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Some properties of differential operator to the subclass of univalent functions with negative coefficients

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Abstract

Various function theorists have successfully defined and investigated different kinds of analytic functions. The applications of such functions have played significant roles in geometry function theory as a field of complex analysis. In this work, therefore, a certain subclass of univalent analytic functions is defined using a generalized differential operator and we have discussed a subclass $TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$ of univalent functions with negative coefficients related to differential operator in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class.

Keywords: Univalent, differential operator, starlike, extreme points, Hadamard product.

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1. Introduction

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order $m, 0 \leqslant m < 1$, if it satisfies

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > m, z \in \mathbb{U}.$$

We denote this class with $S^*(m)$.

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A function $u \in A$ is a convex function of the order $m, 0 \leqslant m < 1$, if it satisfies

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > m, z \in \mathbb{U}.$$

We denote this class with $K(m)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$, respectively.

Let T denote the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqslant 0,\ z \in \mathbb{U} \tag{1.1}$$

and let $T^*(m) = T \cap S^*(m),\ C(m) = T \cap K(m)$. The class $T^*(m)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [17].

Differential operators in a complex domain play a significant role in functions theory and its information. They have used to describe the geometric interpolation of analytic functions in a complex domain. Also, they have utilized to generate new formulas of holomorphic functions. Lately, Lupas [11] presented a amalgamation of two well-known differential operators prearranged by Ruscheweyh [14] and Salagean [15]. Later, these operators are investigated by researchers considering different classes and formulas of analytic functions [8, 12**?** , 13].

The theory of special functions in one variable has a long and ironic past; the rising importance in special functions of several variables is moderately contemporary. Currently, there has been quick progress specifically in the area of special functions with the consideration of symmetries and harmonic analysis connected with root systems. The drive for this work comes from some generalizations of the theory of symmetric spaces, whose functions can be written as special functions depending on definite sets of parameters. A key implementation in the study of special functions with reflection symmetries is Dunkl operators, which are known as a class of differential-difference operators. In this effort, we present a Dunkl differential-difference operator of the first type in a complex domain, under a special class of analytic functions, called a class of normalized analytic functions. This class plays an important role in the field of geometric function theory. Based on this connection between the Dunkl operator and geometric function theory, we impose a major class of geometric presentations called the starlike class of analytic functions. A significant motivation to study Dunkl operators is created by their application in the analysis of quantum many-body systems of a special type. These operators describe integrated systems in one dimension and have seen considerable increased attention in mathematical physics, especially in conformal field theory (see [1–3, 5, 6, 9, 10, 16]).

Let $u \in A$, then the Ruscheweyh formula is indicated by the structure formula

$$\varphi^{\wp} u(z) = z + \sum_{n=2}^{\infty} C^{\wp}_{\wp+n-1} a_n z^n.$$

While, the Salagean operator admits the construction

$$\psi^{\wp} u(z) = z + \sum_{n=2}^{\infty} n^{\wp} a_n z^n.$$

Lupas operator is formulated by the structure

$$\lambda^{\wp}_{\sigma} = z + \sum_{n=2}^{\infty} \left(\sigma n^{\wp} + (1-\sigma) C^{\wp}_{\wp+n-1}\right) a_n z^n, \quad z \in \mathbb{U}, \sigma \in [0,1].$$

Newly, Ibrahim and Darus [7] considered the next differential operator

$$\begin{aligned}
&\Theta_\delta^0 u(z) = u(z),\\
&\Theta_\delta^1 u(z) = zu'(z) + \frac{\delta}{2}(u(z) - u(-z) - 2z), \quad \delta \in \mathbb{R},\\
&\quad \vdots\\
&\Theta_\delta^{\wp} u(z) = \Theta_\delta\left(\Theta_\delta^{\wp-1} u(z)\right) = z + \sum_{n=2}^{\infty}\left(n + \frac{\delta}{2}(1 + (-1)^{n+1})\right)^{\wp} a_n z^n.
\end{aligned}$$

