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Applications of q-Mittag-Leffler type Poisson distribution to subclass of q-starlike functions

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Abstract

In the recent years, the usage of special functions in combinations with the q -series has got attraction of a number of mathematicians. In this paper, we first highlighted and studied some well-known celebrated special functions. We then use the idea of convolution and define a new series, namely the series of q-Mittag-Leffler functions with the q-Poisson distribution. Finally by using the q-Mittag-Leffler type q-Poisson distribution we then define a new subclass of q-Starlike functions associated with the Janwoski functions. We derive a number of useful results like the Fekete-Szeg ö problems, distortion theorems and a sufficient condition.

Keywords: Analytic functions, starlike function, Mitag-Leffer and q-Mittag-Leffler functions, q-Poisson distribution, Janowski function.

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1. Introduction

Let $\mathcal{A}$ denotes the class of all and are holomorpic (analytic) functions λ defined inside the open unit disk

$$\mathbf{U} = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}$$

with the normalization condition

$$\lambda(0) = 0 = \lambda'(0) - 1,$$

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such that for a function $\lambda \in \mathcal{A}$ Taylor-Maclaurin series representation is given by:

$$\lambda(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n \quad (\xi \in \mathbf{U}). \tag{1.1}$$

Let by $\mathcal{S}$ we denote the subclass of the functions class $\mathcal{A}$ containing all univalent functions in $\mathbf{U}$. In addition $\mathcal{S}^*$ represents the class of starlike functions which is the usual subclass of $\mathcal{S}$ such that for $\mathcal{S}^*$ the following inequality holds true for the functions $\lambda \in \mathcal{A}$,

$$\Re\left(\frac{\xi\lambda'(\xi)}{\lambda(\xi)}\right) > 0 \quad (\xi \in \mathbf{U}).$$

Moreover, let λ and g be two analytic functions, then the function λ is called subordinated to the function g and is denoted by

$$\lambda(\xi) \prec g(\xi)$$

if there is a Schwarz function w in $\mathbf{U}$ with the property

$$w(0) = 0 \text{ and } |w(\xi)| < 1$$

such that

$$\lambda(\xi) = g(w(\xi)).$$

Also, the following equivalence is satisfied if the function g is univalent in $\mathbf{U}$,

$$\lambda(\xi) \prec g(\xi) \quad (\xi \in \mathbf{U}) \Longleftrightarrow \lambda(0) = g(0) \text{ and } \lambda(\mathbf{U}) \subset g(\mathbf{U}).$$

Next, we denote the class of analytic functions p by $\mathcal{P}$, which are normalized as

$$p(\xi) = \xi + \sum_{n=2}^{\infty} p_n \xi^n$$

such that

$$\Re(p(\xi)) > 0 \quad (\xi \in \mathbf{U}).$$

For two functions λ and h, where λ is of the form (1.1) and another function h having form

$$h(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n,$$

the convolution of both functions can be represented by $(\lambda * h)(\xi)$ and is defined by

$$(\lambda * h)(\xi) = \xi + \sum_{n=2}^{\infty} a_n b_n \xi^n, \ \xi \in \mathbb{U}.$$

An analytic function h with $h(0) = 1$ is said to belong to the class $\mathcal{P}[T, Y]$ if and only if

$$h(\xi) \prec \frac{1 + T\xi}{1 + Y\xi} \quad (-1 \leqslant Y < T \leqslant 1).$$

This class of analytic function was first time studied by Janowski (see [12]), who also showed that $h(\xi) \in \mathcal{P}[T, Y]$ if and only if there exists a function $p \in \mathcal{P}$ such that

$$h(\xi) = \frac{(T+1)p(\xi) - (T-1)}{(Y+1)p(\xi) - (Y-1)} \quad (-1 \leqslant Y < T \leqslant 1).$$

Definition 1.1 ([12]). A function $\lambda \in \mathcal{A}$ is placed in the class $\mathcal{S}^*[T, Y]$ if and only if

$$\frac{\xi\lambda'(\xi)}{\lambda(\xi)} = \frac{(T+1)p(\xi)-(T-1)}{(Y+1)p(\xi)-(Y-1)} \quad (-1 \leqslant Y < T \leqslant 1).$$

The above-defined functions class of starlike functions associated with the Janwoski functions has been studied and investigated by the many authors see for example [31] see also [14, 23].

