J. Math. Computer Sci., 28 (2023), 252–263

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Asymptotic analysis for the elasticity system with Tresca and maximal monotone graph conditions

Mourad Boudersa[a,*], Mourad Dilmi[b], Hamid Benseridi[b]

[a]Departement of mathematics, Unversity of Batna 2, Algeria.

[b]Departement of mathematics, Unversity of Sétif 1, Algeria.

Abstract

In this paper, we consider the stationary problem in three dimensional thin domain Ω^ε with maximal monotone graph and Tresca conditions. In the first step, we present the problem statement and give the variational formulation. We then study the asymptotic behavior when one dimension of the domain tends to zero. In the latter case a specific Reynolds limit equation is obtained and the uniqueness of the displacement of the limit problem are proved.

Keywords: Asymptotic approach, maximal monotone graph, Tresca law, variational problem, weak solution.

2020 MSC: 35R35, 76F10, 78M35.

1. Introduction

The asymptotic behavior of linear elasticity has been studied by several authors. The study of the non linear problem in a stationary regime in a three dimensional thin domain with non linear friction of Tresca has been considered in [3]. In [13], Lions studied theoretically a problem governed by the Laplace equation with Dirichlet boundary conditions. He proved the existence of a solution based essentially on the method of compactness and the uniqueness of the solution by imposing conditions on the data. The asymptotic analysis of a contact problem in a three dimensional with friction between two elastic bodies was investigated in [12]. The authors in [4] studied the non linear boundary value problem governed by partial differential equations which describe the evolution of linear elastic materials. The asymptotic behavior of a Bingham fluid in a thin domain with boundary conditions non linear was studied in [10]. In [22] the authors worked on the asymptotic convergence of a dynamical problem of a non isothermal linear elasticity with friction of Tresca type. In the last few years, some research papers have been written dealing with the asymptotic analysis of an incompressible fluid in a three-dimensional thin domain, when one dimension of the fluid domain tends to zero (see, e.g., [1, 6, 7] and the references cited therein). More recently, the authors in [2] have studied the asymptotic analysis of a dynamical problem of isothermal elasticity with non linear friction of Tresca type but without the intervention of the nonlinear term. The

*Corresponding author

Email address: mouradboudersa1974@yahoo.fr (Mourad Boudersa)

doi: 10.22436/jmcs.029.03.04

Received: 2022-02-10 *Revised:* 2022-05-11 *Accepted:* 2022-07-29

problem of the junction between three dimensional and two dimensional linearly elastic structures and various asymptotic developments for the junction between plates can be found in [8].

The purpose of this paper is to study the asymptotic analysis of a stationary problem for the linear elasticity represented by a thin domain Ω^ε in $\mathbb{R}^3$ with Tresca and maximal monotone graph boundary conditions. We consider the Dirichlet boundary conditions on $\bar{\Gamma}_1^\varepsilon \cup \bar{\Gamma}_L^\varepsilon$ where $\bar{\Gamma}_L^\varepsilon$ is the lateral one, the maximal monotone graph condition at the top surface $\bar{\Gamma}_1^\varepsilon$, finally, a non-linear Tresca interface condition at the bottom one ω.

This work is organized as follows. In Section 2 we introduce some notations and give the problem statement and variational formulation. In Section 3, we use the change of variable $z = \frac{x_3}{\varepsilon}$, to transform the initial problem posed in the domain Ω^ε into a new problem posed in a fixed domain Ω independent of the parameter ε. We find some estimates on the displacement of the small parameter and prove the convergence theorem by using several inequalities. In Section 4, we investigate the convergence results of the weak problem and its uniqueness.

2. Problem statement and variational formulation

We consider a homogeneous, elastic and isotropic body domain defined by:

$$\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \omega, 0 < x_3 < \varepsilon h(x')\},$$

where ω is a non-empty bounded domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary, $h(.)$ is a Lipschitz continuous function defined on ω such that $0 < h_* \leqslant h(x') \leqslant h^*$, $\forall (x', 0) \in \omega$ and ε is a small parameter that will tend to zero. We decompose the boundary of Ω^ε as $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_L^\varepsilon \cup \omega$ with

$$\begin{aligned}
\omega &= \{(x', x_3) \in \bar{\Omega}^\varepsilon : x_3 = 0\},\\
\Gamma_1^\varepsilon &= \{(x', x_3) \in \bar{\Omega}^\varepsilon : (x', 0) \in \omega,\ x_3 = \varepsilon h\left(x'\right)\},\\
\Gamma_L^\varepsilon &= \{(x', x_3) \in \bar{\Omega}^\varepsilon : x' \in \partial\omega, 0 < x_3 < \varepsilon h(x')\},
\end{aligned}$$

where ω is the bottom of the domain, Γ_1^ε is the upper surface and Γ_L^ε the lateral part of Γ^ε (see Fig. 1). Let $u^\varepsilon(.) : \Omega^\varepsilon \to \mathbb{R}^3$ be the displacement and $\mathbb{S}$ the set of all symmetric 3×3 matrices. Then $D(u^\varepsilon) \in \mathbb{S}$ denotes the symmetric gradient of u^ε whose components are $\frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right)$, $1 \leqslant i, j \leqslant 3$. Let $\sigma^\varepsilon = \left(\sigma_{ij}^\varepsilon\right), i, j = 1, 2, 3$ denotes the stress tensor, with

$$\sigma_{ij}^\varepsilon(u^\varepsilon) = 2\mu d_{ij}(u^\varepsilon) + \lambda d_{kk}(u^\varepsilon)\delta_{ij},$$

where μ and λ are elasticity coefficients with $\lambda > 0$ and $\lambda + \mu \geqslant 0$, δ_{ij} the symbol of Krönecker.

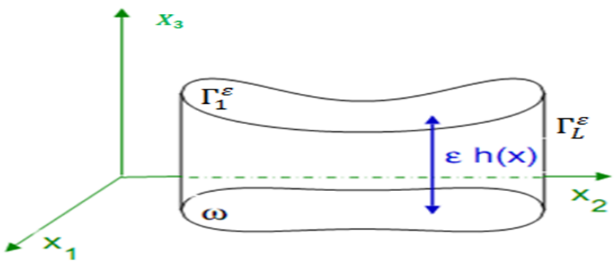

Figure 1: The domain Ω^ε.

We consider now the following mechanical problem.