When $\delta = 0$, we have $\psi^{\wp} u(z)$. In addition, it is a modified formula of the well-known Dunkl operator [4], where δ is known as the Dunkl order. Proceeding, we define $\mathscr{J}_{\sigma,\delta}^{\wp}$ is a generalized formula of $\lambda_\sigma^{\wp}$ as follows:

$$\begin{aligned}
\mathscr{J}_{\sigma,\delta}^{\wp} u(z) &= (1-\sigma)\varphi^{\wp} u(z) + \sigma\Theta_\delta^{\wp} u(z) = z + \sum_{n=2}^{\infty} \phi(n,\wp,\sigma,\delta) a_n z^n,\\
\text{where } \phi(n,\wp,\sigma,\delta) &= \left((1-\sigma)C_{\wp+n-1}^{\wp} + \sigma\left(n + \frac{\delta}{2}(1 + (-1)^{\wp+1})\right)^{\wp}\right).
\end{aligned} \tag{1.2}$$

Clearly, the operator $\mathscr{J}_{\sigma,\delta}^{\wp} u(z) \in S$. And this operator is a mixed operator involving Ruscheweyh derivative and Salagean operator.

The operator $\mathscr{J}_{\sigma,\delta}^{\wp}$ unifies several previously studied operators. Namely as

- $\mathscr{J}_{\sigma,\delta}^{0} u(z) = u(z)$;
- $\mathscr{J}_{\sigma,0}^{\wp} u(z) = \lambda_\sigma^{\wp} u(z)$;
- $\mathscr{J}_{0,\delta}^{\wp} u(z) = \varphi^{\wp} u(z)$;
- $\mathscr{J}_{1,\delta}^{\wp} u(z) = \Theta_\delta^{\wp} u(z)$;
- $\mathscr{J}_{1,0}^{\wp} u(z) = \psi^{\wp} u(z)$.

In this study, we shall define a new differential operator of complex coefficients and study its behaviors based on the properties of the theory of geometric functions. The new operator will be formulated in generalized sub-classes of starlike functions. Therefore, a certain subclass of univalent analytic functions is defined using a generalized differential operator. Furthermore, some geometric properties for the class were established.

Now, by making use of the linear operator $\mathscr{J}_{\sigma,\delta}^{\wp} u$, we define a new subclass of functions belonging to the class A.

Definition 1.1. For $0 \leqslant \vartheta < 1, 0 \leqslant \hbar < 1, 0 < \ell < 1, \delta \geqslant 0, \sigma \in [0,1]$ and $\wp \in \mathbb{N}$, we let $TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$ be the subclass of u consisting of functions of the form (1.1) and its geometrical condition satisfies

$$\left|\frac{\vartheta\left((\mathscr{J}_{\sigma,\delta}^{\wp} u(z))' - \frac{\mathscr{J}_{\sigma,\delta}^{\wp} u(z)}{z}\right)}{\hbar\left(\mathscr{J}_{\sigma,\delta}^{\wp} u(z)\right)' + (1-\vartheta)\frac{\mathscr{J}_{\sigma,\delta}^{\wp} u(z)}{z}}\right| < \ell, \quad z \in \mathbb{U},$$

where $\mathscr{J}_{\sigma,\delta}^{\wp} u(z)$, is given by (1.2).

2. Coefficient inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$.

Theorem 2.1. *Let the function* u *be defined by* (1.1). *Then* $u \in TS^{\wp}_{\sigma,\delta}(\vartheta, \hbar, \ell)$ *if and only if*

$$\sum_{n=2}^{\infty}[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)a_n \leqslant \ell(\hbar+(1-\vartheta)), \tag{2.1}$$

where $0<\ell<1, 0\leqslant\vartheta<1, 0\leqslant\hbar<1, \delta\geqslant 0, \sigma\in[0,1]$ *and* $\wp\in\mathbb{N}$. *The result* (2.1) *is sharp for the function*