The familiar Mittag-Leffler function $E_\alpha(\xi)$ (see [17]) and its two-parameter extension $E_{\alpha;\beta}(\xi)$ having similar properties to those of Mittag-Leffler function (see [33, 34]), which are defined (as usual) by means of the following series:

$$E_\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\alpha n+1)} \quad (\xi \in \mathbb{C}; \alpha > 0)$$

and

$$E_{\alpha;\beta}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma(\alpha n+\beta)} \qquad (\xi \in \mathbb{C}; \alpha > 0; \beta > 0), \tag{1.2}$$

respectively. For some recent studies about Mittag-Leffler function see [8, 9, 20, 28]. It were Porwal and Dixit (see [19]) who studied and linked the well-known Mittag-leffer function and the Poisson distribution scientifically. The Mitag-leffer type Poisson distribution (see [19]), can be represented by the following series

$$\Upsilon(\psi, \alpha, \beta)(\xi) = \xi + \sum_{n=2}^{\infty} \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta)E_{\alpha;\beta}(\psi)}\xi^n,$$

where $\Upsilon(\psi, \alpha, \beta)(\xi)$ is normalized function of class $\mathcal{S}$ since

$$\Upsilon(\psi, \alpha, \beta)(0) = 0 = \Upsilon'(\psi, \alpha, \beta)(0) - 1.$$

Furthermore, for Mittag-leffer type Poisson distribution series the probability mass function is given by

$$P(\psi, \alpha, \beta; n)(\xi) = \frac{\psi^n}{E_{\alpha;\beta}(\psi)\Gamma(\alpha n+\beta)},$$

where the value of $E_{\alpha;\beta}(\psi)$ is shown by (1.2). It is important to be noted that one of the generalization of Poisson distribution is the Mittag-Lefer type Poisson distribution. Furthermore Bajpai (see [5]) is also studied and obtained some necessary and sufficient condition for Mittag-leffer type Poisson distribution series. Alessa et al. (see [4]) defined the convolution operator using the Mittag-Leffer type of Poisson distribution series as:

$$F(\xi) = \Upsilon(\psi, \alpha, \beta) * \lambda(\xi) = \xi + \sum_{n=2}^{\infty} \phi_\psi^n(\alpha, \beta) a_n \xi^n,$$

where

$$\phi_\psi^n(\alpha, \beta) = \frac{\psi^{n-1}}{\Gamma(\alpha(n-1)+\beta)E_{\alpha;\beta}(\psi)}. \tag{1.3}$$

It was Bansal and Prajapat in (see [6]) who studied and investigated geometric properties for the Mittag-Leffler function $E_{\alpha;\beta}(\xi)$ which includes starlikeness, convexity and close-to-convexity. In (see [21]) results of differential subordination were also obtained which are associated with generalized Mittag-Leffler function.

We now define some basic definitions and terminilogies of the q-calculus that has been used in our present paper. Let $0 < q < 1$ should be supposed throughout the paper and

$$\mathbb{N} = \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\} \quad (\mathbb{N}_0 = \{0, 1, 2, \ldots\}).$$

Definition 1.2. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q}, & (\lambda \in \mathbb{C}), \\ \sum\limits_{k=1}^{n-1} q^k = 1+q+q^2+\cdots+q^{n-1}, & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 1.3. Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1, & n = 0, \\ \prod\limits_{k=1}^{n} [k]_q, & n \in \mathbb{N}. \end{cases}$$

Definition 1.4. For $t > 0$, let the q-gamma function is defined as

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.$$

Definition 1.5. The q-derivative (or q-difference) D_q of a function $\lambda \in \mathcal{A}$ is given subset of $\mathbb{C}$ by

$$D_q\lambda(\xi) = \begin{cases} \frac{\lambda(\xi)-\lambda(q\xi)}{(1-q)\xi}, & (\xi \in \mathbf{U}\backslash\{0\}; q \in (0,1)), \\ \lambda'(0), & (\xi = 0; q \in (0,1)). \end{cases} \tag{1.4}$$