Problem 2.1. Find a displacement field $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ such that

$$-\operatorname{div} \sigma^\varepsilon + (\alpha^\varepsilon)^2 u^\varepsilon = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{2.1}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon,$$

$$P(u^\varepsilon) - \sigma^\varepsilon(u^\varepsilon).n = \beta_\xi(u^\varepsilon) \quad \text{on } \Gamma_1^\varepsilon, \tag{2.2}$$

$$u^\varepsilon.n = 0 \quad \text{on } \Gamma_1^\varepsilon \cup \omega,$$

$$\left.\begin{array}{r} |\sigma_T^\varepsilon| < k^\varepsilon \implies u_T^\varepsilon = 0 \\ |\sigma_T^\varepsilon| = k^\varepsilon \implies \exists \gamma \geqslant 0 : u_T^\varepsilon = -\gamma \sigma_T^\varepsilon \end{array}\right\} \quad \text{on } \omega, \tag{2.3}$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to Γ^ε, and

$$u_n^\varepsilon = u^\varepsilon.n, \quad u_T^\varepsilon = u^\varepsilon - (u_n^\varepsilon)n, \quad \sigma_n^\varepsilon = (\sigma^\varepsilon.n).n, \quad \sigma_T^\varepsilon = \sigma^\varepsilon.n - (\sigma_n^\varepsilon)n,$$

are, respectively the normal and the tangential components of u^ε on the boundary ω and the components of the normal and the tangential stress tensor on ω. We denote by P the differential operator of the first order to coefficients lipschitzians and β_ξ defined by $\beta_\xi = \xi^{-1}(I + J_\xi)$ with $\xi > 0$, the Yosida's approach of β where $J_\xi = -(I + \xi\beta)^{-1}$ is the resolvante of β which is the maximal monotone graph such that $0 \in \beta(0)$. Boundary condition (2.2) is the regularized condition of Yosida. Condition (2.3) represents a Tresca friction law on ω where k^ε is the friction coefficient and $f^\varepsilon = ({f_i}^\varepsilon)_{1 \leqslant i \leqslant 3}$ is the body forces. Furthermore, the equation (2.1) represents the equilibrium equation, here $\alpha^\varepsilon \geqslant 0$.

To get a weak formulation, we introduce the closed convex

$$V^\varepsilon = \left\{ v \in H^1(\Omega^\varepsilon)^3 : v = 0 \text{ on } \Gamma_L^\varepsilon \text{ and } v.n = 0 \text{ on } \omega \cup \Gamma_1^\varepsilon \right\}.$$

By standard calculations, the variational formulation of the problem 2.1 is given by following.

Problem 2.2. Find a displacement field $u^\varepsilon \in V^\varepsilon$ such that

$$\begin{aligned} a(u^\varepsilon, \varphi - u^\varepsilon) - \int_{\Gamma_1^\varepsilon} P(u^\varepsilon)(\varphi - u^\varepsilon)\, d\tau + \int_{\Gamma_1^\varepsilon} \beta_\xi(u^\varepsilon)(\varphi - u^\varepsilon)\, d\tau \\ + j^\varepsilon(\varphi) - j^\varepsilon(u^\varepsilon) \geqslant (f^\varepsilon, \varphi - u^\varepsilon) \quad \forall \varphi \in V^\varepsilon, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} a(u^\varepsilon, \varphi) &= 2\mu \int_{\Omega^\varepsilon} d_{ij}(u^\varepsilon)\, d_{ij}(\varphi)\, dx + \lambda \int_{\Omega^\varepsilon} \operatorname{div}(u^\varepsilon) \operatorname{div}(\varphi)\, dx + (\alpha^\varepsilon)^2 \int_{\Omega^\varepsilon} u^\varepsilon \varphi\, dx, \\ j^\varepsilon(\varphi) &= \int_\omega k^\varepsilon |\varphi| dx', \\ (f, \varphi) &= \int_{\Omega^\varepsilon} f_i \varphi_i\, dx. \end{aligned}$$

Theorem 2.3. *Problems 2.1 and 2.2 are equivalent.*

Proof. See [3]. □

3. Change of the domain and some estimates

For the asymptotic analysis of problem 2.1, we use the approach which consist in transporting the initially posed problem in the domain Ω^ε which depends on a small parameter ε to an equivalent problem with a fixed domain Ω which is independent of ε. For that, we introduce the change of the variable $z = \dfrac{x_3}{\varepsilon}$, which changes (x', x_3) in Ω^ε to (x', z) in Ω where

$$\Omega = \{(x', z) \in \mathbb{R}^3, (x', 0) \in \omega \text{ and } 0 < z < h(x')\},$$

and we denote by $\Gamma = \bar{\omega} \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_L$ its boundary, then we define the following functions in Ω,

$$\begin{cases} \hat{u}_i^\varepsilon\left(x', z\right) = u_i^\varepsilon\left(x', x_3\right), \quad i = 1,2, \\ \hat{u}_3^\varepsilon\left(x', z\right) = \varepsilon^{-1} u_3^\varepsilon\left(x', x_3\right). \end{cases}$$

Let us assume the following

$$\hat{f}\left(x', z\right) = \varepsilon^2 f^\varepsilon\left(x', x_3\right), \ \hat{k} = \varepsilon k^\varepsilon \text{ and } \hat{\alpha} = \varepsilon \alpha^\varepsilon.$$

Let

$$\begin{aligned} V &= \left\{ \hat{\varphi} \in H^1\left(\Omega\right)^3 : \hat{\varphi} = 0 \text{ on } \Gamma_L \ ; \ \hat{\varphi}.n = 0 \text{ on } \omega \cup \Gamma_1 \right\}, \\ V_z &= \left\{ v = (v_1, v_2) \in L^2\left(\Omega\right)^2 : \frac{\partial v_i}{\partial z} \in L^2\left(\Omega\right), i = 1,2 \text{ and } v = 0 \text{ on } \Gamma_L \right\}, \\ \Pi\left(V\right) &= \left\{ \varphi = (\varphi_1, \varphi_2) \in H^1\left(\Omega\right)^2 : \varphi = 0 \text{ on } \Gamma_L \right\}. \end{aligned}$$

V_z is the Banach space with norm

$$\|v\|_{V_z} = \left(\sum_{i=1}^{2} \left(\|v_i\|^2_{L^2(\Omega)} + \left\| \frac{\partial v_i}{\partial z} \right\|^2_{L^2(\Omega)} \right) \right)^{\frac{1}{2}}.$$

