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On some weaker forms of soft continuity and their decomposition theorems

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Abstract

In this paper, we employ soft ω^0-open sets to establish four new classes of soft functions in STSs: soft ω^0 -continuity, soft weak ω^0-continuity, soft w^* -continuity, and soft w^*-ω^0-continuity. We show that soft weak ω^0-continuity and soft w^*-ω^0 -continuity are distinct notions, each of which is strictly weaker than soft ω^0-continuity. Furthermore, we get a soft ω^0 -continuity decomposition theorem via both weak ω^0-continuity and soft w^*-ω^0-continuity. In addition, we demonstrate that soft w^*-continuity is precisely between soft continuity and soft w^*-ω^0-continuity. We further show that soft w^* -continuity and soft weak continuity are distinct concepts. In addition, we develop a soft continuity decomposition theorem via soft w^* -continuity and soft weak continuity. Finally, we examine the connection between our new soft topological ideas and their corresponding topological concepts. Include keywords, mathematical subject classification numbers as needed.

Keywords: Soft ω^0-openness, soft continuity, soft weak continuity, generated soft topology.

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1. Introduction and preliminaries

This work adheres to the principles and nomenclature presented in [8, 9]. STS and TS will be used in this study to signify soft topological space and topological space, respectively. Molodtsov [26] developed the concept of soft sets as a generic mathematical tool for coping with uncertainty in 1999. Let Y represent a universal set and E represent a set of parameters. A function $H : E \longrightarrow \mathcal{P}(Y)$ is a soft set over Y relative to E. $SS(Y, E)$ denotes the family of all soft sets over Y relative to E. The null soft set and the absolute soft set shall be represented by 0_E and 1_E, respectively, in this work. STS was defined as a contemporary mathematical structure in [30] as follows: A STS is a triplet (Y, δ, E), where δ comprises 0_E and 1_E, and is closed under finite soft intersection and arbitrary soft union. If (Y, δ, E) is a STS and $K \in SS(Y, E)$, then K is a soft open set in (Y, δ, E) if $K \in \delta$ and K is a soft closed set in (Y, δ, E) if $1_E - K \in \delta$. Soft topology principles and applications are still a hot field of research ([1–9, 12–21, 23, 27, 31]).

Generalizations of soft open sets play an effective role in soft topology through their use to improve on some known results or to open the door to redefine and investigate some of the soft topological concepts such as soft compactness, soft correlation, soft class axioms, soft assignments, etc. Authors in [10] defined

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and investigated ω^0-open sets as a generalization of open sets, which are a strong form of ω-open sets. The author in [3] has extended ω^0-open sets to include STSs.

In this paper, we employ soft ω^0-open sets to establish four new classes of soft functions in STSs: soft ω^0-continuity, soft weak ω^0-continuity, soft w^*-continuity, and soft w^*-ω^0-continuity. We show that soft weak ω^0-continuity and soft w^*-ω^0-continuity are distinct notions, each of which is strictly weaker than soft ω^0-continuity. Furthermore, we get a soft ω^0-continuity decomposition theorem via both weak ω^0-continuity and soft w^*-ω^0-continuity. In addition, we demonstrate that soft w^*-continuity is precisely between soft continuity and soft w^*-ω^0-continuity. We further show that soft w^*-continuity and soft weak continuity are distinct concepts. In addition, we develop a soft continuity decomposition theorem via soft w^*-continuity and soft weak continuity. Finally, we examine the connection between the new soft continuity concepts and their corresponding continuity concepts.

In the next work, we hope to find an application for our new soft topological notions in a decision-making problem.

Let (Y, δ, E) be a STS, $(Y, \Im)$ be a TS, $K \in SS(Y, E)$, and $W \subseteq Y$. Throughout this paper, $Cl_\delta(K)$, $Int_\delta(K)$, $Bd_\delta(K)$, $Cl_\Im(W)$, $Int_\Im(W)$, and $Bd_\Im(W)$ will denote the soft closure of K in (Y, δ, E), the soft interior of K in (Y, δ, E), the soft boundary of K in (Y, δ, E), the closure of W in $(Y, \Im)$, the interior of W in $(Y, \Im)$, and the boundary of W in $(Y, \Im)$, respectively. Also, the family of all soft closed sets in (Y, δ, E) (resp. closed sets in $(Y, \Im)$) will be denoted by δ^c (resp. $\Im^c$).

The following definitions will be used in the sequel.

Definition 1.1 ([10]). Let $(Y, \Im)$ be a TS and $U \subseteq Y$. Then

(a) U is said to be an ω^0-open set in $(Y, \Im)$ if for every $y \in U$, there exist $V \in \Im$ and a countable subset $B \subseteq Y$ such that $y \in V$ and $V - B \subseteq Int_\Im(U)$;
(b) U is said to be an ω^0-closed subset of $(Y, \Im)$ if $Y - U$ is an ω^0-open subset of $(Y, \Im)$;
(c) the collection of all ω^0-open subsets of $(Y, \Im)$ will be denoted by $\Im_{\omega^0}$.

Definition 1.2. A function $g : (Y, \Im) \longrightarrow (Z, \aleph)$ is said to be

(a) weakly continuous at $y \in Y$ if for every $N \in \aleph$ such that $g(y) \in N$, there exists $W \in \Im$ such that $y \in W$ and $g(W) \subseteq Cl_\aleph(N)$. g is said to be weakly continuous if g is weakly continuous at each $y \in Y$, [25];
(b) w^*-continuous if $g^{-1}(Bd_\aleph(W)) \in \delta^c$ for each $W \in \aleph$, [25];
(c) ω-continuous at $y \in Y$ if for every $N \in \aleph$ such that $g(y) \in N$, there exists $W \in \Im_\omega$ such that $y \in W$ and $g(W) \subseteq N$. g is said to be ω-continuous if g is ω-continuous at each $y \in Y$, [24];
(d) ω^0-continuous at $y \in Y$ if for every $N \in \aleph$ such that $g(y) \in N$, there exists $W \in \Im_{\omega^0}$ such that $y \in W$ and $g(W) \subseteq N$. g is said to be ω^0-continuous if g is ω^0-continuous at each $y \in Y$, [11];
(e) weakly ω^0-continuous at $y \in Y$ if for every $N \in \aleph$ such that $g(y) \in N$, there exists $W \in \Im_{\omega^0}$ such that $y \in W$ and $g(W) \subseteq Cl_\aleph(N)$. g is said to be weakly ω^0-continuous if g is weakly ω^0-continuous at each $y \in Y$, [11];
(f) w^*-ω^0-continuous if $g^{-1}(Bd_\aleph(W)) \in \Im^c_{\omega^0}$ for each $W \in \aleph$, [11].

Definition 1.3 ([22]). Let Y be a non-empty set and E be a set of parameters. A soft set $F \in SS(Y, E)$ is called a soft point over Y relative to E if there exist $e \in E$ and $y \in Y$ such that

$$F(a) = \begin{cases} \{y\}, & \text{if } a = e, \\ \emptyset, & \text{if } a \neq e. \end{cases}$$

We denote F by e_y. The family of all soft points over Y relative to E is denoted by $SP(Y, E)$.

Definition 1.4 ([3]). A soft set K of a STS (Y, δ, E) is said to be a soft ω^0-open set in (Y, δ, E) if for any $e_y \widetilde{\in} K$, there exists $S \in \delta$ and $C \in CSS(Y, B)$ such that $e_y \widetilde{\in} S$ and $S - C \widetilde{\subseteq} Int_\delta(K)$. Soft complements of soft ω^0-open sets in (Y, δ, E) are said to be soft ω^0-closed sets in (Y, δ, E).