We deduce from Definition 1.5 that

$$\lim_{q\to 1^-} (D_q\lambda)(\xi) = \lim_{q\to 1-} \left(\frac{\lambda(\xi)-\lambda(\xi q)}{(1-q)\xi}\right) = \lambda'(\xi)$$

for a given subset $\mathbb{C}$ of a differentiable function λ. It can be easily seen from (1.1) and (1.4) that

$$(D_q\lambda)(\xi) = 1 + \sum_{n=2}^{\infty} [n]_q a_n \xi^{n-1}.$$

In Geometric Function Theory of Complex Analysis the usage of the operator D_q is quite significant. Many new subclasses of analytic functions have been investigated and discussed by using the operator D_q. In Geometric Function Fheory (GFT), Srivastava [30, pp. 347 et seq.] was first time used the basic (or q-) hypergeometric functions. Inspiring by [30] of Srivastava published paper, many researchers have started works on this direction. For example, the generalization of the class of starlike functions to q-stralike functions were did by Ismail et al. (see [11]). Next, Kanas and Reducanu [13], Muhammad and Sokol [16], and Noor et al. [18, 25] have used the q-operator and have defined and investigated a number of subclasses of the class of q-starlike functions. Also in [2], Ahmad et al. (see also [27]), have used the q-derivative operator to define a new subclass of q-meromorphic starlike functions. For more details on this subject we refer the reader to see [1, 3, 35]. They also developed some remarkable results for their defined functions class.

The class of q-starlike function is defined as follow.

Definition 1.6. A function $\lambda \in \mathcal{A}$ is placed in the class $\mathcal{S}_q^*$ if

$$\lambda(0) = 0 = \lambda'(0) - 1 \tag{1.5}$$

and

$$\left|\frac{\xi}{\lambda(\xi)}(D_q\lambda)(\xi) - \frac{1}{1-q}\right| \leqslant \frac{1}{1-q}, \qquad \xi \in \mathbf{U}. \tag{1.6}$$

It is readly observe that as $q \to 1-$ the closed disk

$$\left|w - \frac{1}{1-q}\right| \leqslant \frac{1}{1-q}$$

becomes the right-half plane and the class $\mathcal{S}_q^*$ reduces to the well known class $\mathcal{S}^*$. Note that the symbolocally $\mathcal{S}_q^*$ was first used by Sahoo et al. (see [24]). Equivalently (1.5) and (1.6) can be rewritten as follows by using the principle of subordination between the anaytic functions

$$\frac{\xi}{\lambda(\xi)}(D_q\lambda)\xi \prec \hat{p}(\xi) \qquad (\hat{p}(\xi) := \frac{1+\xi}{1-q\xi}). \tag{1.7}$$

The q-Mittag-Leffler functions was defined and normalized by Sharma et al. (see [26]) as follows. Some special cases of $\mathfrak{F}_{\alpha,\beta}(\xi; q)$ are:

$$\begin{cases} \mathfrak{F}_{0,\beta}(\xi;q) = \frac{\xi}{1-\xi}, \\ \mathfrak{F}_{1,1}(\xi;q) = \xi e_q^{\xi}, \\ \mathfrak{F}_{1,2}(\xi;q) = e_q^{\xi} - 1, \\ \mathfrak{F}_{1,3}(\xi;q) = \frac{(e_q^{\xi}-\xi-1)(1+q)}{\xi}, \\ \mathfrak{F}_{1,4}(\xi;q) = \frac{(1+q)(1+q+q^2)}{\xi^2}\left(e_q^{\xi} - \xi - 1 - \frac{\xi^2}{1+q}\right), \end{cases}$$

where e_q^{ξ} is one of the q-analogues of e^{ξ}, the classical exponential function given by

$$e_q^{\xi} = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma_q(n+1)} \quad \text{(see [32, p. 488, Eq. 6.3 (7)]).}$$

The q-Mitag-leffer type Poisson distribution can be represented as:

$$\mathfrak{F}_{\alpha,\beta}(\xi;q) = \xi + \sum_{n=2}^{\infty} \frac{\psi^{n-1}}{\Gamma_q(\alpha(n-1)+\beta)\mathfrak{F}_{\alpha,\beta}(\xi;q)}\xi^n,$$

where $\Upsilon(\psi,\alpha,\beta)(\xi)$ is normalized function of class $\mathcal{S}$ since

$$\mathfrak{F}_{\alpha,\beta}(\xi;q)(0) = 0 = \Upsilon' = (\psi,\alpha,\beta)(0) - 1.$$