By injecting the new data and unknown factor in (2.4) and after multiplication by ε, we deduce

$$\begin{aligned} &\mu\varepsilon^2 \sum_{i,j=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i} \right) \frac{\partial}{\partial x_j} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) dx' dz \\ &\quad + \mu \sum_{i=1}^{2} \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right) \left[\frac{\partial}{\partial z} \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) + \varepsilon^2 \frac{\partial}{\partial x_i} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) \right] dx' dz \\ &\quad + 2\mu\varepsilon^2 \int_\Omega \frac{\partial \hat{u}_3^\varepsilon}{\partial z} . \frac{\partial}{\partial z} \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) dx' dz + \lambda \varepsilon^2 \int_\Omega \operatorname{div}\left(\hat{u}^\varepsilon \right) \operatorname{div}\left(\hat{\varphi} - \hat{u}^\varepsilon \right) dx' dz \\ &\quad + \hat{\alpha}^2 \sum_{i=1}^{2} \int_\Omega \hat{u}_i^\varepsilon \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) dx' dz + \varepsilon^2 \hat{\alpha}^2 \int_\Omega \hat{u}_3^\varepsilon \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) dx' dz + \int_\omega \hat{k} \left(|\hat{\varphi}| - |\hat{u}^\varepsilon| \right) dx' \\ &\quad - \varepsilon \sum_{i=1}^{2} \int_\omega P_1(\hat{u}_i^\varepsilon) \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' - \varepsilon^3 \int_\omega P_1(\hat{u}_3^\varepsilon) \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' \\ &\quad + \sum_{i=1}^{2} \int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon) \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' + \varepsilon \int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon) \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) \sqrt{1 + |\nabla h^\varepsilon(x')|^2} dx' \\ &\geqslant \sum_{i=1}^{2} \int_\Omega \hat{f}_i \left(\hat{\varphi}_i - \hat{u}_i^\varepsilon \right) dx' dz + \varepsilon \int_\Omega \hat{f}_3 \left(\hat{\varphi}_3 - \hat{u}_3^\varepsilon \right) dx' dz, \quad \forall \hat{\varphi} \in V, \end{aligned}$$

where

$$P_1 = \sum_{i=1}^{2} a_i(x') \frac{\partial}{\partial x_i} + a_0(x'),$$

P_1 is differential operator of the first order and

$$\hat{\beta}_\xi(\hat{u}^\varepsilon) = \varepsilon \beta_\xi(u^\varepsilon).$$

We introduce some results which will be used later.

Lemma 3.1 ([4, Poincaré inequality]). *We have*

$$\int_{\Omega^\varepsilon} |u^\varepsilon|^2 \, dx \leqslant 2\varepsilon h^* \int_{\Gamma_1^\varepsilon} |u^\varepsilon|^2 \, d\tau + 2(\varepsilon h^*)^2 \int_{\Omega^\varepsilon} \left|\frac{\partial u^\varepsilon}{\partial x_3}\right|^2 dx. \tag{3.1}$$

Lemma 3.2 ([4, Korn inequality]). *We have*

$$\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} \leqslant 2\int_{\Omega^\varepsilon} |D(u^\varepsilon)|^2 \, dx + C(\Gamma_1^\varepsilon)\int_{\Gamma_1^\varepsilon} |u^\varepsilon|^2 \, d\tau, \tag{3.2}$$

where

$$C(\Gamma_1^\varepsilon) = 2\left\|\frac{\partial}{\partial x_2} h^\varepsilon\right\|_{C(\bar{\omega})}\left(1 + \left\|\frac{\partial}{\partial x_1} h^\varepsilon\right\|^2_{C(\bar{\omega})}\right).$$

Lemma 3.3 ([4]). *We have*

$$\int_{\Gamma_1} |\hat{u}^\varepsilon|^2 \, d\tau' \leqslant C(\Omega)\int_{\Omega} \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2\right) dx' dz, \tag{3.3}$$

where $C(\Omega)$ *is a constant independent of* ε.

Lemma 3.4 ([12]). *Let* $a_k(x') \in C^{0,1}$ *for* $k = 0,1,2$, *we get*

$$\int_{\omega} P_1(\hat{u}^\varepsilon) dx' \leqslant c^* \int_{\omega} |\hat{u}^\varepsilon| \, dx',$$

where c^* *is a constant independent of* ε.

Now we will obtain a priori estimates for the displacement field $\hat{u}^\varepsilon$ in the domain Ω.

Theorem 3.5. *Assuming that* $\hat{f} \in L^2(\Omega)^3$, *the friction coefficient* $\hat{k} \in L^\infty(\omega)$ *and under the assumpions*

$$\varepsilon C(\Gamma_1^\varepsilon) \leqslant \frac{1}{\mu}, \quad \frac{15}{16}\mu > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \hat{c}, \quad \hat{\alpha}^2 > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \hat{c},$$

there exists a constant $c_i > 0$, $i = 1,2,3$ *independent of* ε *such that*

$$\varepsilon^2 \sum_{i,j=1}^{2}\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j}\right\|^2_{L^2(\Omega)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} + \varepsilon^4\left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right\|^2_{L^2(\Omega)}\right) + \varepsilon^2\left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right\|^2_{L^2(\Omega)} \leqslant c_1, \tag{3.4}$$

$$\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)} \leqslant c_2, \quad \textit{for } i = 1,2, \tag{3.5}$$

$$\|\varepsilon \hat{u}_3^\varepsilon\|_{L^2(\Omega)} \leqslant c_3. \tag{3.6}$$

Proof. Let u^ε be a solution of the variational problem (2.4). Choosing $\varphi = 0$ as test function in (2.4), we get

$$a(u^\varepsilon, u^\varepsilon) - \int_{\Gamma_1^\varepsilon} P(u^\varepsilon) u^\varepsilon \, d\tau + \int_{\Gamma_1^\varepsilon} \beta_\xi(u^\varepsilon) u^\varepsilon \, d\tau + j^\varepsilon(u^\varepsilon) \leqslant (f^\varepsilon, u^\varepsilon).$$