The collection of all soft ω^0-open sets in (Y, δ, E) will be denoted by δ_{ω^0}.

Definition 1.5. A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is said to be

(a) soft weakly continuous at $e_y \in SP(Y, E)$ if for every $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$, there exists $K \in \delta$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(S)$. f_{pu} is said to be soft weakly continuous if f_{pu} is soft weakly continuous at each $e_y \in SP(Y, E)$, [29];

(b) soft ω-continuous at $e_y \in SP(Y, E)$ if for every $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$, there exists $K \in \delta_\omega$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}S$. f_{pu} is said to be soft ω-continuous if f_{pu} is soft ω-continuous at each $e_y \in SP(Y, E)$, [7].

Definition 1.6 ([28])**.** A STS (Y, δ, E) is said to be soft Lindelof if for every $\mathcal{K} \subseteq \circ$ such that $1_E = \widetilde{\cup}_{K\in\mathcal{K}}K$, there exists a countable subfamily $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $1_E = \widetilde{\cup}_{K_1\in\mathcal{K}}K$.

2. Soft ω^0-continuity

In this section, we introduce the notion of soft ω^0-continuous soft functions and establish their main properties. We characterize them by several methods. With the help of examples, we show the relationships between soft ω^0-continuous functions and some types of soft functions, such as soft continuous functions and soft ω-continuous functions. Also, we examine the connection between a soft ω^0-continuous soft function and its corresponding topological concept.

Definition 2.1. A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is said to be soft ω^0-continuous at $e_y \in SP(Y, E)$ if for every $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$, there exists $K \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}S$. f_{pu} is said to be soft ω^0-continuous if f_{pu} is soft ω^0-continuous at each $e_y \in SP(Y, E)$.

Theorem 2.2. *For a soft function* $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$*, the following conditions are equivalent:*

(a) f_{pu} *is soft* ω^0*-continuous;*
(b) $f^{-1}_{pu}(K) \in \delta_{\omega^0}$ *for each* $K \in \beta$*;*
(c) *for a soft base* $\mathcal{H}$ *of* (Z, β, D), $f^{-1}_{pu}(H) \in \delta_{\omega^0}$ *for each* $H \in \mathcal{H}$*;*
(d) *for a soft subbase* $\mathcal{T}$ *of* (Z, β, D), $f^{-1}_{pu}(T) \in \delta_{\omega^0}$ *for each* $T \in \mathcal{T}$*;*
(e) $f^{-1}_{pu}(R) \in \delta^c_{\omega^0}$ *for each* $R \in \beta^c$*;*
(f) $f_{pu}(Cl_{\delta_{\omega^0}}(M))\widetilde{\subseteq}Cl_\beta(f_{pu}(M))$ *for each* $M \in SS(Y, E)$*;*
(g) $Cl_{\delta_{\omega^0}}(f^{-1}_{pu}(F))\widetilde{\subseteq}f^{-1}_{pu}(Cl_\beta(F))$ *for each* $F \in SS(Z, D)$*.*

Proof.

(a) $\Longrightarrow$ (b): Let $K \in \beta$. Let $e_y\widetilde{\in}f^{-1}_{pu}(K)$, then $f_{pu}(e_y)\widetilde{\in}K$. By (a), we find $J \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}J$ and $f_{pu}(J)\widetilde{\subseteq}K$. Therefore, we have $e_y\widetilde{\in}J\widetilde{\subseteq}f^{-1}_{pu}(f_{pu}(J))\widetilde{\subseteq}f^{-1}_{pu}(K)$. Hence, $f^{-1}_{pu}(K) \in \delta_{\omega^0}$.

(b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (d): They are clear.

(d) $\Longrightarrow$ (e): Let $R \in \beta^c$. It is sufficient to show that $1_E - f^{-1}_{pu}(R) \in \delta_{\omega^0}$. Let $e_y\widetilde{\in}1_E - f^{-1}_{pu}(R) = f^{-1}_{pu}(1_D - R)$. Then $f_{pu}(e_y)\widetilde{\in}1_D - R \in \beta$. By (d), we find $T_1, T_2, \ldots, T_m \in \mathcal{T}$ such that $f_{pu}(e_y)\widetilde{\in}T_1\widetilde{\cap}T_2\widetilde{\cap}\cdots\widetilde{\cap}T_m\widetilde{\subseteq}1_D - R$. Thus,

$$e_y\widetilde{\in}f^{-1}_{pu}(T_1)\widetilde{\cap}f^{-1}_{pu}(T_2)\widetilde{\cap}\cdots\widetilde{\cap}f^{-1}_{pu}(T_m)\widetilde{\subseteq}f^{-1}_{pu}(1_D - R) = 1_E - f^{-1}_{pu}(R).$$

By (d), $f^{-1}_{pu}(T_1)\widetilde{\cap}f^{-1}_{pu}(T_2)\widetilde{\cap}\cdots\widetilde{\cap}f^{-1}_{pu}(T_m) \in \delta_{\omega^0}$ and so, $1_E - f^{-1}_{pu}(R) \in \delta_{\omega^0}$.

(e) $\Longrightarrow$ (f): Let $M \in SS(Y, E)$, then $Cl_\beta(f_{pu}(M)) \in \beta^c$ and by (e), $f^{-1}_{pu}(Cl_\beta(f_{pu}(K))) \in \delta^c_{\omega^0}$. As $M\widetilde{\subseteq}f^{-1}_{pu}(f_{pu}(M))\widetilde{\subseteq}f^{-1}_{pu}(Cl_\beta(f_{pu}(M)))$, then $Cl_{\delta_{\omega^0}}(M)\widetilde{\subseteq}f^{-1}_{pu}(Cl_\beta(f_{pu}(M)))$ and thus,

$$f_{pu}\left(Cl_{\delta_{\omega^0}}(M)\right)\widetilde{\subseteq}f_{pu}\left(f^{-1}_{pu}(Cl_\beta(f_{pu}(M)))\right)\widetilde{\subseteq}Cl_\beta(f_{pu}(M)).$$

(f) $\Longrightarrow$ (g): Let $F \in SS(Z,D)$, then by (f), $f_{pu}(Cl_{\delta_{\omega^0}}(f_{pu}^{-1}(F)))\widetilde{\subseteq}Cl_\beta(f_{pu}(f_{pu}^{-1}(F)))\widetilde{\subseteq}Cl_\beta(F)$. Thus,

$$Cl_{\delta_{\omega^0}}(f_{pu}^{-1}(F))\widetilde{\subseteq}f_{pu}^{-1}\left(f_{pu}(Cl_{\delta_{\omega^0}}(f_{pu}^{-1}(F)))\right)\widetilde{\subseteq}f_{pu}^{-1}(Cl_\beta(F)).$$

(g) $\Longrightarrow$ (a): Let $e_y \in SP(Y,E)$ and $S \in \beta$ with $f_{pu}(e_y) \in S$. Then by (g),

$$Cl_{\delta_{\omega^0}}\left(1_E - f_{pu}^{-1}(S)\right) = Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(1_D - S)\right)\widetilde{\subseteq}f_{pu}^{-1}(Cl_\beta(1_D - S)) = f_{pu}^{-1}(1_D - S) = 1_E - f_{pu}^{-1}(S).$$

