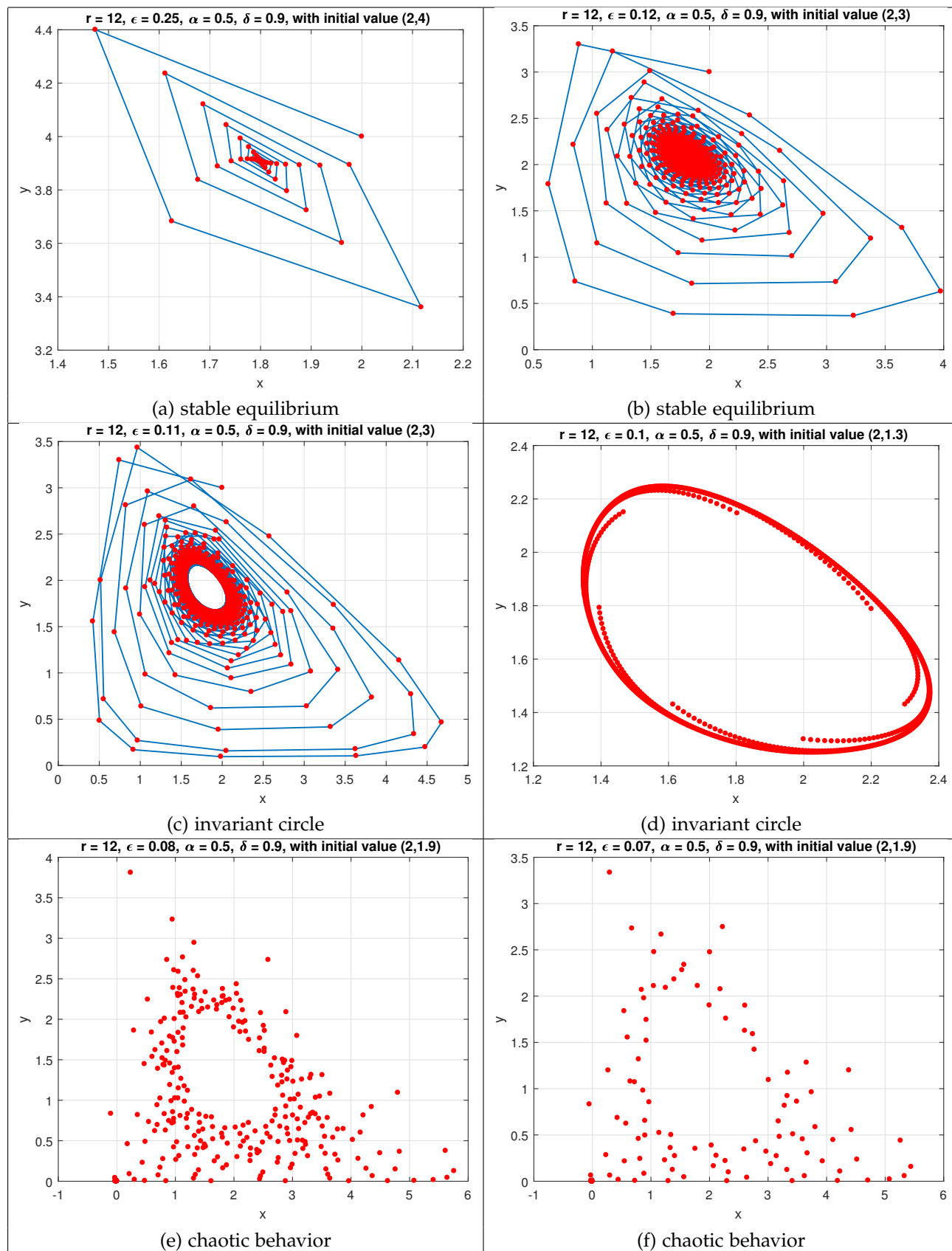
$$\rho_{1,2} = 0.54602388448 \pm 0.8377693009i, \tag{4.1}$$