As $j^\varepsilon(u^\varepsilon)$ is positive and β_ξ is increasing lipschitzian with $\beta_\xi(0) = 0$, we have

$$2\mu\int_{\Omega^\varepsilon} |D(u^\varepsilon)|^2 \, dx + (\alpha^\varepsilon)^2 \int_{\Omega^\varepsilon} |u^\varepsilon|^2 \, dx - \int_{\Gamma_1^\varepsilon} P(u^\varepsilon) u^\varepsilon \, d\tau \leqslant (f^\varepsilon, u^\varepsilon). \tag{3.7}$$

By (3.2) and (3.3), we have

$$2\mu\int_{\Omega^\varepsilon} |D(u^\varepsilon)|^2 \, dx \geqslant \mu\|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} - \mu C(\Gamma_1^\varepsilon)\int_{\Gamma_1^\varepsilon} |u^\varepsilon|^2 \, d\tau \tag{3.8}$$

$$
\begin{aligned}
&\geqslant \mu \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} - \mu C(\Gamma_1^{\varepsilon}) \int_{\omega} |u^{\varepsilon}|^2 \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} dx' \\
&\geqslant \mu \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} - \mu C(\Gamma_1^{\varepsilon}) \max_{x'\in\omega} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} \int_{\omega} |u^{\varepsilon}|^2 \, dx' \\
&\geqslant \mu \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} - \mu C(\Gamma_1^{\varepsilon}) \max_{x'\in\omega} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} \int_{\Gamma_1} |\hat{u}^{\varepsilon}|^2 \, d\tau' \\
&\geqslant \mu \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} - \mu C(\Gamma_1^{\varepsilon}) C(\Omega, h) \int_{\Omega} \left(|\hat{u}^{\varepsilon}|^2 + |\nabla \hat{u}^{\varepsilon}|^2\right) dx' dz,
\end{aligned}
$$

where

$$
C(\Omega, h) = C(\Omega) \max_{x'\in\omega} \sqrt{1+|\nabla h(x')|^2},
$$

as

$$
(f^{\varepsilon}, u^{\varepsilon}) \leqslant \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \|u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})},
$$

by (3.1) (Poincaré inequality), we obtain

$$
(f^{\varepsilon}, u^{\varepsilon}) \leqslant \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \left(\sqrt{2\varepsilon h^*} \left(\int_{\Gamma_1^{\varepsilon}} |u^{\varepsilon}|^2 \, d\tau \right)^{\frac{1}{2}} + \sqrt{2}\varepsilon h^* \|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \right),
$$

we have

$$
\sqrt{2}\varepsilon h^* \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} = \sqrt{2}\varepsilon h^* \sqrt{\frac{8}{\mu}} \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \sqrt{\frac{\mu}{8}} \|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}.
$$

Using Young's inequality, we get

$$
\sqrt{2}\varepsilon h^* \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \leqslant \frac{8\varepsilon^2 h^{*2}}{\mu} \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{16} \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})}.
$$

In the same way, we have

$$
\begin{aligned}
\sqrt{2\varepsilon h^*} \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \left(\int_{\Gamma_1^{\varepsilon}} |u^{\varepsilon}|^2 \, d\tau \right)^{\frac{1}{2}} &= \sqrt{2\varepsilon h^*} \sqrt{\frac{\varepsilon}{\mu}} \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \sqrt{\frac{\mu}{\varepsilon}} \left(\int_{\Gamma_1^{\varepsilon}} |u^{\varepsilon}|^2 \, d\tau \right)^{\frac{1}{2}} \\
&\leqslant \frac{\varepsilon^2 h^*}{\mu} \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{2\varepsilon} \int_{\Gamma_1^{\varepsilon}} |u^{\varepsilon}|^2 \, d\tau \\
&\leqslant \frac{\varepsilon^2 h^*}{\mu} \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{2\varepsilon} \max_{x'\in\omega} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} \int_{\omega} |u^{\varepsilon}|^2 \, dx' \\
&\leqslant \frac{\varepsilon^2 h^*}{\mu} \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{2\varepsilon} C(\Omega, h) \int_{\Omega} \left(|\hat{u}^{\varepsilon}|^2 + |\nabla \hat{u}^{\varepsilon}|^2\right) dx' dz.
\end{aligned}
$$

Thus

$$
(f^{\varepsilon}, u^{\varepsilon}) \leqslant \left[\frac{8\varepsilon^2 h^{*2}}{\mu} + \frac{\varepsilon^2 h^*}{\mu} \right] \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{16} \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \frac{\mu}{2\varepsilon} C(\Omega, h) \int_{\Omega} \left(|\hat{u}^{\varepsilon}|^2 + |\nabla \hat{u}^{\varepsilon}|^2\right) dx' dz. \tag{3.9}
$$

On the other hand, by Lemma 3.4, we have

$$
\begin{aligned}
\int_{\Gamma_1^{\varepsilon}} P(u^{\varepsilon}) u^{\varepsilon} \, d\tau &= \sum_{i=1}^{2} \int_{\omega} P_1(\hat{u}_i^{\varepsilon}) \hat{u}_i^{\varepsilon} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} dx' + \varepsilon^2 \int_{\omega} P_1(\hat{u}_3^{\varepsilon}) \hat{u}_3^{\varepsilon} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} dx' \\
&\leqslant \max_{x'\in\omega} \sqrt{1+|\nabla h^{\varepsilon}(x')|^2} \left(\sum_{i=1}^{2} \int_{\omega} P_1(\hat{u}_i^{\varepsilon}) \hat{u}_i^{\varepsilon} \, dx' + \varepsilon^2 \int_{\omega} P_1(\hat{u}_3^{\varepsilon}) \hat{u}_3^{\varepsilon} \, dx' \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant c^* \max_{x'\in\omega} \sqrt{1+|\nabla h^\varepsilon(x')|^2} \left(\sum_{i=1}^{2} \int_\omega |\hat{u}_i^\varepsilon|^2 \, dx' + \varepsilon^2 \int_\omega |\hat{u}_3^\varepsilon|^2 \, dx' \right) \\
&\leqslant c^* \max_{x'\in\omega} \sqrt{1+|\nabla h^\varepsilon(x')|^2} \int_\omega |\hat{u}^\varepsilon|^2 \, dx' \\
&\leqslant c^* \max_{x'\in\omega} \sqrt{1+|\nabla h^\varepsilon(x')|^2} \int_{\Gamma_1} |\hat{u}^\varepsilon|^2 \, d\tau' \\
&\leqslant c^* C(\Omega, h) \int_\Omega \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2 \right) dx' dz.
\end{aligned} \tag{3.10}
$$