$$u(z) = z - \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}z^n, \quad n\geqslant 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z|=1$. Then we obtain

$$\begin{aligned}
&\left|\vartheta\left((\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' - \frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}\right)\right| - \ell\left|\hbar\left(\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' + (1-\vartheta)\frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}\right)\right|\\
&= \left|-\vartheta\sum_{n=2}^{\infty}(n-1)\phi(n,\wp,\sigma,\delta)a_nz^{n-1}\right| - \ell\left|\hbar+(1-\vartheta)-\sum_{n=2}^{\infty}(n\hbar+1-\vartheta)\phi(n,\wp,\sigma,\delta)a_nz^{n-1}\right|\\
&\leqslant \sum_{n=2}^{\infty}[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)a_n - \ell(\hbar+(1-\vartheta)) \leqslant 0.
\end{aligned}$$

Hence, by maximum modulus principle, $u\in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$.

Now assume that $u\in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$ so that

$$\left|\frac{\vartheta\left((\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' - \frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}\right)}{\hbar(\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' + (1-\vartheta)\frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}}\right| < \ell, \quad z\in\mathbb{U}.$$

Hence

$$\left|\vartheta\left((\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' - \frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}\right)\right| < \ell\left|\hbar\left(\mathcal{J}^{\wp}_{\sigma,\delta}u(z))' + (1-\vartheta)\frac{\mathcal{J}^{\wp}_{\sigma,\delta}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{n=2}^{\infty}\vartheta(n-1)\phi(n,\wp,\sigma,\delta)a_nz^{n-1}\right| < \ell\left|\hbar+(1-\vartheta)-\sum_{n=2}^{\infty}(n\hbar+1-\vartheta)\phi(n,\wp,\sigma,\delta)a_nz^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)a_n \leqslant \ell(\hbar+(1-\vartheta)),$$

and this completes the proof. □

Corollary 2.2. *Let the function* $u\in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$. *Then*

$$a_n \leqslant \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}z^n, \quad n\geqslant 2.$$

3. Distortion and covering theorem

We introduce the growth and distortion theorems for the functions in the class $TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$.

Theorem 3.1. *Let the function* $u\in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$. *Then*

$$|z| - \frac{\ell(\hbar+(1-\vartheta))}{\phi(2,\wp,\sigma,\delta)[\vartheta+\ell(2\hbar+1-\vartheta)]}|z|^2 \leqslant |u(z)| \leqslant |z| + \frac{\ell(\hbar+(1-\vartheta))}{\phi(2,\wp,\sigma,\delta)[\vartheta+\ell(2\hbar+1-\vartheta)]}|z|^2.$$

The result is sharp and attained,

$$u(z) = z - \frac{\ell(\hbar+(1-\vartheta))}{\phi(2,\wp,\sigma,\delta)[\vartheta+\ell(2\hbar+1-\vartheta)]}z^2.$$

Proof.

$$|u(z)| = \left|z - \sum_{n=2}^{\infty} a_n z^n\right| \leqslant |z| + \sum_{n=2}^{\infty} a_n |z|^n \leqslant |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}.$$

Thus

$$|u(z)| \leqslant |z| + \frac{\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|z|^2.$$

Also

$$|u(z)| \geqslant |z| - \sum_{n=2}^{\infty} a_n |z|^n \geqslant |z| - |z|^2 \sum_{n=2}^{\infty} a_n \geqslant |z| - \frac{\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|z|^2.$$

Then proof of the theorem follows. □

Theorem 3.2. *Let* $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$. *Then*

$$1 - \frac{2\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|z| \leqslant |u'(z)| \leqslant 1 + \frac{2\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}|z|$$

with equality for

$$u(z) = z - \frac{2\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}z^2.$$