The probability mass function for q-Mitag-leffer type q-Poisson distribution series is given by

$$\mathfrak{F}_{\alpha,\beta}(\xi;q) = \frac{\psi^n}{\mathfrak{F}_{\alpha,\beta}(\xi;q)(\psi)\Gamma_q(\alpha n+\beta)},$$

where $\mathfrak{F}_{\alpha,\beta}(\xi;q)$ is given by (1.2). We now define the following

$$F(\xi) = \Upsilon(\psi,\alpha,\beta,q) * \lambda(\xi) = \xi + \sum_{n=2}^{\infty} \phi_{\Psi}^n(\alpha,\beta,q)a_n\xi^n, \tag{1.8}$$

where

$$\Phi_{\Psi}^n(\alpha,\beta,q) = \frac{\psi^{n-1}}{\Gamma_q(\alpha(n-1)+\beta)\mathfrak{F}_{\alpha,\beta}(\xi;q)}.$$

Definition 1.7. A function $F \in \mathcal{A}$ is said to belong to class $S_q^*[T, B, q]$ if and only if

$$\frac{\xi(D_qF)\xi}{F(\xi)} = \frac{(T+1)(1+q)[p(\xi)-1] + 2[p(\xi)+1-q(p(\xi)-1)]}{(Y+1)(1+q)[p(\xi)-1] + 2[p(\xi)+1-q(p(\xi)-1)]} \quad (-1 \leqslant B < T \leqslant 1;\ q \in (0,1)),$$

which by using the principle of Subordination between analytic function can be written as

$$\frac{\xi(D_qF)\xi}{F(\xi)} \prec \Phi(\xi),$$

where

$$\phi(\xi) = \frac{(T+1)\xi + 2 + (T-1)q\xi}{(Y+1)\xi + 2 + (Y-1)q\xi} \quad (-1 \leqslant Y < T \leqslant 1; q \in (0,1)). \tag{1.9}$$

The aim of the present investigation is to make use of certain special functions along with some probabilistic distributions to define a new subclass of analytic functions. We then obtain some useful results for our defined functions class. In the introductory section we have first given some remarkable known consequences and then by motivating from these recent research going on we have define a new subclass of q-starlike functions. In Section 2, we give some known results which will help in order to obtain our main results. In Section 3 we give our main results. Our first result is related to the well-known Fekete-Szegö inequality. The second result is related to find out a sufficient condition for our defined functions class. Some closure theorem for the defined function class is also included. Our last two results are related to distortion theorems.

2. Set of lemmas

The following Lemmas will be used in our present study.

Lemma 2.1 ([15])**.** *If the function* $p \in \mathcal{P}$*, then*

$$\left|p_2 - \upsilon p_1^2\right| \leqq \begin{cases} -4\upsilon + 2, & (\upsilon \leqq 0), \\ 2, & (0 \leqq \upsilon \leqq 1), \\ 4\upsilon - 2, & (\upsilon \geqq 1). \end{cases} \tag{2.1}$$

Remark 2.2. Although the upper bound in (2.1) is sharp, it can be improved as follows in the case when $0 < \upsilon < 1$:

$$\left|p_2 - \upsilon p_1^2\right| + \upsilon \left|p_1\right|^2 \leqq 2 \quad \left(0 < \upsilon \leqq \frac{1}{2}\right), \quad \text{and} \quad \left|p_2 - \upsilon p_1^2\right| + (1-\upsilon) \left|p_1\right|^2 \leqq 2 \quad \left(\frac{1}{2} \leqq \upsilon < 1\right).$$

Lemma 2.3 ([22])**.** *Let the function* p *take the form of the series as*

$$p(\xi) = 1 + \sum_{n=1}^{\infty} p_n \xi^n$$

is subordinated to the function H *represented by*

$$H(\xi) = 1 + \sum_{n=1}^{\infty} C_n \xi^n.$$

If the function $H(\xi)$ *is holomorphic in* $\mathbf{U}$ *and* $\mathcal{H}(\mathbf{U})$ *is convex, then*

$$|p_n| \leqq |C_1| \qquad (n \in \mathbb{N}).$$

Lemma 2.4 ([7])**.** *Let the function* $p \in \mathcal{P}$*. Then*

$$|p_n| \leqq 2 \qquad (n \in \mathbb{N}).$$

This last inequality is sharp.