By injecting (3.8), (3.9), and (3.10) in (3.7) it becomes

$$
\begin{aligned}
&\frac{15}{16}\mu \|\nabla u^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} + (\alpha^\varepsilon)^2 \int_{\Omega^\varepsilon} |u^\varepsilon|^2 \, dx - \left(\mu C(\Gamma_1^\varepsilon) + c^* + \frac{\mu}{2\varepsilon} \right) C(\Omega, h) \int_\Omega \left(|\hat{u}^\varepsilon|^2 + |\nabla \hat{u}^\varepsilon|^2 \right) dx' dz \\
&\leqslant \frac{\varepsilon^2 h^*}{\mu} (8h^* + 1) \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)}.
\end{aligned} \tag{3.11}
$$

Multiplying inequality (3.11) by ε and using

$$
\varepsilon^2 \|f^\varepsilon\|^2_{L^2(\Omega^\varepsilon)} = \varepsilon^{-1} \|\hat{f}\|^2_{L^2(\Omega)},
$$

we obtain

$$
\begin{aligned}
\left[\frac{15}{16}\mu - \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) \right] \|\nabla \hat{u}^\varepsilon\|^2_{L^2(\Omega)} + \left[\hat{\alpha}^2 - \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) \right] \|\hat{u}^\varepsilon\|^2_{L^2(\Omega)} \\
\leqslant \frac{h^*}{\mu} (8h^* + 1) \|\hat{f}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Assume that

$$
\frac{15}{16}\mu > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \hat{c}, \qquad \hat{\alpha}^2 > \left(1 + c^* + \frac{\mu}{2}\right) C(\Omega, h) = \hat{c},
$$

so

$$
\|\nabla \hat{u}^\varepsilon\|^2_{L^2(\Omega)} \leqslant c_1,
$$

we deduce (3.4) with

$$
c_1 = \frac{h^*}{\mu \hat{c}} (8h^* + 1) \|\hat{f}\|^2_{L^2(\Omega)},
$$

and

$$
\|\hat{u}_i^\varepsilon\|^2_{L^2(\Omega)} \leqslant c_2, \text{ for } i = 1, 2, \qquad \|\varepsilon \hat{u}_3^\varepsilon\|^2_{L^2(\Omega)} \leqslant c_3.
$$

□

4. Convergence results and the limit problem

Theorem 4.1. *Under the same assumptions as in Theorem 3.5, there exists $u^* = (u_1^*, u_2^*) \in V_z$ such that*

$$
\hat{u}_i^\varepsilon \rightharpoonup u_i^*, \quad i = 1, 2, \quad \textit{weakly in } V_z, \tag{4.1}
$$

$$
\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0, \quad i, j = 1, 2, \quad \textit{weakly in } L^2(\Omega), \tag{4.2}
$$

$$
\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0, \quad \textit{weakly in } L^2(\Omega),
$$

$$
\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0, \quad i = 1, 2, \quad \textit{weakly in } L^2(\Omega), \tag{4.3}
$$

$$
\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0, \quad \textit{weakly in } L^2(\Omega).
$$

Proof. From (3.4) and (3.5), we deduce (4.1). Also (4.2) follows from (3.4) and (4.1). From (3.4) and (3.6) we obtain (4.3). □

Theorem 4.2. *With the same assumptions of Theorem 3.5, the solution u^* satisfies the following relations*

$$-\mu\frac{\partial^2 u_i^*}{\partial z^2}+\hat{\alpha}^2 u_i^*=\hat{f}_i, \quad \textit{for } i=1,2, \ \textit{in } L^2(\Omega), \tag{4.4}$$

$$\begin{aligned}\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}(\hat{\varphi}_i-u_i^*)\,dx'dz+\hat{\alpha}^2\sum_{i=1}^{2}\int_\Omega u_i^*(\hat{\varphi}_i-u_i^*)\,dx'dz+\int_\omega \hat{k}(|\hat{\varphi}|-|u^*|)\,dx'\\ \geqslant\sum_{i=1}^{2}\int_\Omega \hat{f}_i(\hat{\varphi}_i-u_i^*)\,dx'dz, \quad \forall\hat{\varphi}\in\Pi(V).\end{aligned} \tag{4.5}$$

Proof. The variational inequality (3.2) can be written as

$$\begin{aligned}&\sum_{i=1}^{4}I_i(\varepsilon)+\lambda\varepsilon^2\int_\Omega \operatorname{div}(\hat{u}^\varepsilon)\operatorname{div}(\hat{\varphi}-\hat{u}^\varepsilon)\,dx'dz+\int_\omega \hat{k}|\hat{\varphi}|\,dx'\\ &\quad+\hat{\alpha}^2\sum_{i=1}^{2}\int_\Omega \hat{u}_i^\xi(\hat{\varphi}_i-\hat{u}_i^\xi)\,dx'dz+\varepsilon^2\hat{\alpha}^2\int_\Omega \hat{u}_3^\xi(\hat{\varphi}_3-\hat{u}_3^\xi)\,dx'dz\\ &\quad-\varepsilon\sum_{i=1}^{2}\int_\omega P_1(\hat{u}_i^\xi)(\hat{\varphi}_i-\hat{u}_i^\xi)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'-\varepsilon^3\int_\omega P_1(\hat{u}_3^\xi)(\hat{\varphi}_3-\hat{u}_3^\xi)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'\\ &\quad+\sum_{i=1}^{2}\int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon)(\hat{\varphi}_i-\hat{u}_i^\xi)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'+\varepsilon\int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon)(\hat{\varphi}_3-\hat{u}_3^\xi)\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx'\\ &\quad-\int_\omega \hat{k}|\hat{u}^\varepsilon|\,dx'\geqslant\sum_{i=1}^{2}\int_\Omega \hat{f}_i(\hat{\varphi}_i-\hat{u}_i^\xi)\,dx'dz+\varepsilon\int_\Omega \hat{f}_3(\hat{\varphi}_3-\hat{u}_3^\xi)\,dx'dz,\end{aligned}$$

where

$$\begin{aligned}I_1&=\mu\varepsilon^2\sum_{i,j=1}^{2}\int_\Omega\left(\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}+\frac{\partial\hat{u}_j^\xi}{\partial x_i}\right)\frac{\partial}{\partial x_j}(\hat{\varphi}_i-\hat{u}_i^\xi)\,dx'dz,\\ I_2&=\mu\sum_{i=1}^{2}\int_\Omega\left(\frac{\partial\hat{u}_i^\xi}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\xi}{\partial x_i}\right)\frac{\partial}{\partial z}(\hat{\varphi}_i-\hat{u}_i^\xi)\,dx'dz,\\ I_3&=\mu\sum_{i=1}^{2}\int_\Omega\varepsilon^2\left(\frac{\partial\hat{u}_i^\xi}{\partial z}+\varepsilon^2\frac{\partial\hat{u}_3^\xi}{\partial x_i}\right)\frac{\partial}{\partial x_i}(\hat{\varphi}_3-\hat{u}_3^\xi)\,dx'dz,\\ I_4&=2\mu\int_\Omega\varepsilon^2\frac{\partial\hat{u}_3^\xi}{\partial z}\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{u}_3^\xi)\,dx'dz.\end{aligned}$$