Proof. Notice that

$$\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)] \sum_{n=2}^{\infty} n a_n \leqslant \sum_{n=2}^{\infty} n[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta) a_n \leqslant \ell(\hbar + (1-\vartheta)),$$

from Theorem 2.1. Thus

$$\begin{aligned} |u'(z)| = \left|1 - \sum_{n=2}^{\infty} n a_n z^{n-1}\right| &\leqslant 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\leqslant 1 + |z| \sum_{n=2}^{\infty} n a_n \leqslant 1 + |z| \frac{2\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}. \end{aligned} \tag{3.1}$$

On the other hand

$$\begin{aligned} |u'(z)| = \left|1 - \sum_{n=2}^{\infty} n a_n z^{n-1}\right| &\geqslant 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \\ &\geqslant 1 - |z| \sum_{n=2}^{\infty} n a_n \geqslant 1 - |z| \frac{2\ell(\hbar + (1-\vartheta))}{\phi(2, \wp, \sigma, \delta)[\vartheta + \ell(2\hbar + 1 - \vartheta)]}. \end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we get the result. □

4. Radii of starlikeness, convexity, and close-to-convexity

In the following theorems, we obtain the radii of starlikeness, convexity, and close-to-convexity for the class $TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$.

Theorem 4.1. *Let* $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$. *Then* u *is starlike in* $|z| < R_1$ *of order* ϑ, $0 \leqslant \vartheta < 1$, *where*

$$R_1 = \inf_n \left\{ \frac{(1-\vartheta)(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \lambda, \vartheta)}{(n-\vartheta)\ell(\hbar + (1-\vartheta))} \right\}^{\frac{1}{n-1}}, \quad n \geqslant 2.$$

Proof. u is starlike of order $\vartheta, 0 \leqslant \vartheta < 1$ if

$$\Re\left\{ \frac{zu'(z)}{u(z)} \right\} > \vartheta.$$

Thus it is enough to show that

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| = \left| \frac{-\sum_{n=2}^{\infty} (n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right| \leqslant \frac{\sum_{n=2}^{\infty} (n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

Thus

$$\left| \frac{zu'(z)}{u(z)} - 1 \right| \leqslant 1 - \vartheta \quad \text{if} \quad \sum_{n=2}^{\infty} \frac{(n-\vartheta)}{(1-\vartheta)} a_n |z|^{n-1} \leqslant 1. \tag{4.1}$$

Hence by Theorem 2.1, (4.1) will be true if

$$\frac{n-\vartheta}{1-\vartheta} |z|^{n-1} \leqslant \frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta)}$$

or if

$$|z| \leqslant \left[\frac{(1-\vartheta)(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{(n-\vartheta)\ell(\hbar + (1-\vartheta))} \right]^{\frac{1}{n-1}}, n \geqslant 2. \tag{4.2}$$

The theorem follows easily from (4.2). □

Theorem 4.2. *Let* $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$. *Then* u *is convex in* $|z| < R_2$ *of order* $\vartheta, 0 \leqslant \vartheta < 1$, *where*

$$R_2 = \inf_n \left\{ \frac{(1-\vartheta)(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{n(n-\vartheta)\ell(\hbar + (1-\vartheta))} \right\}^{\frac{1}{n-1}}, \quad n \geqslant 2.$$

Proof. u is convex of order $\vartheta, 0 \leqslant \vartheta < 1$ if

$$\Re\left\{ 1 + \frac{zu''(z)}{u'(z)} \right\} > \vartheta.$$

Thus it is enough to show that

$$\left| \frac{zu''(z)}{u'(z)} \right| = \left| \frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n z^{n-1}} \right| \leqslant \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1}}.$$

Thus
$$\left|\frac{zu''(z)}{u'(z)}\right| \leqslant 1-\vartheta \quad \text{if} \quad \sum_{n=2}^{\infty} \frac{n(n-\vartheta)}{(1-\vartheta)} a_n |z|^{n-1} \leqslant 1. \tag{4.3}$$
Hence by Theorem 2.1, (4.3) will be true if
$$\frac{n(n-\vartheta)}{1-\vartheta}|z|^{n-1} \leqslant \frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta)}$$
or if
$$|z| \leqslant \left[\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,\wp,\sigma,\delta)}{n(n-\vartheta)\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{n-1}}, n \geqslant 2. \tag{4.4}$$
The theorem follows easily from (4.4). □