So, $1_E - f_{pu}^{-1}(S) \in \delta_{\omega^0}^c$. Thus, we have $e_y\widetilde{\in}f_{pu}^{-1}(S) \in \delta_{\omega^0}$ and $f_{pu}(f_{pu}^{-1}(S))\widetilde{\subseteq}S$. This ends the proof. □

Theorem 2.3. *If $f_{pu} : (Y,\delta,E) \longrightarrow (Z,\beta,D)$ is soft ω^0-continuous at e_y, then $p : (Y,\delta_e) \longrightarrow (Z,\beta_{u(e)})$ is ω^0-continuous at y.*

Proof. Let $f_{pu} : (Y,\delta,E) \longrightarrow (Z,\beta,D)$ be soft ω^0-continuous at e_y. Let $W \in \beta_{u(e)}$ with $p(y) \in W$. There exists $K \in \beta$ such that $K(u(e)) = W$. Then we have $f_{pu}(e_y) = (u(e))_{p(y)}\widetilde{\in}K \in \beta$. So by soft ω^0-continuity of f_{pu} at e_y, we find $M \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}M$ and $f_{pu}(M)\widetilde{\subseteq}K$. Hence, we have $y \in M(e) \in (\delta_{\omega^0})_e$. Also, by Theorem 12 of [3], $(\delta_{\omega^0})_e \subseteq (\delta_e)_{\omega^0}$ and thus, $M(e) \in (\delta_e)_{\omega^0}$. We are going to show that $p(M(e)) \subseteq W$. Let $x \in M(e)$, then $e_x\widetilde{\in}M$. As $f_{pu}(M)\widetilde{\subseteq}K$, $f_{pu}(e_x) = (u(e))_{p(x)}\widetilde{\in}K$. Thus, $p(x) \in K(u(e)) = W$. □

We leave the validity of the converse of Theorem 2.3 as an open question.

Corollary 2.4. *If $f_{pu} : (Y,\delta,E) \longrightarrow (Z,\beta,D)$ is soft ω^0-continuous, then $p : (Y,\delta_e) \longrightarrow (Z,\beta_{u(e)})$ is ω^0-continuous for every $e \in E$.*

Theorem 2.5. *For any function $p : (Y,\mathfrak{I}) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y,\mathfrak{I}) \longrightarrow (Z,\aleph)$ is ω^0-continuous at a point $y \in Y$ if and only if $f_{pu} : (Y,\tau(\mathfrak{I}),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft ω^0-continuous at e_y for each $e \in E$.*

Proof.

Necessity. Let $p : (Y,\mathfrak{I}) \longrightarrow (Z,\aleph)$ be ω^0-continuous at y. Let $e \in E$ and $K \in \tau(\aleph)$ with $f_{pu}(e_y)\widetilde{\in}K$. Then $p(y) \in K(u(e)) \in \aleph$ and by ω^0-continuity of p at y, we find $W \in \mathfrak{I}_{\omega^0}$ such that $y \in W$ and $p(W) \subseteq K(u(e))$. Since by Corollary 4 of [3], $\tau(\mathfrak{I}_{\omega^0}) = (\tau(\mathfrak{I}))_{\omega^0}$ and $e_W \in \tau(\mathfrak{I}_{\omega^0})$, then $e_W \in (\tau(\mathfrak{I}))_{\omega^0}$. So, we have $e_y\widetilde{\in}e_W \in (\tau(\mathfrak{I}))_{\omega^0}$ with $f_{pu}(e_W) = (u(e))_{p(W)}\widetilde{\subseteq}K$. It follows that $f_{pu} : (Y,\tau(\mathfrak{I}),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft ω^0-continuous at e_y.

Sufficiency. Suppose that $f_{pu} : (Y,\tau(\mathfrak{I}),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft ω^0-continuous at e_y for every $e \in E$. Then, according to Theorem 2.3, $p : (Y,(\tau(\mathfrak{I}))_e) \longrightarrow \left(Z,(\tau(\aleph))_{u(e)}\right)$ is ω^0-continuous at y. Since $(\tau(\mathfrak{I}))_e = \mathfrak{I}$ and $(\tau(\aleph))_{u(e)} = \aleph$, we get the result. □

Corollary 2.6. *For any function $p : (Y,\mathfrak{I}) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y,\mathfrak{I}) \longrightarrow (Z,\aleph)$ is ω^0-continuous if and only if $f_{pu} : (Y,\tau(\mathfrak{I}),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft ω^0-continuous.*

Theorem 2.7.

(a) *Every soft continuous soft function is soft ω^0-continuous.*
(b) *Every soft ω^0-continuous soft function is soft ω-continuous.*

Proof. The proof of each of (a) and (b) follows from the definitions and Theorem 5 of [3]. □

The two examples below demonstrate that the class of soft ω^0-continuous functions is strictly between the classes of soft continuous functions and soft ω^0-continuous functions.

Example 2.8. Let $Y = \{1, 2\}$, $\Im = \{\emptyset, Y, \{1\}\}$, $\aleph = \{\emptyset, Y, \{2\}\}$, and $E = \mathbb{Q}$. Consider the identities functions $p : Y \longrightarrow Y$ and $u : E \longrightarrow E$. Then $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Y, \tau(\aleph), E)$ is soft ω^0-continuous but not soft continuous.

Example 2.9. Let $Y = \mathbb{R}$, $Z = \{1, 2\}$, $\Im$ be the usual topology on Y, $\aleph = \{\emptyset, Z, \{1\}\}$, and $E = \{a, b\}$. Define $p : (Y, \Im) \longrightarrow (Z, \aleph)$ as follows:

$$p(y) = \begin{cases} 2, & \text{if } y \in \mathbb{Q}, \\ 1, & \text{if } y \in \mathbb{R} - \mathbb{Q}. \end{cases}$$

Let $u : E \longrightarrow E$ be the identity function. Then $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Z, \tau(\aleph), E)$ is soft ω-continuous but not soft ω^0-continuous.

Theorem 2.10. *If $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous and $f_{qv} : (Z, \beta, D) \longrightarrow (W, \gamma, L)$ is soft continuous, then $f_{(q\circ p)(v\circ u)} : (Y, \delta, E) \longrightarrow (W, \gamma, L)$ is soft ω^0-continuous.*

Proof. Let $G \in \gamma$. By soft continuity of f_{qv}, $f_{qv}^{-1}(G) \in \beta$. Because of the soft ω^0-continuity of f_{pu}, $f_{pu}^{-1}(f_{qv}^{-1}(G)) = f_{(q\circ p)(v\circ u)}^{-1}(G) \in \delta_{\omega^0}$. This ends the proof. □

Theorem 2.11. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft ω^0-continuous and let $W \subseteq Y$ such that $C_W \in \delta_{\omega^0} - \{0_E\}$. Then $f_{(p_{|W})u} : (W, \delta_W, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous.*

Proof. Let $K \in \beta$. Then by soft ω^0-continuity of f_{pu}, $f_{pu}^{-1}(K) \in \delta_{\omega^0}$. Thus, $f_{(p_{|W})u}^{-1}(K) = f_{pu}^{-1}(K)\widetilde{\cap}C_W \in (\delta_{\omega^0})_W$. Hence, by Theorem 11 of [3], $f_{(p_{|W})u}^{-1}(K) \in (\delta_W)_{\omega^0}$. Therefore, $f_{(p_{|W})u} : (W, \delta_W, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous. □