By the Theorem 4.1, we have

$$\lim_{\varepsilon\to0}\sum_{i=1}^{4}I_i(\varepsilon)=\mu\sum_{i=1}^{2}\int_\Omega\frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}(\hat{\varphi}_i-u_i^*)\,dx'dz,$$

$$\lim_{\varepsilon\to0}\int_\Omega \varepsilon\hat{f}_3\hat{\varphi}_3\,dx'dz=0,$$

$$\lim_{\varepsilon\to 0} \varepsilon \int_\omega P_1(\hat{u}_i^\varepsilon)\,(\hat{\varphi}_i - \hat{u}_i^\varepsilon)\,\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx' = 0, \quad i=1,2,$$
$$\lim_{\varepsilon\to 0} \varepsilon^3 \int_\omega P_1(\hat{u}_3^\xi)\,(\hat{\varphi}_3 - \hat{u}_3^\xi)\,\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx' = 0,$$
$$\lim_{\varepsilon\to 0} \sum_{i=1}^{2} \int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon)\,(\hat{\varphi}_i - \hat{u}_i^\xi)\,\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx' = 0, \quad i=1,2,$$
$$\lim_{\varepsilon\to 0} \varepsilon \int_\omega \hat{\beta}_\xi(\hat{u}^\varepsilon)\,(\hat{\varphi}_3 - \hat{u}_3^\xi)\,\sqrt{1+|\nabla h^\varepsilon(x')|^2}dx' = 0,$$

because

$$\left\|\hat{\beta}_\xi(\hat{u}^\varepsilon)\right\|^2_{L^2(\omega)} = \|\varepsilon\beta_\xi(u^\varepsilon)\|^2_{L^2(\omega)} \leqslant \frac{\varepsilon^2}{\xi^2}\|u^\varepsilon\|^2_{L^2(\omega)} = \frac{\varepsilon^2}{\xi^2}\int_\omega \left[\sum_{i=1}^{2}|\hat{u}_i^\xi|^2 + \varepsilon^2|\hat{u}_3^\xi|^2\right]dx' \underset{\varepsilon\to 0}{\to} 0.$$

And as j is convex and lower semi-continuous $\left(\lim_{\varepsilon\to 0}\left(\inf\int_\omega \hat{k}\,|\hat{u}^\varepsilon|\,dx'\right) \geqslant \int_\omega \hat{k}\,|u^*|\,dx'\right)$, we obtain

$$\begin{aligned}\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial}{\partial z}\,(\hat{\varphi}_i - u_i^*)\,dx'dz + \hat{\alpha}^2\sum_{i=1}^{2}\int_\Omega u_i^*\,(\hat{\varphi}_i - u_i^*)\,dx'dz + \int_\omega \hat{k}\,(|\hat{\varphi}| - |u^*|)\,dx' \\ \geqslant \sum_{i=1}^{2}\int_\Omega \hat{f}_i\,(\hat{\varphi}_i - \hat{u}_i^\xi)\,dx'dz.\end{aligned} \tag{4.6}$$

We now choose in the variational inequality (4.6),

$$\hat{\varphi}_i = u_i^* \pm \psi_i,\ \psi_i \in H_0^1(\Omega),\ \ i=1,2,$$

we get

$$\mu\sum_{i=1}^{2}\int_\Omega \frac{\partial u_i^*}{\partial z}\frac{\partial \psi_i}{\partial z}dx'dz + \hat{\alpha}^2\sum_{i=1}^{2}\int_\Omega u_i^*\psi_i dx'dz = \sum_{i=1}^{2}\int_\Omega \hat{f}_i\psi_i dx'dz.$$

Using now the Green formula and choosing $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Omega)$, we obtain

$$-\int_\Omega \mu\frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial z}\right)\psi_i dx'dz + \hat{\alpha}^2\int_\Omega u_i^*\psi_i dx'dz = \int_\Omega \hat{f}_i\psi_i dx'dz.$$

Thus

$$-\mu\frac{\partial^2 u_i^*}{\partial z^2} + \hat{\alpha}^2 u_i^* = \hat{f}_i, \quad \text{for } i=1,2, \text{ in } H^{-1}(\Omega), \tag{4.7}$$

and as $\hat{f} \in L^2(\Omega)$, then (4.7) is valid in $L^2(\Omega)$. □

Theorem 4.3. *Under the same assumptions as in Theorem 4.2, we have the following inequality*

$$\int_\omega \hat{k}(|\psi + s^*| - |s^*|)dx' - \int_\omega \mu\tau^*\psi dx' \geqslant 0 \qquad \forall\psi \in L^2(\omega)^2, \tag{4.8}$$

$$\begin{cases} \mu|\tau^*| < \hat{k} \Rightarrow s^* = 0, \\ \mu|\tau^*| = \hat{k} \Rightarrow \exists\gamma > 0 \textit{ such that } s^* = \gamma\tau^*, \end{cases} \tag{4.9}$$

where

$$\tau^* = \frac{\partial u^*}{\partial z}(x',0), \quad s^*(x') = u^*(x',0), \ \textit{and } s_h^*(x') = u^*(x',h(x')).$$