3. Main results

In this section, we will prove our main results. Throughout our discussion, we assume that

$$-1 \leqq Y < A \leqq 1 \qquad \text{and} \qquad q \in (0,1).$$

Theorem 3.1. *For the function* F *having the form* (1.8)*, if* $F \in \mathcal{S}_q^*[A, Y, q]$*, then*

$$|a_3 - \mu a_2^2| \leqq \begin{cases} \left(\dfrac{A-Y}{4q^2\Phi_\Psi^7(\alpha,\beta,q)}\right)\Lambda(q), & (\mu < \sigma_1), \\ \dfrac{A-Y}{2q\Phi_\Psi^3(\alpha,\beta,q)}, & (\sigma_1 \leqq \mu \leqq \sigma_3), \\ \left(\dfrac{Y-A}{4q^2\Phi_\Psi^7(\alpha,\beta,q)}\right)\Lambda(q), & (\mu > \sigma_3), \end{cases}$$

where

$$\Lambda(q) := (A-Y) + (A-2Y-1)\,q + (1-Y)\,q^2 - \mu\,(A-Y)\,(q+1)^2,$$

$$\sigma_1 = \frac{(A-Y)-(A-2Y-3)\,q+(1-Y)\,q^2\Phi_\Psi^4(\alpha,\beta,q)}{(A-Y)\,(1+q)^2},$$

$$\sigma_2 = \frac{(A-Y)-(T-2Y-5)\,q+(1-Y)\,q^2\Phi_\Psi^4(\alpha,\beta,q)}{(T-Y)\,(1+q)^2},$$

and

$$\sigma_3 = \frac{(T-Y)-(T-2Y-7)\,q+(1-Y)\,q^2\Phi_\Psi^4(\alpha,\beta,q)}{(T-Y)\,(1+q)^2}.$$

It is asserted also that

$$|a_3 - \mu a_2^2| + \left(\mu - \frac{(T-Y)-(T-2Y-3)\,q+(1-Y)\,q^2\Phi_\Psi^4(\alpha,\beta,q)}{(T-Y)\,(1+q)^2}\right)|a_2|^2 \leqq \frac{T-Y}{2q\Phi_\Psi^3(\alpha,\beta,q)} \quad (\sigma_1 \leqq \mu \leqq \sigma_2)$$

and

$$|a_3 - \mu a_2^2| - \left(\mu - \frac{(T-Y)-(T-2Y-7)\,q+(1-Y)\,q^2\Phi_\Psi^4(\alpha,\beta,q)}{(T-Y)\,(1+q)^2}\right)|a_2|^2 \leqq \frac{T-Y}{2q\Phi_\Psi^3(\alpha,\beta,q)} \quad (\sigma_2 \leqq \mu \leqq \sigma_3),$$

where $\Phi_\Psi^n(\alpha,\beta,q)$ *is given by* (1.3)*.*

Proof. If $F \in \mathcal{S}_q^*[T, Y, q]$, then from (1.7) we have

$$\frac{\xi(D_qF)(\xi)}{F(\xi)} \prec \phi(\xi), \tag{3.1}$$

where $\phi(\xi)$ is defined by (1.9). We now let a function p as:

$$p(\xi) = \frac{1+w(\xi)}{1-w(\xi)} = 1 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots.$$