Corollary 2.12. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft ω^0-continuous and let $W \subseteq Y$ such that $C_W \in \delta - \{0_E\}$. Then $f_{(p_{|W})u} : (W, \delta_W, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous.*

Theorem 2.13. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be a soft function and let $e_y \in SP(Y, E)$. If there is a $W \subseteq Y$ such that $C_W \in \delta_{\omega^0}$, $e_y\widetilde{\in}C_W$, and $f_{(p_{|W})u} : (W, \delta_W, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous at e_y, then $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous at e_y.*

Proof. Let $K \in \beta$ such that $f_{pu}(e_y) \in K$. As $f_{(p_{|W})u}$ is soft ω^0-continuous at e_y, we find $M \in (\delta_W)_{\omega^0}$ such that $e_y\widetilde{\in}M$ and $f_{(p_{|W})u}(M)\widetilde{\subseteq}K$. Since $C_W \in \delta_{\omega^0}$ and $M \in (\delta_W)_{\omega^0}$, then $M \in \delta_{\omega^0}$. It follows that $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous at e_y. □

Corollary 2.14. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be a soft function. Let $\{C_{Y_i} : i \in I\} \subseteq \delta_{\omega^0}$ such that $1_E = \widetilde{\cup}_{i\in I}C_{Y_i}$. If for each $i \in I$, $f_{(p_{|Y_i})u} : (Y_i, \delta_{Y_i}, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous, then $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous.*

Proof. Let $e_y \in SP(Y, E)$. It is sufficient to see that $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous at e_y. Choose $j \in I$ such that $e_y\widetilde{\in}C_{Y_j}$. Then, by Theorem 2.13, $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft ω^0-continuous. □

Theorem 2.15. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft ω-continuous and soft surjective. If (Y, δ, E) is soft Lindelof, then (Z, β, D) is soft Lindelof.*

Proof. Let $\mathcal{K} \subseteq \odot$ such that $1_D = \widetilde{\cup}_{K\in\mathcal{K}}K$. Then $1_E = \widetilde{\cup}_{K\in\mathcal{K}}f_{pu}^{-1}(K)$. Also, since f_{pu} is soft ω-continuous, then $\left\{f_{pu}^{-1}(K) : K \in \mathcal{K}\right\} \subseteq \delta_\omega$. Since (Y, δ, E) is soft Lindelof, then by Theorem 35 of [9], (Y, δ_ω, E). Thus, there exists a countable subset $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $1_E = \widetilde{\cup}_{K\in\mathcal{K}_1}f_{pu}^{-1}(K)$. Hence, $1_D = f_{pu}\left(\widetilde{\cup}_{K\in\mathcal{K}_1}f_{pu}^{-1}(K)\right) = \widetilde{\cup}_{K\in\mathcal{K}_1}f_{pu}\left(f_{pu}^{-1}(K)\right)\widetilde{\subseteq}\widetilde{\cup}_{K\in\mathcal{K}_1}K$. It follows that (Z, β, D) is soft Lindelof. □

Corollary 2.16. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft ω^0-continuous and soft surjective. If (Y, δ, E) is soft Lindelof, then (Z, β, D) is soft Lindelof.*

Proof. Follows from Theorems 2.7 (b) and 2.15. □

Corollary 2.17. *Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft continuous and soft surjective. If (Y, δ, E) is soft Lindelof, then (Z, β, D) is soft Lindelof.*

Proof. Follows from Theorems 2.7 (a) and Corollary 2.16. □

3. Decomposition theorems of soft continuity and soft ω^0-continuity

In this section, we offer three new forms of soft functions: soft weak ω^0-continuity, soft w^*-continuity, and soft w^*-ω^0-continuity. We demonstrate that soft w^*-ω^0-continuity and soft weak ω^0-continuity are separate concepts, each of which is strictly weaker than soft ω^0-continuity. Furthermore, using both weak ω^0-continuity and soft w^*-ω^0-continuity, we get a soft ω^0-continuity decomposition theorem. Furthermore, we show that soft w^*-continuity is precisely between soft continuity and soft w^*-ω^0-continuity. Furthermore, we demonstrate that soft w^*-continuity and soft weak continuity are separate notions. Furthermore, we prove a soft continuity decomposition theorem using soft w^*-continuity and soft weak continuity. Finally, we investigate the relationship between our novel soft topological notions and their related topological concepts.

Definition 3.1. A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is said to be soft weakly ω^0-continuous at $e_y \in SP(Y, E)$ if for every $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$, there exists $K \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(S)$. f_{pu} is said to be soft weakly ω^0-continuous if f_{pu} is soft weakly ω^0-continuous at each $e_y \in SP(Y, E)$.

Theorem 3.2. *For a soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$, the following conditions are equivalent:*

(a) *f_{pu} is soft weakly ω^0-continuous;*
(b) *for each $S \in \beta$, $Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(S)\right)\widetilde{\subseteq}f_{pu}^{-1}(Cl_\beta(S))$;*
(c) *for each $S \in \beta$, $f_{pu}^{-1}(S)\widetilde{\subseteq}Int_{\delta_{\omega^0}}\left(f_{pu}^{-1}(Cl_\beta(S))\right)$.*

Proof.
(a) $\Longrightarrow$ (b): Let $S \in \beta$. Suppose to the contrary that there exists $e_y\widetilde{\in}Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(S)\right) - f_{pu}^{-1}(Cl_\beta(S))$. Since $f_{pu}(e_y)\widetilde{\notin}Cl_\beta(S)$, then there exists $G \in \beta$ such that $f_{pu}(e_y) \in G$ and $G\widetilde{\cap}S = 0_D$. By (a), there exists $K \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(G)$. Since $e_y\widetilde{\in}Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(S)\right)$ and $e_y\widetilde{\in}K \in \delta_{\omega^0}$, then $K\widetilde{\cap}f_{pu}^{-1}(S) \neq 0_E$. Choose $b_z\widetilde{\in}K$ such that $f_{pu}(b_z)\widetilde{\in}S$. Since $b_z\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(G)$, then $f_{pu}(b_z)\widetilde{\in}Cl_\beta(G)$. Since $f_{pu}(b_z)\widetilde{\in}S \in \beta$ and $f_{pu}(b_z)\widetilde{\in}Cl_\beta(G)$, then $G\widetilde{\cap}S \neq 0_D$, a contradiction.
(b) $\Longrightarrow$ (c): Let $S \in \beta$. Then $1_D - Cl_\beta(S) \in \beta$ and by (b),

$$Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(1_D - Cl_\beta(S))\right)\widetilde{\subseteq}f_{pu}^{-1}(Cl_\beta(1_D - Cl_\beta(S))),$$

and so

$$1_E - f_{pu}^{-1}(Cl_\beta(1_D - Cl_\beta(S)))\widetilde{\subseteq}1_E - Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(1_D - Cl_\beta(S))\right).$$

Now,

$$1_E - f_{pu}^{-1}(Cl_\beta(1_D - Cl_\beta(S))) = f_{pu}^{-1}(1_D - Cl_\beta(1_D - Cl_\beta(S))) = f_{pu}^{-1}(Int_\beta(Cl_\beta(S))).$$