Also u^* and s^* satisfy the following equality of Reynolds

$$\int_\omega \left(\int_0^h u^*\left(x',z\right)dz - \frac{h}{2}s^*(x') - \frac{h}{2}s_h^*\left(x'\right) + \tilde{F}(x') + \tilde{U}^*(x')\right)\nabla\psi = 0\,, \quad \forall\psi\in H^1\left(\omega\right), \tag{4.10}$$

where

$$\tilde{F}\left(x'\right) = \frac{1}{\mu}\int_0^h F\left(x',z\right)dz - \frac{h}{2\mu}F\left(x',h\right), \quad F\left(x',z\right) = \int_0^z\int_0^\zeta \hat{f}\left(x',\eta\right)d\zeta d\eta,$$

and

$$\tilde{U}^*(x') = \frac{h\hat{\alpha}^2}{2\mu}U^*\left(x',h\right) - \frac{\hat{\alpha}^2}{\mu}\int_0^h U^*\left(x',z\right)dz, \quad U^*\left(x',z\right) = \int_0^z\int_0^\zeta u^*\left(x',\eta\right)d\zeta d\eta.$$

Proof. Passing to the limit in (3.2) and using the Green formula then choosing $\hat{\varphi}_i = u_i^* + \psi_i$, $\psi_i \in H^1_{\Gamma_1\cup\Gamma_L}(\Omega)$ $i=1,2$, where

$$H^1_{\Gamma_1\cup\Gamma_L}\left(\Omega\right) = \left\{\varphi\in H^1\left(\Omega\right) : \varphi = 0 \text{ on } \Gamma_1\cup\Gamma_L\right\},$$

we obtain

$$-\sum_{i=1}^2\int_\Omega \mu\frac{\partial^2 u_i^*}{\partial z^2}\psi_i dx'dz + \sum_{i=1}^2\int_\Gamma \mu\frac{\partial u_i^*}{\partial z}n\psi_i d\sigma + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^*\psi_i dx'dz + \int_\omega \hat{k}\left|u^*+\psi\right|dx' - \int_\omega \hat{k}\left|u^*\right|dx'$$
$$\geqslant \sum_{i=1}^2\int_\Omega \hat{f}_i\psi_i dx'dz.$$

But

$$\int_\Gamma \mu\frac{\partial u_i^*}{\partial z}n\psi_i d\sigma = -\int_\omega \mu\frac{\partial u_i^*}{\partial z}\left(x',0\right)\psi_i dx',$$

then

$$-\sum_{i=1}^2\int_\Omega \mu\frac{\partial^2 u_i^*}{\partial z^2}\psi_i dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^*\psi_i dx'dz + \int_\omega \hat{k}\left(\left|\psi+s^*\right| - \left|s^*\right|\right)dx' - \int_\omega \mu\tau_i^*\psi_i dx'$$
$$\geqslant \sum_{i=1}^2\int_\Omega \hat{f}_i\psi_i dx'dz.$$

By (4.4), we deduce

$$\int_\omega \hat{k}(|\psi+s^*| - |s^*|)dx' - \int_\omega \mu\tau^*\psi dx' \geqslant 0.$$

This inequality remain valid for any $\psi\in\left(\mathcal{D}\left(\omega\right)\right)^2$ and by density of $\mathcal{D}\left(\omega\right)$ in $L^2\left(\omega\right)$, we deduce (4.8). We also obtain (4.9) as in [1].

To prove (4.10), we integrate twice (4.4) between 0 and z, we obtain

$$u^*\left(x',z\right) = s^*\left(x'\right) + z\tau^* - \frac{1}{\mu}F(x',z) + \frac{\hat{\alpha}^2}{\mu}U^*(x',z), \tag{4.11}$$

where

$$U^*(x',z) = \int_0^z\int_0^\zeta u^*\left(x',\eta\right)d\zeta d\eta,$$

replacing z by h, we obtain

$$u^*\left(x',h(x')\right) = s^*\left(x'\right) + h\tau^* - \frac{1}{\mu}F(x',h(x')) + \frac{\hat{\alpha}^2}{\mu}U^*(x',h(x')),$$

whence

$$h\tau^* = s_h^*\left(x'\right) - s^*\left(x'\right) + \frac{1}{\mu}F(x',h(x')) - \frac{\hat{\alpha}^2}{\mu}U^*(x',h(x')), \tag{4.12}$$

integrating (4.11) with respect to z in the interval $(0,h(x'))$, we obtain

$$\int_0^h u^*\left(x',z\right) dz = hs^*\left(x'\right) + \frac{h^2}{2}\tau^* - \frac{1}{\mu}\int_0^h F(x',z)dz + \frac{\hat{\alpha}^2}{\mu}\int_0^h U^*(x',z)dz.$$

From (4.11) and (4.12), we deduce that

$$\int_0^h u^*\left(x',z\right) dz - \frac{h}{2}s^*(x') - \frac{h}{2}s_h^*\left(x'\right) + \tilde{F}(x') + \tilde{U}^*(x') = 0,$$

with

$$\tilde{F}\left(x'\right) = \frac{1}{\mu}\int_0^h F\left(x',z\right) dz - \frac{h}{2\mu}F\left(x',h\right), \quad F\left(x',z\right) = \int_0^z\int_0^\zeta \hat{f}\left(x',\eta\right) d\zeta d\eta,$$

and

$$\tilde{U}^*(x') = \frac{h\hat{\alpha}^2}{2\mu}U^*\left(x',h\right) - \frac{\hat{\alpha}^2}{\mu}\int_0^h U^*\left(x',z\right) dz, \quad U^*\left(x',z\right) = \int_0^z\int_0^\zeta u^*\left(x',\eta\right) d\zeta d\eta.$$

Then

$$\int_\omega \left(\int_0^h u^*\left(x',z\right) dz - \frac{h}{2}s^*(x') - \frac{h}{2}s_h^*\left(x'\right) + \tilde{F}(x') + \tilde{U}^*(x')\right)\nabla\psi = 0.$$

□

Theorem 4.4. *The solution u^* of our limit problem is unique.*

Proof. Let u^1 and u^2 be two solutions of (4.5), then

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial u_i^1}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i - u_i^1\right) dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^1\left(\hat{\varphi}_i - u_i^1\right) dx'dz + j\left(\hat{\varphi}\right) - j\left(u^1\right) \geqslant \sum_{i=1}^2\left(\hat{f}_i, \hat{\varphi}_i - u_i^1\right), \tag{4.13}$$

and

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial u_i^2}{\partial z}\frac{\partial}{\partial z}\left(\hat{\varphi}_i - u_i^2\right) dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^2\left(\hat{\varphi}_i - u_i^2\right) dx'dz + j\left(\hat{\varphi}\right) - j\left(u^2\right) \geqslant \sum_{i=1}^2\left(\hat{f}_i, \hat{\varphi}_i - u_i^2\right), \tag{4.14}$$

where

$$j\left(\hat{\varphi}\right) = \int_\omega \hat{k}\left|\hat{\varphi}\right| dx'.$$