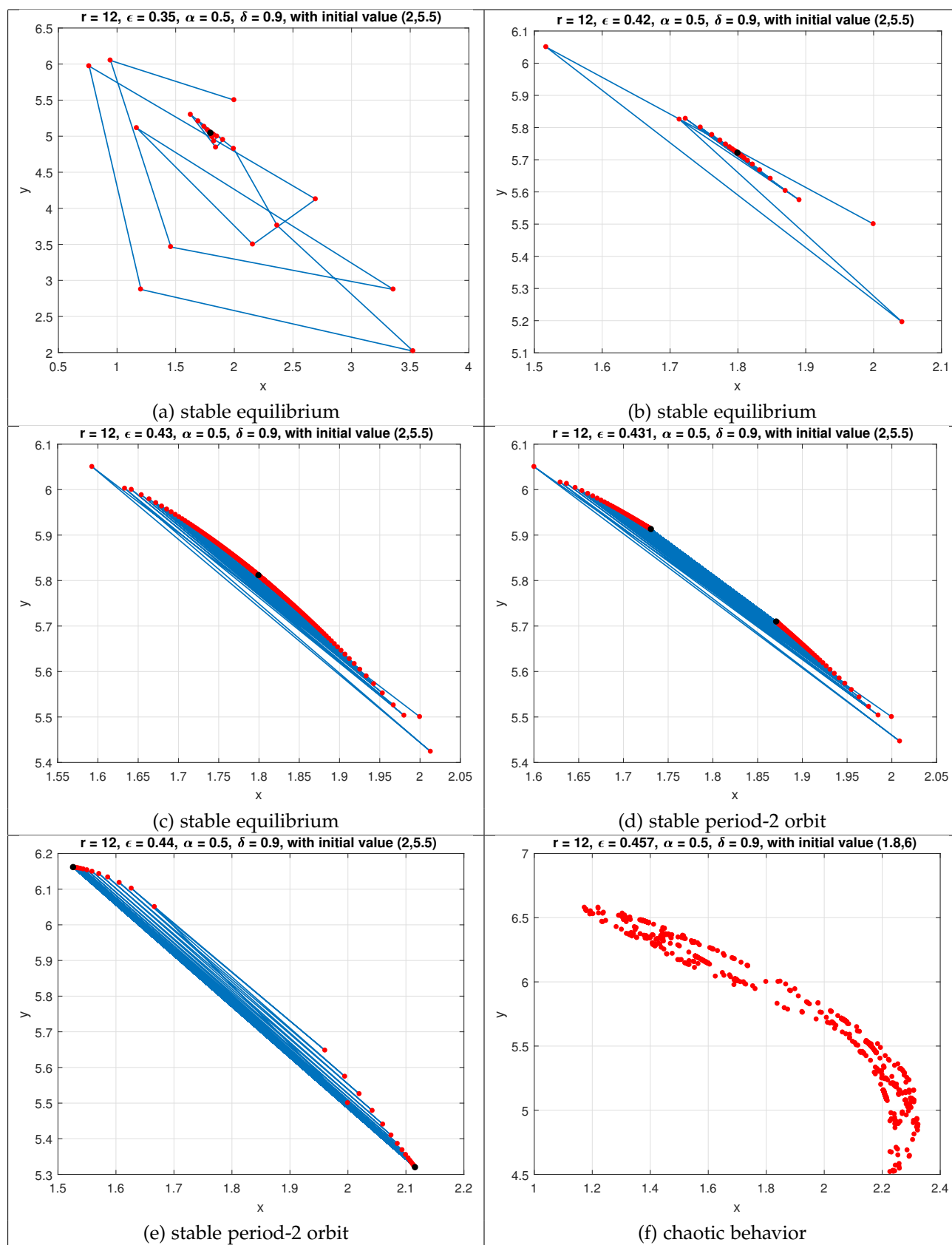
after performing some computations in Maple,

$$\tau_{02} = -0.0620712713i, \tau_{11} = -0.1387147809 - 0.3335848680i, \tau_{20} = -0.135756798i, \tau_{21} = 0. \tag{4.2}$$