Also, since $S \in \beta$, then $S\widetilde{\subseteq}Int_\beta(Cl_\beta(S))$ and so $f_{pu}^{-1}(S)\widetilde{\subseteq}f_{pu}^{-1}(Int_\beta(Cl_\beta(S)))$. Thus, we have

$$\begin{aligned} f_{pu}^{-1}(S)\widetilde{\subseteq}1_E - f_{pu}^{-1}(Cl_\beta(1_D - Cl_\beta(S)))\widetilde{\subseteq}1_E - Cl_{\delta_{\omega^0}}\left(f_{pu}^{-1}(1_D - Cl_\beta(S))\right) \\ = 1_E - Cl_{\delta_{\omega^0}}\left((1_E - f_{pu}^{-1}(Cl_\beta(S)))\right) = Int_{\delta_{\omega^0}}\left(f_{pu}^{-1}(Cl_\beta(S))\right). \end{aligned}$$

(c) $\Longrightarrow$ (a): Let $e_y \in SP(Y,E)$ and let $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$. Then by (c), $f_{pu}^{-1}(S)\widetilde{\subseteq}Int_{\delta_{\omega^0}}\left(f_{pu}^{-1}(Cl_\beta(S))\right)$. Put $K = Int_{\delta_{\omega^0}}\left(f_{pu}^{-1}(Cl_\beta(S))\right)$. Then $f_{pu}(e_y)\widetilde{\in}K \in \delta_{\omega^0}$ and

$$f_{pu}(K) = f_{pu}(Int_{\delta_{\omega^0}}\left(f_{pu}^{-1}(Cl_\beta(S))\right))\widetilde{\subseteq}f_{pu}(\left(f_{pu}^{-1}(Cl_\beta(S))\right))\widetilde{\subseteq}Cl_\beta(S).$$

This ends the proof. □

Theorem 3.3. *For any function $p : (Y,\Im) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y,\Im) \longrightarrow (Z,\aleph)$ is weakly continuous at a point $y \in Y$ if and only if $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly continuous at e_y for every $e \in E$.*

Proof.

Necessity. Let $p : (Y,\Im) \longrightarrow (Z,\aleph)$ be weakly continuous at y. Let $e \in E$ and $K \in \tau(\aleph)$ with $f_{pu}(e_y)\widetilde{\in}K$. Then $p(y) \in K(u(e)) \in \aleph$ and by weak continuity of p at y, we find $W \in \Im$ such that $y \in W$ and $p(W) \subseteq Cl_\aleph(K(u(e)))$. So, we have $e_y\widetilde{\in}e_W \in \tau(\Im)$ with $f_{pu}(e_W) = (u(e))_{p(W)} \widetilde{\subseteq} (u(e))_{Cl_\aleph(K(u(e)))} \widetilde{\subseteq}Cl_{\tau(\aleph)}(K)$. It follows that $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly continuous at e_y.

Sufficiency. Let $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ be soft weakly continuous at e_y for every $e \in E$. Let $W \in \aleph$ such that $p(y) \in W$. Choose $e \in E$. Then we have $f_{pu}(e_y) = u(e)_{p(y)} \widetilde{\in} (u(e))_W \in \tau(\aleph)$. By soft weak continuity of f_{pu} at e_y, there exists $G \in \tau(\Im)$ such that $e_y\widetilde{\in}G$ and $f_{pu}(G)\widetilde{\subseteq}Cl_{\tau(\aleph)}((u(e))_W) = u(e)_{Cl_\aleph(W)}$. Thus, we have $y \in G(e) \in \Im$ and $p(G(e)) \subseteq (f_{pu}(G))(u(e)) \subseteq \left(u(e)_{Cl_\aleph(W)}\right)(u(e)) = Cl_\aleph(W)$. □

Corollary 3.4. *For any function $p : (Y,\Im) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y,\Im) \longrightarrow (Z,\aleph)$ is weakly continuous if and only if $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly continuous.*

Theorem 3.5. *For any function $p : (Y,\Im) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y,\Im) \longrightarrow (Z,\aleph)$ is weakly ω^0-continuous at a point $y \in Y$ if and only if $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly ω^0-continuous at e_y for every $e \in E$.*

Proof.

Necessity. Let $p : (Y,\Im) \longrightarrow (Z,\aleph)$ be weakly ω^0-continuous at y. Let $e \in E$ and $K \in \tau(\aleph)$ with $f_{pu}(e_y)\widetilde{\in}K$. Then $p(y) \in K(u(e)) \in \aleph$ and by weak ω^0-continuity of p at y, we find $W \in \Im_{\omega^0}$ such that $y \in W$ and $p(W) \subseteq Cl_\aleph(K(u(e)))$. Since by Corollary 4 of [3], $\tau(\Im_{\omega^0}) = (\tau(\Im))_{\omega^0}$ and $e_W \in \tau(\Im_{\omega^0})$, then $e_W \in (\tau(\Im))_{\omega^0}$. So, we have $e_y\widetilde{\in}e_W \in (\tau(\Im))_{\omega^0}$ with $f_{pu}(e_W) = (u(e))_{p(W)} \widetilde{\subseteq} (u(e))_{Cl_\aleph(K(u(e)))} \widetilde{\subseteq}Cl_{\tau(\aleph)}(K)$. It follows that $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly ω^0-continuous at e_y.

Sufficiency. Let $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ be soft weakly ω^0-continuous at e_y for every $e \in E$. Let $W \in \aleph$ such that $p(y) \in W$. Choose $e \in E$. Then we have $f_{pu}(e_y) = u(e)_{p(y)} \widetilde{\in} (u(e))_W \in \tau(\aleph)$. By soft weak ω^0-continuity of f_{pu} at e_y, there exists $G \in (\tau(\Im))_{\omega^0}$ such that $e_y\widetilde{\in}G$ and $f_{pu}(G)\widetilde{\subseteq}Cl_{\tau(\aleph)}((u(e))_W) = u(e)_{Cl_\aleph(W)}$. Since by Corollary 4 of [3], $\tau(\Im_{\omega^0}) = (\tau(\Im))_{\omega^0}$, then $G \in \tau(\Im_{\omega^0})$. Thus, we have $y \in G(e) \in \Im_{\omega^0}$ and $p(G(e)) \subseteq (f_{pu}(G))(u(e)) \subseteq \left(u(e)_{Cl_\aleph(W)}\right)(u(e)) = Cl_\aleph(W)$. □

Corollary 3.6. *For any function $p : (Y,\Im) \longrightarrow (Z,\aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $(Y,\Im) \longrightarrow (Z,\aleph)$ is weakly ω^0-continuous if and only if $f_{pu} : (Y,\tau(\Im),E) \longrightarrow (Z,\tau(\aleph),D)$ is soft weakly ω^0-continuous.*

Theorem 3.7. *Every soft ω^0-continuous soft function is soft weakly ω^0-continuous.*

Proof. Straightforward. □

The following example shows that the converse of Theorem 3.7 is not always true.