Theorem 4.3. *Let $u \in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$. Then u is close-to-convex in $|z| < R_3$ of order ϑ, $0 \leqslant \vartheta < 1$, where*
$$R_3 = \inf_n \left\{\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,\lambda,\vartheta)}{n\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{n-1}}, \quad n \geqslant 2.$$

Proof. u is close-to-convex of order $\vartheta, 0 \leqslant \vartheta < 1$ if
$$\Re\left\{u'(z)\right\} > \vartheta.$$
Thus it is enough to show that
$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \leqslant \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$
Thus
$$|u'(z)-1| \leqslant 1-\vartheta \quad \text{if} \quad \sum_{n=2}^{\infty} \frac{n}{(1-\vartheta)} a_n |z|^{n-1} \leqslant 1. \tag{4.5}$$
Hence by Theorem 2.1, (4.5) will be true if
$$\frac{n}{1-\vartheta}|z|^{n-1} \leqslant \frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta)}$$
or if
$$|z| \leqslant \left[\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,\wp,\sigma,\delta)}{n\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{n-1}}, n \geqslant 2. \tag{4.6}$$
The theorem follows easily from (4.6). □

5. Extreme points

In the following theorem, we obtain extreme points for the class $TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$.

Theorem 5.1. *Let $u_1(z) = z$ and*
$$u_n(z) = z - \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)} z^n, \quad \textit{for} \quad n = 2,3,\ldots.$$
Then $u \in TS^{\wp}_{\sigma,\delta}(\vartheta,\hbar,\ell)$ if and only if it can be expressed in the form
$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \quad \textit{where} \quad \theta_n \geqslant 0 \quad \textit{and} \quad \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\ell(\hbar + (1-\vartheta))\theta_n}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)} z^n.$$

Now, $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$, since

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))} \frac{\ell(\hbar + (1-\vartheta))\theta_n}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)} = \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leqslant 1.$$

Conversely, suppose $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.

Now $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$ implies from Theorem 2.1,

$$a_n \leqslant \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}.$$

Setting $\theta_n = \frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta))} a_n, n = 2, 3, \ldots,$ and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$. □

6. Hadamard product

In the following theorem, we obtain the convolution result for functions that belong to the class $TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$.

Theorem 6.1. *Let* $u, g \in TS(\vartheta, \hbar, \ell, \lambda)$. *Then* $u * g \in TS(\vartheta, \hbar, \zeta, \lambda)$ *for*

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n, \textit{ and } (u * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geqslant \frac{\ell^2(\hbar + (1-\vartheta))\vartheta(n-1)}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]^2\phi(n, \wp, \sigma, \delta) - \ell^2(\hbar + (1-\vartheta))(n\hbar + 1 - \vartheta)}.$$

Proof. $u \in TS_{\sigma,\delta}^{\wp}(\vartheta, \hbar, \ell)$ and so

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))} a_n \leqslant 1,$$

and

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))} b_n \leqslant 1.$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \zeta(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\zeta(\hbar + (1-\vartheta))} a_n b_n \leqslant 1.$$

By Cauchy-Schwarz inequality,

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))} \sqrt{a_n b_n} \leqslant 1. \tag{6.1}$$

Therefore it is enough to show that

$$\frac{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\zeta(\hbar+(1-\vartheta))}a_n b_n \leqslant \frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta))}\sqrt{a_n b_n}.$$

That is

$$\sqrt{a_n b_n} \leqslant \frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\zeta}{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\ell}.$$

From (6.1),

$$\sqrt{a_n b_n} \leqslant \frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}.$$

Thus it is enough to show that

$$\frac{\ell(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)} \leqslant \frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\zeta}{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\ell},$$

which simplifies to

$$\zeta \geqslant \frac{\ell^2(\hbar+(1-\vartheta))\vartheta(n-1)}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]^2\phi(n,\wp,\sigma,\delta)-\ell^2(\hbar+(1-\vartheta))(n\hbar+1-\vartheta)}.$$

□

7. Closure theorems

We shall prove the following closure theorems for the class $TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$.