Taking $\hat{\varphi} = u^2$ in (4.13) and $\hat{\varphi} = u^1$ in (4.14), we get

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial u_i^1}{\partial z}\frac{\partial}{\partial z}\left(u_i^2 - u_i^1\right) dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^1\left(u_i^2 - u_i^1\right) dx'dz + j\left(u^2\right) - j\left(u^1\right) \geqslant \sum_{i=1}^2\left(\hat{f}_i, u_i^2 - u_i^1\right), \tag{4.15}$$

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial u_i^2}{\partial z}\frac{\partial}{\partial z}\left(u_i^1 - u_i^2\right) dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega u_i^2\left(u_i^1 - u_i^2\right) dx'dz + j\left(u^1\right) - j\left(u^2\right) \geqslant \sum_{i=1}^2\left(\hat{f}_i, u_i^1 - u_i^2\right). \tag{4.16}$$

By adding the two inequalities (4.15) and (4.16), we obtain

$$\mu\sum_{i=1}^2\int_\Omega \frac{\partial}{\partial z}\left(u_i^1 - u_i^2\right)\frac{\partial}{\partial z}\left(u_i^1 - u_i^2\right) dx'dz + \hat{\alpha}^2\sum_{i=1}^2\int_\Omega \left(u_i^1 - u_i^2\right)\left(u_i^1 - u_i^2\right) dx'dz \leqslant 0,$$

this impllies

$$\mu \left\| \frac{\partial}{\partial z} \left(u_i^1 - u_i^2 \right) \right\|_{L^2(\Omega)}^2 + \hat{\alpha}^2 \left\| u_i^1 - u_i^2 \right\|_{L^2(\Omega)}^2 = 0.$$

By Poincaré inequality, we get

$$\left\| u_i^1 - u_i^2 \right\|_{V_z} = 0,$$

so

$$u^1 = u^2.$$

□

References

[1] G. Bayada, M. Boukrouche, *On a free boundary problem for the Reynolds equation derived from the Stokes system with Tresca boundary conditions*, J. Math. Anal. Appl., **282** (2003), 212–231. 1

[2] H. Benseridi, M. Dilmi, *Some inequalities and asymptotic behaviour of dynamic problem of linear elasticity*, Georgian Math. J., **20** (2013), 25–41. 1

[3] H. Benseridi, M. Dilmi, A. Saadallah, *Asymptotic behaviour of a nonlinear boundary value problem with friction*, Proc. Nat. Acad. Sci. India Sect. A, **88** (2018), 55–63. 1, 2

[4] D. Benterki, H. Benseridi, M. Dilmi, *Asymptotic study of a boundary value problem governed by the elasticity operator with nonlinear*, Adv. Appl. Math. Mech., **6** (2014), 191–202. 1, 3.1, 3.2, 3.3

[5] D. Benterki, H. Benseridi, M. Dilmi, *Asymptotic behavior of solutions to a boundary value problem with mixed boundary conditions and friction law*, Bound. Value Probl., **2017** (2017), 17 pages.

[6] M. Boukrouche, R. El mir, *Asymptotic analysis of non-Newtonian fluid in a thin domain with Tresca law*, Nonlinear Anal., **59** (2004), 85–105. 1

[7] M. Boukrouche, G. Lukaszewicz, *On a lubrication problem with Fourier and Tresca boundary conditions*, Math. Models Methods Appl. Sci., **14** (2004), 913–941. 1

[8] P. G. Ciarlet, *Plates and junctions in elastic multi-structures asymptotic analysis*, Springer-Verlag, Berlin, (1990). 1

[9] M. Dilmi, *Problémes aux limites obliques et non linéaires pour les équations de Lamé*, Thése de Doctorat, Université Ferhat Abbes -Sétif 1, (2018).

[10] R. El Mir, *Comportement asymptotique d'un fluide de Bingham dans un film mince avec des conditions non-linéaires sur le bord*, Thése de Doctorat, France, (2006). 1

[11] M. A. Ezzat, M. I. A. Othman, El-Karamany, *The dependence of the modulus of elasticity on the reference temperature in generalized thermoelasticity*, J. Thermal Stres., **24** (2001), 1159–1176.

[12] Y. Letoufa, H. Benseridi, M. Dilmi, *Asymptotic study of a frictionless contact problem between two elastic bodies*, J. Math. Computer Sci., **16** (2016), 336–350. 1, 3.4

[13] J. L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Paris, (1969). 1

[14] J.-L. Lions, E. Magenes, *Problémes aux limites non homogénes et applications*, Dunod, Paris, (1968).

[15] M. Marin, *Generalized solutions in elasticity of micropolar bodies with voids*, Rev. Acad. Canaria Cienc., **8** (1996), 101–106.

[16] M. Marin, *An uniqueness result for body with voids in linear thermoelasticity*, Rend. Mat. Appl. (7), **17** (1997), 103–113.

[17] M. Marin, *On the domain of influence in thermoelasticity of bodies with voids*, Arch. Math. (Brno), **33** (1997), 301–308.

[18] J. Necas, *Les méthodes directes en théorie des équations elliptiques*, Masson, Paris, (1967).

[19] M. I. A. Othman, *The uniqueness and reciprocity theorem for generalized thermo-viscoelasticity with thermal relaxation times*, Mech. Mech. Eng., **7** (2004), 77–87.

[20] M. I. A. Othman, *Generalized electromagneto-thermoelastic plane waves by thermal shock problem in a finite sonductivity half-space with one relaxation time*, Multidiscip. Model. Materials Struct., **1** (2005), 231–250.

[21] M. I. A. Othman, S. Y. Atwa, *Response of micropolar thermoelastic solid with voids due to various sources under Green Naghdi theory*, Acta Mech. Solida Sinica, **25** (2012), 197–209.

[22] A. Saadallah, H. Benseridi, M. Dilmi, S. Drabla, *Estimates for the asymptotic convergence of a non-isothermal linear elasticity with friction*, Georgian Math. J., **23** (2016), 435–446. 1