Clearly we see that $p \in \mathcal{P}$. Or

$$w(\xi) = \frac{p(\xi)-1}{p(\xi)+1}.$$

By applying (3.1), we have

$$\frac{\xi(D_qF)(\xi)}{F(\xi)} = \phi(w(\xi))$$

with
$$\phi(w(\xi)) = \frac{(T+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}{(Y+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}.$$
Now
$$\frac{\xi(D_qF)(\xi)}{F(\xi)} = \frac{(T+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}{(Y+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}.$$
Thus, if
$$p(\xi) = 1+p_1\xi+p_2\xi^2+\cdots,$$
then after some straightforward simplification, we have
$$\begin{aligned}&\frac{(T+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}{(Y+1)(1+q)[p(\xi)-1]+2[p(\xi)+1-q(p(\xi)-1)]}\\ &\qquad = 1+\frac{1}{4}(T-Y)(q+1)p_1\xi+\frac{1}{16}(T-Y)(q+1)\left[4p_2-(3-q+(q+1)Y)p_1^2\right]\xi^2+\cdots.\end{aligned}$$
Similarly, we have
$$\frac{\xi(D_qF)(\xi)}{F(\xi)} = 1+qa_2\xi+\left[\left(q+q^2\right)a_3-qa_2^2\right]\xi^2+\cdots.$$
Therefore, we obtain
$$a_2 = \left(\frac{(T-Y)(q+1)}{4q\Phi_\Psi^2(\alpha,\beta,q)}\right)p_1$$
and
$$a_3 = \left(\frac{T-Y}{4q\Phi_\Psi^3(\alpha,\beta,q)}\right)p_2-\left(\frac{T-Y}{16q^2\Phi_\Psi^3(\alpha,\beta,q)}\right)\Omega(q)p_1^2,$$
where
$$\Omega(q) = (T-Y)+(T-2Y-3)q+(1-Y)q^2.$$
Thus, clearly, we find that
$$\left|a_3-\mu a_2^2\right| = \left(\frac{T-Y}{4\Phi_\Psi^3(\alpha,\beta,q)}\right)\left|p_2-\kappa p_1^2\right|, \tag{3.2}$$
where
$$\kappa = \frac{(q+1)^2(T-Y)\mu-(T-Y)+(T-2Y-3)q+(1-Y)q^2}{4q\Phi_\Psi^4(\alpha,\beta,q)}.$$
Finally, by using Lemma (2.1) in conjunction with (3.2), we obtain the result given by (3.1). □

Theorem 3.2. *A function* F *is said to be in the class* $\mathcal{S}_q^*[T,Y,q]$ *if it satisfies the following condition:*
$$\sum_{n=2}^{\infty}\left(2q[n-1]_q+\left|(Y+1)[n]_q-(T+1)\right|\right)\Phi_\Psi^n(\alpha,\beta,q)\cdot|a_n| < |Y-T|. \tag{3.3}$$

Proof. Suppose that (3.3) is satisfied, it is sufficient to prove that
$$\left|\frac{(Y-1)\frac{\xi(D_qF)(\xi)}{F(\xi)}-(T-1)}{(Y+1)\frac{\xi(D_qF)(\xi)}{F(\xi)}-(T+1)}-\frac{1}{1-q}\right| < \frac{1}{1-q}.$$

We have
$$\left|\frac{(Y-1)\frac{\xi(D_qF)(\xi)}{F(\xi)}-(T-1)}{(Y+1)\frac{\xi(D_qF)(\xi)}{F(\xi)}-(T+1)}-\frac{1}{1-q}\right|$$

$$
\begin{aligned}
&\leqq \left|\frac{(Y-1)\left(D_q F\right)(\xi)-(T-1) F(\xi)}{(Y+1) \xi\left(D_q F\right)(\xi)-(T+1) F(\xi)}-1\right|+\frac{q}{1-q} \\
&=2\left|\frac{F(\xi)-\xi\left(D_q F\right)(\xi)}{(Y+1) \xi\left(D_q F\right)(\xi)-(T+1) F(\xi)}\right|+\frac{q}{1-q} \\
&=2\left|\frac{\sum_{n=2}^{\infty}\left(1-[n]_q\right) \Phi_\Psi^n(\alpha, \beta, q) a_n \xi^n}{(Y-T)+\sum_{n=2}^{\infty}(Y+1)\left([n]_q-(T+1)\right) \Phi_\Psi^n(\alpha, \beta, q) a_n \xi^n}\right|+\frac{q}{1-q} \\
&\leqq 2 \frac{\sum_{n=2}^{\infty}\left|1-[n]_q\right| \Phi_\Psi^n(\alpha, \beta, q) \cdot\left|a_n\right|}{|(Y-T)|-\sum_{n=2}^{\infty}\left|(Y+1)[n]_q \Phi_\Psi^n(\alpha, \beta, q)-(T+1) \Phi_\Psi^n(\alpha, \beta, q)\right| \cdot\left|a_n\right|}+\frac{q}{1-q} .
\end{aligned}
$$