Example 3.8. Let $Y = \mathbb{R}$, $Z = \{a, b, c\}$, $\Im$ be the co-countable topology on Y, $\aleph = \{\emptyset, Z, \{a\}, \{b\}, \{a, b\}\}$, and $E = \mathbb{N}$. Define $p : (Y, \Im) \longrightarrow (Z, \aleph)$ and $u : E \longrightarrow E$ by

$$p(y) = \begin{cases} a, & \text{if } y \in \mathbb{Q}, \\ c, & \text{if } y \in \mathbb{R} - \mathbb{Q}, \end{cases}$$

and $u(e) = e$ for all $e \in E$. To see that p is weakly ω^0-continuous, let $y \in Y$ and $N \in \aleph$ such that $p(y) \in N$. If $y \in \mathbb{Q}$, there exists $\mathbb{R} \in \Im_{\omega^0}$ such that $y \in \mathbb{R}$ and $p(\mathbb{R}) = \{a, c\} = Cl_\aleph(\{a\}) \subseteq Cl_\aleph(N)$. If $y \in \mathbb{R} - \mathbb{Q}$, then $N = Z$ and so there exists $\mathbb{R} \in \Im_{\omega^0}$ such that $y \in \mathbb{R}$ and $p(\mathbb{R}) = \{a, c\} \subseteq Cl_\aleph(N) = Z$. On the other hand, since $\{a\} \in \aleph$ but $p^{-1}(\{a\}) = \mathbb{Q} \notin \Im_{\omega^0}$, then p is not ω^0-continuous. Thus, by Corollaries 2.6 and 3.6, $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Z, \tau(\aleph), E)$ is soft weakly ω^0-continuous but not soft ω^0-continuous.

Theorem 3.9. *If $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft weakly ω^0-continuous such that (Z, β, D) is soft regular, then f_{pu} is soft ω^0-continuous.*

Proof. Let $e_y \in SP(Y, E)$ and let $S \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}S$. Since (Z, β, D) is soft regular, then there exists $T \in \beta$ such that $f_{pu}(e_y)\widetilde{\in}T\widetilde{\subseteq}Cl_\beta(T)\widetilde{\subseteq}S$. By soft weak ω^0-continuity of f_{pu}, there exists $K \in \delta_{\omega^0}$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(T)\widetilde{\subseteq}S$. Therefore, f_{pu} is soft ω^0-continuous. □

Theorem 3.10. *Every soft weakly continuous soft function is soft weakly ω^0-continuous.*

Proof. Straightforward. □

The following example will show that the converse of Theorem 3.10 need not to be true in general.

Example 3.11. Let $Y = \{a, b, c\}$, $\Im = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$, $\aleph = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$, and $E = \mathbb{N}$. Let $p : Y \longrightarrow Y$ and $u : E \longrightarrow E$ be the identities functions. Since $\Im_{\omega^0}$ is the discrete topology on Y, then $p : (Y, \Im) \longrightarrow (Y, \aleph)$ is ω^0-continuous and by Theorem 3.7 it is weakly ω^0-continuous. Suppose that $p : (Y, \Im) \longrightarrow (Y, \aleph)$ is weakly continuous, then it is weakly continuous at b. Since $p(b) = b \in \{b\} \in \aleph$, then there exists $U \in \Im$ such that $b \in U$ and $p(U) = U = Y \subseteq Cl_\aleph(\{b\}) = \{b, c\}$ which is not true. Hence, $p : (Y, \Im) \longrightarrow (Y, \aleph)$ is not weakly continuous. Therefore, by Corollaries 3.4 and 3.6, $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Z, \tau(\aleph), E)$ is soft weakly ω^0-continuous but not soft weakly ω^0-continuous.

Definition 3.12. A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is said to be soft w^*-continuous if $f^{-1}_{pu}(Bd_\beta(K)) \in \delta^c$ for each $K \in \beta$.

Theorem 3.13. *For any function $p : (Y, \Im) \longrightarrow (Z, \aleph)$ between TSs and any function $u : E \longrightarrow D$ between sets of parameters, $p : (Y, \Im) \longrightarrow (Z, \aleph)$ is w^*-continuous if and only if $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Z, \tau(\aleph), D)$ is soft w^*-continuous.*

Proof.

Necessity. Let $p : (Y, \Im) \longrightarrow (Z, \aleph)$ be w^*-continuous. Let $K \in \tau(\aleph)$. Then for every $e \in E$, $K(u(e)) \in \aleph$ and by w^*-continuity of p, $p^{-1}(Bd_\aleph(K(u(e)))) \in \Im^c$. Now, for every $e \in E$ we have

$$\begin{aligned} f^{-1}_{pu}(Bd_{\tau(\aleph)}(K))(e) &= p^{-1}\left(\left(Bd_{\tau(\aleph)}(K)\right)(u(e))\right) \\ &= p^{-1}\left((Cl_{\tau(\aleph)}(K) - K)(u(e))\right) \\ &= p^{-1}\left(\left(\left(Cl_{\tau(\aleph)}(K)\right)(u(e)) - K(u(e))\right)\right) \\ &= p^{-1}(Cl_\aleph(K(u(e))) - K(u(e))) \\ &= p^{-1}(Bd_\aleph K(u(e))) \in \Im^c. \end{aligned}$$

Therefore, $f^{-1}_{pu}(Bd_{\tau(\aleph)}(K)) \in (\tau(\Im))^c$. Hence, $f_{pu} : (Y, \tau(\Im), E) \longrightarrow (Z, \tau(\aleph), D)$ is soft w^*-continuous.

Sufficiency. Suppose that $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Z, \tau(\aleph), D)$ is soft w^*-continuous. Let $V \in \aleph$. Choose $e \in E$. Then we have $(u(e))_V \in \tau(\aleph)$ and so, $f_{pu}^{-1}\left(Bd_{\tau(\aleph)}((u(e))_V\right) \in (\tau(\mathfrak{I}))^c$. Thus,

$$\begin{aligned}
\left(f_{pu}^{-1}\left(Bd_{\tau(\aleph)}((u(e))_V)\right)(e)\right. &= \left(f_{pu}^{-1}\left(Cl_{\tau(\aleph)}\left(((u(e))_V) - (u(e))_V\right)\right)(e)\right. \\
&= \left(f_{pu}^{-1}\left(((u(e))_{Cl_{\aleph}(V)} - (u(e))_V\right)\right)(e) \\
&= \left(f_{pu}^{-1}\left(((u(e))_{Cl_{\aleph}(V)-V}\right)\right)(e) \\
&= p^{-1}\left(\left(((u(e))_{Cl_{\aleph}(V)-V}\right)(u(e))\right) \\
&= p^{-1}\left(Cl_{\aleph}(V) - V\right) \\
&= p^{-1}\left(Bd_{\aleph}(V)\right) \in \mathfrak{I}^c.
\end{aligned}$$

Therefore, $p : (Y, \mathfrak{I}) \longrightarrow (Z, \aleph)$ is w^*-continuous. □

The following two examples will show that soft weak continuity and soft w^*-continuity are independent concepts.

Example 3.14. Let $Y = \{1, 2\}$, $\mathfrak{I} = \{\emptyset, Y, \{1\}\}$, $\aleph = \{\emptyset, Y, \{2\}\}$, and $E = \mathbb{N}$. Consider the identities functions $p : Y \longrightarrow Y$ and $u : E \longrightarrow E$. Then clearly that $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is weakly continuous. On the other hand, since $\{2\} \in \aleph$, while $p^{-1}(Bd_{\aleph}(\{2\})) = p^{-1}(\{1\}) = \{1\} \notin \mathfrak{I}^c$, then $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is not w^*-continuous. Therefore, by Corollary 3.4 and Theorem 3.13, $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Y, \tau(\aleph), E)$ is soft weakly continuous but not soft w^*-continuous.