Theorem 7.1. *Let* $u_j \in TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$, $j=1,2,\ldots,s$. *Then*

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$$

for $u_j(z)=z-\sum\limits_{n=2}^{\infty} a_{n,j}z^n$, *where* $\sum\limits_{j=1}^{s} c_j = 1$.

Proof.

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) = z-\sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n = z-\sum_{n=2}^{\infty} e_n z^n,$$

where $e_n=\sum\limits_{j=1}^{s} c_j a_{n,j}$. Thus $g(z)\in TS_{\sigma,\delta}^{\wp}(\vartheta,\hbar,\ell)$ if

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta))}e_n \leqslant 1,$$

that is, if

$$\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta))}c_j a_{n,j} = \sum_{j=1}^{s} c_j \sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,\wp,\sigma,\delta)}{\ell(\hbar+(1-\vartheta))}a_{n,j} \leqslant \sum_{j=1}^{s} c_j = 1.$$

□

Theorem 7.2. *Let* $u, g \in TS^{\wp}_{\sigma,\delta}(\vartheta, \hbar, \ell)$. *Then*

$$h(z) = z - \sum_{n=2}^{\infty}(a_n^2 + b_n^2)z^n \in TS^{\wp}_{\sigma,\delta}(\vartheta, \hbar, \ell, \zeta),$$

where

$$\zeta \geqslant \frac{2\vartheta(n-1)\ell^2(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]^2\phi(n, \wp, \sigma, \delta) - 2\ell^2(\hbar + (1-\vartheta))(n\hbar + 1 - \vartheta)}.$$

Proof. Since $u, g \in TS^{\wp}_{\sigma,\delta}(\vartheta, \hbar, \ell)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty}\left[\frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))}a_n\right]^2 \leqslant 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))}b_n\right]^2 \leqslant 1.$$

We obtain from the last two inequalities that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))}\right]^2(a_n^2 + b_n^2) \leqslant 1. \tag{7.1}$$

But $h(z) \in TS^{\wp}_{\sigma,\delta}(\vartheta, \hbar, \ell, \zeta)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1) + \zeta(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\zeta(\hbar + (1-\vartheta))}(a_n^2 + b_n^2) \leqslant 1, \tag{7.2}$$

where $0 < \zeta < 1$, however (7.1) implies (7.2) if

$$\frac{[\vartheta(n-1) + \zeta(n\hbar + 1 - \vartheta)]\phi(n, \wp, \sigma, \delta)}{\zeta(\hbar + (1-\vartheta))} \leqslant \frac{1}{2}\left[\frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n, \wp, \sigma, \delta)}{\ell(\hbar + (1-\vartheta))}\right]^2.$$

Simplifying, we get

$$\zeta \geqslant \frac{2\vartheta(n-1)\ell^2(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]^2\phi(n, \wp, \sigma, \delta) - 2\ell^2(\hbar + (1-\vartheta))(n\hbar + 1 - \vartheta)}.$$

□

8. Concluding remarks and observations

Recently, the Dunkel operator is a center of attraction for many well-known mathematicians, because of its diverse applications in many areas of Mathematics and Physics. In our present investigations, we were essentially motivated by the recent research going on in this field of study, and we have introduced a new sub class of univalent functions with negative coefficients. We next investigate some useful properties such as coefficient estimates, distortion theorem, growth theorem, radius of starlikness, and radius of convexity for this class. The authors sincerely hope this article will revive this concept and encourage the other researchers to work in the near future in the area of complex function theory.

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