The value of the discriminatory quantity is $\xi = 0.01289064811 > 0$ in light of (4.1) and (4.2). As a result, if $\epsilon < 0.115082$, the discrete-time model (1.1) undergoes a subcritical Neimark-Sacker bifurcation, resulting in an unstable invariant close curve, as shown in Figure 1 (c)-(f). The stable fixed point E_1 becomes unstable, allowing preys and predators to coexist by a persistent positive periodic oscillation as time passes.

Case 2: Varying ϵ in range $0.35 \leq \epsilon \leq 5.2$ and fixing $r = 12, \alpha = 0.5, \delta = 0.9$. Figure 2 shows that equilibrium E_1 is stable for $\epsilon < 0.430471$, and loses its stability when $\epsilon = 0.430471$ via a period doubling bifurcation. Further, when $\epsilon > 0.430471$ a chaotic set is emerged with the increasing of r .

Figure 1: Phase portraits of fixed point E_1 and its bifurcation curve.

Figure 2: Phase portraits of fixed point E_1 and its bifurcation curve.

5. Conclusions

Allee effect is considered as an important ecological phenomenon. It might cause a sudden and unexpected extinction. In this work, we investigated the impact of the Allee effect on the stability of a predator-prey model. The local stability and bifurcation diagrams of the model (1.1) have been studied by combining stability analysis, phase-plane analysis, and bifurcation diagram analysis. The stability of the equilibrium point could be changed from stable to unstable or vice versa according to the values of the parameters. Furthermore, the solution will spend less time to reach the stable state when it is stable. It is also worth noting that the Allee effect can help to eliminate the chaos. Because of the natural systems are more complex than models, the Allee effect should not be neglected [9]. The global qualitative analysis of model (1.1) has not been obtained yet, we will leave it for our future work.