It could be seen that the last expression is bounded above by $\frac{1}{1-q}$ if

$$
\sum_{n=2}^{\infty}\left(2 q[n-1]_q+\left|[n]_q(Y+1)-(T+1)\right|\right) \Phi_\Psi^n(\alpha, \beta) \cdot\left|a_n\right|<|Y-T|,
$$

which represents the Proof of the given Theorem. □

Theorem 3.3. *The class $S_q^*[T, Y, q]$ is closed under convex combination.*

Proof. Let $\lambda_k(\xi) \in S_q^*[T, Y, q]$ such that

$$
\lambda_k(\xi)=\xi+\sum_{n=2}^{\infty} a_{n, k} \xi^n, \quad k \in\{1,2\}
$$

it is enough for us that we may show

$$
t \lambda_1 \xi+(1-t) \lambda_2 \xi \in S_q^*[T, Y, q], \quad(t \in[0,1])
$$

as

$$
t \lambda_1 \xi+(1-t) \lambda_2 \xi=\xi+\sum_{n=2}^{\infty}\left[t a_{n, 1}+(1-t) a_{n, 2}\right] \Phi_\Psi^n(\alpha, \beta, q) \xi^n,
$$

so by (3.3), we have

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left[2 q[n-1]_q+[n]_q(Y+1)-(T+1)\right] \Phi_\Psi^n(\alpha, \beta, q)\left[t a_{n, 1}+(1-t) a_{n, 2}\right] \\
&\quad \leqslant \sum_{n=2}^{\infty}\left[2 q[n-1]_q+[n]_q(Y+1)-(T+1)\right] \Phi_\Psi^n(\alpha, \beta, q)\left|t a_{n, 1}\right| \\
&\quad\quad +(1-t) \sum_{n=2}^{\infty}\left[2 q[n-1]_q+[n]_q(Y+1)-(T+1)\right] \Phi_\Psi^n(\alpha, \beta, q)\left|a_{n, 2}\right| \\
&\quad <t|Y-T|+(1-t)|Y-T|=|Y-T|,
\end{aligned}
$$

hence

$$
t \lambda_1 \xi+(1-t) \lambda_2 \xi \in S^*[T, Y, q],
$$

which completes the proof. □

Theorem 3.4. *Let* $F \in S^*[T, Y, q]$, *then for* $|\xi| = r$,

$$r - \frac{|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r^2 \leqslant |\lambda(\xi)| \leqslant r + \frac{|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r^2,$$

the function is sharp for $n = 2$.

Proof. Let $F \in S^*[T, Y, q]$, using (1.1) the following inequity can be deduced,

$$|\lambda(\xi)| \leqslant |\xi| + \sum_{n=2}^{\infty} |a_n|\,|\xi^n| \leqslant |\xi| + |\xi|^2 \sum_{n=2}^{\infty} |a_n| \leqslant r + \frac{|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r^2,$$

similarly

$$|\lambda(\xi)| \geqslant |\xi| - \sum_{n=2}^{\infty} |a_n|\,|\xi^n| \geqslant |\xi| - |\xi|^2 \sum_{n=2}^{\infty} |a_n| \geqslant r - \frac{|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r^2.$$

□

Theorem 3.5. *Let* $F \in S^*[T, Y, q]$, *then for* $|\xi| = r$,

$$1 - \frac{2\,|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r \leqslant \left|\lambda'(\xi)\right| \leqslant 1 + \frac{2\,|Y-T|}{2q\,[1]_q + |Y-T+(Y+1)q|\,\Phi^2_{\Psi}(\alpha,\beta,q)} r.$$

Proof. The proof is similar to that of the Theorem 3.4, therefore omitted. □

4. Conclusion

The usage of the special functions are quite significant in many diverse areas of mathematics, physics, and other sciences. This direction of study has got a dramatic attraction of a number of mathematicians. In our present investigation, we have first highlighted and studied some well-known special functions. We are motivated by the ongoing research on this subject, and have used the idea of convolution and defined a new series, namely the series of q-Mitag-Leffer functions with the q-Poisson distribution. Finally, using the q-Mitag-Leffer type q-Poisson distribution we have then defined a new subclass of q-Starlike functions associated with the Janwoski functions. We have derived a number of useful results like the Fekete-Szegö problems, distortion Theorems, and a sufficient condition.

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