Example 3.15. Let $Y = \{1, 2\}$, $\mathfrak{I} = \{\emptyset, Y\}$, $\aleph = \{\emptyset, Y, \{1\}, \{2\}\}$, and $E = \mathbb{N}$. Consider the identities functions $p : Y \longrightarrow Y$ and $u : E \longrightarrow E$. Since for each $V \in \aleph$, $p^{-1}(Bd_{\aleph}(V)) = p^{-1}(\emptyset) = \emptyset \in \mathfrak{I}^c$, then $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is w^*-continuous. Suppose that $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is weakly continuous. Since $p(2) = 2 \in \{2\} \in \aleph$, then there exists $U \in \mathfrak{I}$ such that $p(U) = p(Y) = Y \subseteq Cl_{\aleph}(\{2\}) = \{2\}$ which is impossible. Thus, $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is not weakly continuous. Therefore, by Corollary 3.4 and Theorem 3.13, $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Y, \tau(\aleph), E)$ is soft w^*-continuous but not soft weakly continuous.

Theorem 3.16. *Every soft continuous soft function is soft w^*-continuous.*

Proof. Let $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ be soft continuous and let $K \in \beta$. Since f_{pu} is soft continuous and $Bd_{\beta}(K) \in \beta^c$, then $f_{pu}^{-1}(Bd_{\beta}(K)) \in \delta^c$. Therefore, f_{pu} is soft w^*-continuous. □

Since every soft continuous function is soft weakly continuous, then Example 3.15 shows that the converse of Theorem 3.16 need not to be true in general.

The following theorem is a decomposition of soft continuity.

Theorem 3.17. *A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft continuous if and only if it is both soft weakly continuous and soft w^*-continuous.*

Proof.

Necessity. Suppose that $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is soft continuous. Then by Proposition 5.3 of [11], f_{pu} is soft weakly continuous. Also, by Theorem 3.16, f_{pu} is soft w^*-continuous.

Sufficiency. Suppose that $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is both soft weakly continuous and soft w^*-continuous. Let $e_y \in SP(Y, E)$ and let $S \in \beta$ such that $f_{pu}(e_y) \widetilde{\in} S$. Since f_{pu} is soft weakly continuous, there exists $K \in \delta$ such that $e_y \widetilde{\in} K$ and $f_{pu}(K) \widetilde{\subseteq} Cl_{\beta}(S)$. Since $f_{pu}(e_y) \widetilde{\in} S$ and $Bd_{\beta}(S) = Cl_{\beta}(S) - S$, then $f_{pu}(e_y) \widetilde{\notin} Bd_{\beta}(S)$ and so $e_y \widetilde{\notin} f_{pu}^{-1}\left(Bd_{\beta}(S)\right)$. Also, since Since f_{pu} is soft w^*-continuous, then $f_{pu}^{-1}\left(Bd_{\beta}(S)\right) \in \delta^c$. Therefore, we have $e_y \widetilde{\in} K - f_{pu}^{-1}\left(Bd_{\beta}(S)\right) \in \delta$.

Claim. $f_{pu}(K - f_{pu}^{-1}\left(Bd_{\beta}(S)\right)) \subseteq S$, which ends the proof.

Proof of claim. Suppose to the contrary that there exists $d_z \widetilde{\in} f_{pu}(K - f_{pu}^{-1}(Bd_\beta(S))) - S$. Choose $b_t \widetilde{\in} K - f_{pu}^{-1}(Bd_\beta(S))$ such that $d_z = f_{pu}(b_t)$. Since $b_t \widetilde{\in} K$ and $f_{pu}(K) \widetilde{\subseteq} Cl_\beta(S)$, then $d_z = f_{pu}(b_t) \widetilde{\in} Cl_\beta(S)$ and so $b_t \widetilde{\in} f_{pu}^{-1}(Cl_\beta(S))$. Since $b_t \widetilde{\notin} f_{pu}^{-1}(Bd_\beta(S)) = f_{pu}^{-1}(Cl_\beta(S) - S) = f_{pu}^{-1}(Cl_\beta(S)) - f_{pu}^{-1}(S)$ and $b_t \widetilde{\in} f_{pu}^{-1}(Cl_\beta(S))$, then $b_t \widetilde{\in} f_{pu}^{-1}(S)$. But since $f_{pu}(b_t) = d_z \widetilde{\notin} S$, then $b_t \widetilde{\notin} f_{pu}^{-1}(S)$. This is a contradiction. □

Definition 3.18. A soft function $f_{pu} : (Y, \delta, E) \longrightarrow (Z, \beta, D)$ is said to be soft w^*-ω^0-continuous if $f_{pu}^{-1}(Bd_\beta(K)) \in \delta^c_{\omega^0}$ for each $K \in \beta$.

Theorem 3.19. *For any function* $p : (Y, \mathfrak{I}) \longrightarrow (Z, \aleph)$ *between TSs and any function* $u : E \longrightarrow D$ *between sets of parameters,* $p : (Y, \mathfrak{I}) \longrightarrow (Z, \aleph)$ *is* w^*-ω^0*-continuous if and only if* $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Z, \tau(\aleph), D)$ *is soft* w^*-ω^0*-continuous.*

Proof.

Necessity. Let $p : (Y, \mathfrak{I}) \longrightarrow (Z, \aleph)$ be w^*-ω^0-continuous. Let $K \in \tau(\aleph)$. Then for every $e \in E$, $K(u(e)) \in \aleph$ and by w^*-ω^0-continuity of p, $p^{-1}(Bd_\aleph(K(u(e)))) \in \mathfrak{I}^c_{\omega^0}$. Now, for every $e \in E$ we have

$$\begin{aligned} f_{pu}^{-1}(Bd_{\tau(\aleph)}(K))(e) &= p^{-1}\left(\left(Bd_{\tau(\aleph)}(K)\right)(u(e))\right) \\ &= p^{-1}\left((Cl_{\tau(\aleph)}(K) - K)(u(e))\right) \\ &= p^{-1}\left(\left(\left(Cl_{\tau(\aleph)}(K)\right)(u(e)) - K(u(e))\right)\right) \\ &= p^{-1}\left((Cl_\aleph(K(u(e))) - K(u(e)))\right) \\ &= p^{-1}(Bd_\aleph(K(u(e))) \in \mathfrak{I}^c_{\omega^0}. \end{aligned}$$

Therefore, $f_{pu}^{-1}(Bd_{\tau(\aleph)}(K)) \in \tau\left(\mathfrak{I}^c_{\omega^0}\right)$. Hence, by Corollary 4 of [3], $f_{pu}^{-1}(Bd_{\tau(\aleph)}(K)) \in (\tau(\mathfrak{I}))^c_{\omega^0}$. It follows that $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Z, \tau(\aleph), D)$ is soft w^*-ω^0-continuous.