References

- [1] W. C. Allee, *Co-operation among animals*, Amer. J. Sociol., **37** (1931), 386–398. 1
- [2] E. S. Allman, J. A. Rhodes, *Mathematical models in biology: an introduction*, Cambridge University Press, Cambridge, (2004).
- [3] J. Carr, *Applications of center Manifold Theory*, Springer-Verlag, New York-Berlin, (2012). 3.2
- [4] C. Celik, O. Duman, *Allee effect in a discrete-time predator-prey system*, Chaos Solitons Fractals, **40** (2009), 1956–1962. 1
- [5] X.-W. Chen, X.-L. Fu, Z.-J. Jing, *Dynamics in a discrete-time predator-prey system with allee effect*, Acta Math. Appl. Sin. Engl. Ser., **29** (2013), 143–164. 1
- [6] F. Courchamp, L. Berec, J. Gascoigne, *Allee effects in ecology and conservation*, OUP Oxford, Oxford, (2008). 1
- [7] J. D. Crawford, *Introduction to bifurcation theory*, Rev. Modern Phys., **63** (1991), 991–1037. 3.1, 3.2
- [8] B. Dennis, *Allee effects: population growth, critical density, and the chance of extinction*, Natur. Resource Modeling, **3** (1989), 481–538. 1
- [9] A. M. De Roos, L. Persson, *Size-dependent life-history traits promote catastrophic collapses of top predators*, Proc. Nat. Acad. Sci., **99** (2002), 12907–12912. 5
- [10] L. Edelstein-Keshet, *Mathematical models in biology*, SIAM, Philadelphia, (2005). 1
- [11] S. N. Elaydi, *Discrete chaos: with applications in science and engineering*, Chapman and Hall/CRC, Boston, (2007). 2, 2, 2, 3.1
- [12] A. M. Fisher, S. J. Cornell, G. I. Holwell, T. A. Price, *Mate-finding allee effects can be exacerbated or relieved by sexual cannibalism*, J. Animal Ecol., **89** (2020), 1581–1592. 1
- [13] G. Fulford, P. Forrester, A. Jones, *Modelling with differential and difference equations*, Cambridge University Press, Cambridge, (1997). 1
- [14] S. J. Gao, L. S. Chen, *The effect of seasonal harvesting on a single-species discrete population model with stage structure and birth pulses*, Chaos Solitons Fractals, **24** (2005), 1013–1023. 1
- [15] J. Gascoigne, R. N. Lipcius, *Allee effects in marine systems*, Marine Ecology Progress Series, **269** (2004), 49–59. 1
- [16] J. Gleick, *Chaos: making a new science*, Viking Press, New York, (2008). 1
- [17] J. Guckenheimer, P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, Springer-Verlag, New York, (2013). 3.1, 3.2, 3.2
- [18] M. Y. Hamada, T. El-Azab, H. El-Metwally, *Bifurcations and dynamics of a discrete predator-prey model of ricker type*, J. Appl. Math. Comput., **2022** (2022), 23 pages. 1
- [19] G. Iooss, *Bifurcation of maps and applications*, North-Holland Publishing Co., Amsterdam-New York, (1979). 3.1, 3.2
- [20] K. N. Keya, M. Kamrujjaman, M. S. Islam, *The influence of density in population dynamics with strong and weak allee effect*, J. Egyptian Math. Soc., **29** (2021), 26 pages. 1
- [21] A. Q. Khan, *Bifurcations of a two-dimensional discrete-time predator-prey model*, Adv. Difference Equ., **2019** (2019), 23 pages. 3.2
- [22] Y. A. Kuznetsov, *Elements of applied bifurcation theory*, Springer-Verlag, New York, (2004). 2, 3.1, 3.2, 3.2
- [23] Y. A. Kuznetsov, H. G. E. Meijer, *Numerical Bifurcation Analysis of Maps: From Theory to Software*, Cambridge University Press, Cambridge, (2019). 3.1, 3.2, 3.2, 3.2
- [24] X. L. Liu, D. M. Xiao, *Complex dynamic behaviors of a discrete-time predator-prey system*, Chaos Solitons Fractals, **32** (2007), 80–94. 3.1
- [25] A. J. Lotka, *Elements of physical biology*, Dover Publications, New York, (1925). 1
- [26] W. E. Ricker, *Stock and recruitment*, J. Fisheries Board Canad., **11** (1954), 559–623. 1
- [27] M. Robert, *Simple mathematical models with complicated dynamics*, Nature, **261** (1976), 459–467. 1
- [28] M. S. Shabbir, Q. Din, K. Ahmad, A. Tassaddiq, A. H. Soori, M. A. Khan, *Stability, bifurcation, and chaos control of a novel discrete-time model involving allee effect and cannibalism*, Adv. Difference Equ., **2020** (2020), 28 pages. 1
- [29] A. Tassaddiq, M. S. Shabbir, Q. Din, H. Naaz, *Discretization, bifurcation, and control for a class of predator-prey interactions*, Fractal and Fractional, **6** (2022), 18 pages. 1

- [30] V. Volterra, *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*, Societa anonima tipografica "Leonardo da Vinci", Italy, (1927). 1
- [31] W.-X. Wang, Y.-B. Zhang, C.-Z. Liu, *Analysis of a discrete-time predator-prey system with allee effect*, *Ecolog. Complex.*, **8** (2011), 81–85. 1
- [32] S. Wiggins, *Introduction to applied nonlinear dynamical systems and chaos*, Springer-Verlag, New York, (1990). 2
- [33] W.-B. Zhang, *Discrete dynamical systems, bifurcations and chaos in economics*, Elsevier, New York, (2006). 3.2