Sufficiency. Suppose that $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Z, \tau(\aleph), D)$ is soft w^*-ω^0-continuous. Let $V \in \aleph$. Choose $e \in E$. Then we have $(u(e))_V \in \tau(\aleph)$ and so, $f_{pu}^{-1}\left(Bd_{\tau(\aleph)}((u(e))_V\right) \in (\tau(\mathfrak{I}))^c_{\omega^0}$. Thus, by [3, Corollary 4], $f_{pu}^{-1}\left(Bd_{\tau(\aleph)}((u(e))_V\right) \in \tau\left(\mathfrak{I}^c_{\omega^0}\right)$ and

$$\begin{aligned} \left(f_{pu}^{-1}\left(Bd_{\tau(\aleph)}((u(e))_V\right)\right)(e) &= \left(f_{pu}^{-1}\left(Cl_{\tau(\aleph)}(((u(e))_V) - (u(e))_V\right)\right)(e) \\ &= \left(f_{pu}^{-1}\left(((u(e))_{Cl_\aleph(V)} - (u(e))_V\right)\right)(e) \\ &= \left(f_{pu}^{-1}\left(((u(e))_{Cl_\aleph(V)-V}\right)\right)(e) \\ &= p^{-1}\left(\left(((u(e))_{Cl_\aleph(V)-V}\right)((u(e)))\right) \\ &= p^{-1}(Cl_\aleph(V) - V) \\ &= p^{-1}(Bd_\aleph(V)) \in \mathfrak{I}^c_{\omega^0}. \end{aligned}$$

Therefore, $p : (Y, \mathfrak{I}) \longrightarrow (Z, \aleph)$ is w^*-ω^0-continuous. □

The following two examples will show that soft weak ω^0-continuity and soft w^*-ω^0-continuity are independent concepts:

Example 3.20. Let $Y = \mathbb{R}$, $\mathfrak{I}$ be the usual topology on Y, $\aleph$ be the discrete topology on Y, and $E = \mathbb{Z}$. Consider the identities functions $p : Y \longrightarrow Y$ and $u : E \longrightarrow E$. Since for each $V \in \aleph$, $p^{-1}(Bd_\aleph(V)) = p^{-1}(\emptyset) = \emptyset \in \mathfrak{I}^c_{\omega^0}$, then $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is w^*-ω^0-continuous. To see that p is not weakly ω^0-continuous, suppose to the contrary that p is weakly ω^0-continuous. Since $1 \in p(\{1\}) = \{1\}$, then there exists $U \in \mathfrak{I}_{\omega^0}$ such that $1 \in U$ and $p(U) = U \subseteq Cl_\aleph(\{1\}) = \{1\}$. Thus, $U = \{1\} \in \mathfrak{I}_{\omega^0}$. Thus, there exists $S \in \mathfrak{I}$ and a countable set $C \subseteq Y$ such that $1 \in S$ and $S - C \subseteq Int_\mathfrak{I}(\{1\}) = \emptyset$. Hence S is countable which is a contradiction. Hence, $p : (Y, \mathfrak{I}) \longrightarrow (Y, \aleph)$ is not weakly ω^0-continuous. Therefore, by Theorem 3.19 and Corollary 3.6, $f_{pu} : (Y, \tau(\mathfrak{I}), E) \longrightarrow (Y, \tau(\aleph), E)$ is soft w^*-ω^0-continuous but not soft weakly ω^0-continuous.

Example 3.21. Let $Y=\mathbb{R}$, $Z=\{1,2\}$, $\mathfrak{I}$ be the co-countable topology on Y, $\aleph=\{\emptyset, Z, \{1\}\}$, and $E=\mathbb{Z}$. Define $p:(Y,\mathfrak{I})\longrightarrow(Z,\aleph)$ as follows:

$$p(y)=\begin{cases}1, & \text{if } y\in\mathbb{Q},\\ 2, & \text{if } y\in\mathbb{R}-\mathbb{Q}.\end{cases}$$

Let $u:E\longrightarrow E$ be the identity function. Then $f_{pu}:(Y,\tau(\mathfrak{I}),E)\longrightarrow(Z,\tau(\aleph),E)$ is soft weakly ω^0-continuous but not soft w^*-ω^0-continuous.

Theorem 3.22. *Every soft ω^0-continuous soft function is soft w^*-ω^0-continuous.*

Proof. Let $f_{pu}:(Y,\delta,E)\longrightarrow(Z,\beta,D)$ be soft ω^0-continuous and let $K\in\beta$. Since f_{pu} is soft ω^0-continuous and $Bd_\beta(K)\in\beta^c$, then by Theorem 2.2 (e), $f_{pu}^{-1}(Bd_\beta(K))\in\delta^c_{\omega^0}$. Therefore, f_{pu} is soft w^*-ω^0-continuous. □

Example 3.20 and Theorem 3.7 shows that the converse of Theorem 3.22 need not to be true in general. The following theorem is a decomposition of soft ω^0-continuity.

Theorem 3.23. *A soft function $f_{pu}:(Y,\delta,E)\longrightarrow(Z,\beta,D)$ is soft ω^0-continuous if and only if it is both soft weakly ω^0-continuous and soft w^*-ω^0-continuous.*

Proof.

Necessity. Suppose that $f_{pu}:(Y,\delta,E)\longrightarrow(Z,\beta,D)$ is soft ω^0-continuous. Then by Theorems 3.7 and 3.22, f_{pu} is both soft weakly ω^0-continuous and soft w^*-ω^0-continuous.

Sufficiency. Suppose that $f_{pu}:(Y,\delta,E)\longrightarrow(Z,\beta,D)$ is both soft weakly ω^0-continuous and soft w^*-ω^0-continuous. Let $e_y\in SP(Y,E)$ and let $S\in\beta$ such that $f_{pu}(e_y)\widetilde{\in}S$. Since f_{pu} is soft weakly ω^0-continuous, there exists $K\in\delta_{\omega^0}$ such that $e_y\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(S)$. Since $f_{pu}(e_y)\widetilde{\in}S$ and $Bd_\beta(S)=Cl_\beta(S)-S$, then $f_{pu}(e_y)\widetilde{\notin}Bd_\beta(S)$ and so $e_y\widetilde{\notin}f_{pu}^{-1}(Bd_\beta(S))$. Also, since f_{pu} is soft w^*-ω^0-continuous, then $f_{pu}^{-1}(Bd_\beta(S))\in\delta^c_{\omega^0}$. Therefore, we have $e_y\widetilde{\in}K-f_{pu}^{-1}(Bd_\beta(S))\in\delta_{\omega^0}$.

Claim. $f_{pu}(K-f_{pu}^{-1}(Bd_\beta(S)))\subseteq S$ which ends the proof.

Proof of claim. Suppose to the contrary that there exists $d_z\widetilde{\in}f_{pu}(K-f_{pu}^{-1}(Bd_\beta(S)))-S$. Choose $b_t\widetilde{\in}K-f_{pu}^{-1}(Bd_\beta(S))$ such that $d_z=f_{pu}(b_t)$. Since $b_t\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}Cl_\beta(S)$, then $d_z=f_{pu}(b_t)\widetilde{\in}Cl_\beta(S)$ and so $b_t\widetilde{\in}f_{pu}^{-1}(Cl_\beta(S))$. Since $b_t\widetilde{\notin}f_{pu}^{-1}(Bd_\beta(S))=f_{pu}^{-1}(Cl_\beta(S)-S)=f_{pu}^{-1}(Cl_\beta(S))-f_{pu}^{-1}(S)$ and $b_t\widetilde{\in}f_{pu}^{-1}(Cl_\beta(S))$, then $b_t\widetilde{\in}f_{pu}^{-1}(S)$. But since $f_{pu}(b_t)=d_z\widetilde{\notin}S$, then $b_t\widetilde{\notin}f_{pu}^{-1}(S)$. This is a contradiction. □

4. Conclusion

The continual supply of topological space classes, instances, characteristics, and relationships has aided topology's progress. As a result, it is critical to increase the structure of soft topological spaces in the same way.

The goals of this study are to examine the behaviors of soft ω^0-open via soft topological spaces, to introduce new classes of soft functions, and to pave the way for the definition and investigation of certain new soft topological notions in the future.

In this paper, four new classes of soft functions are introduced. Several characterizations, relationships, and examples are given. Also, two decomposition theorems via them are obtained. The following topics could be considered in future studies:

1) to define soft ω^0- homomorphisms;
2) to define weaker and stronger forms of soft ω^0-continuous functions.

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