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Co-infection dynamics of two short term diseases with effect of recovery delay

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Abstract

A co-infection model with two short-term diseases with delay in recovery is proposed. Here, we consider the simultaneous transmission of infection does not happen but of simultaneous recovery from both illnesses. The system consists of four epidemiological classes populations, namely: susceptible (S), an infected class with the first disease (I_1), an infected class with the second disease (I_2), co-infected class (I_{12}). We have found all possible equilibrium states, and the basic reproduction number also examined their stability without and with delay. Analytically, we have established that the local stability of equilibrium points depends on the basic reproduction number in the absence of recovery delay. But with delay, it requires some additional conditions. We have also checked the effect of delay on stability of endemic steady state numerically and showed that beyond a critical threshold value of delay parameter, the system loses its stability, and Hopf bifurcation occurs. Finally, a numerical simulation presented supports the analytical findings.

Keywords: Epidemic co-infection model, recovery delay, asymptotic stability, Hopf-bifurcation.

2020 MSC: 34A34, 34C23, 34C25, 34D20, 92D30.

1. Introduction

The co-infection is defined as the aggression of a single host by two pathogens, which may cause various diseases or different parasite variants (Allen et al. [1], Courchamp et al. [6] etc). A relation between superinfection and co-infection with various strains is discussed in [25], as well as studied the effect of both on virulence coexistence and evolution.

Simultaneously, it is natural to establish a fundamental model that can explain the general characteristics of a co-infection epidemic and analyze its dynamics to better understand qualitative behavior in this system. In the case of two diseases, co-infection can be classified into many types: (i) both are short term diseases; (ii) one short term and long term diseases; (iii) one short term, and other is nonrecoverable (i.e., permanent) diseases; (iv) both are nonrecoverable diseases. For example, co-infections of the third type are typical for people infected with permanent illnesses like the human immunodeficiency virus (HIV). HIV weakens the immune system, making it susceptible to other infections such as opportunistic infections [14]. The case of HIV-HSV (herpes simplex virus) co-infection, for example, has been well

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known. Such co-infection commonly results in HSV reactivation, hastening the progression of HIV disease to AIDS [34]. Co-infections can also occur when a patient already has a long-term, slow-progressing disease. For example, in tuberculosis (TB), even a minor illness may cause the infection to reactivate. Recent years it has been observed that the COVID-19 disease becomes more deadly to those having already long-term infection like tuberculosis (TB) [9] also HIV with TB shows deadly effect as co-infection [33]. Simultaneously, few studies look at the interaction of two or more diseases. The interactions between M. tuberculosis and HIV-1 are studied on an immunological basis in [16]. [7] focuses on the statistical implications of mapping two diseases. [24] examined co-infection from a large viewpoint.

Numerous mathematical models describe disease nature, treatment optimization, and population vaccination. Some studies look at models that combine two infections, such as tuberculosis and AIDS [16, 23], or identify two type of strains for a single disease that are prevalent in the people, such as tuberculosis or influenza [5, 21, 26]. The case of sexually transmitted (STDs) such as gonorrhea and AIDS [15] is an example of a practical scenario in which two diseases coexist.

Mathematical models with delay are commonly used for studies and forecasts in numerous areas in life sciences, such as epidemiology, ecology, eco-epidemiology, neuroscience, immunology, and neural networks [2, 4, 20, 27, 29]. In such models, time lags or time delays can be linked with the length of many secret cycles, such as the life cycle periods, the duration between development of new viruses, the span of the infection growth, the recovery process, etc. [28]. The inclusion of time delays increases the complexity of the model. Thus It is important to analyze the model's qualitative behavior, using stability or bifurcation approach [10]. In recent years, the non-linearity and sensitivity analysis of DDEs has been intensively analyzed in numerous scientific and technological fields, especially in the sense of chaotic dynamics [19, 30]. Before this study, Hao and Fan [13], integrated double delays in the mathematical model of HIV and using the traditional analysis proposed by Beretta and Kuang [3], they achieve adequate conditions for the presence of Hopf bifurcation. Dynamical systems with single and double delay analysis with limiting conditions for two delays studied [31, 32]. Later, Gu et al. discussed multiple delays, especially two delay problems with limiting characteristic equation of the variational matrix [11]. Lin and Wang, generalized the characteristic equation of two delay problems and given more effective analysis [22]. Recently many researchers used the single and multiple delays in the eco-epidemic model and demonstrated the stabilization effect of the disease [17, 18]. They studied the dynamics of the interior steady-state's stability for different pairs of delay factors and showed that delay could lead to oscillation through a hop bifurcation.

The paper aims to look at a co-infection model induced by two short-term infections and the effect of the recovery delay in co-infection dynamics. It may be two different infections or two distinct strains of the same disease. Further, it is assumed that each human is thought to be susceptible to one or both diseases, and no vaccine is available.

2. Formulation of the mathematical model

We have divided the total population in four compartments, i.e., susceptible population (S), an infected class population with the first disease (I_1), an infected class population with the second disease (I_2), and Co-infected population (I_{12}). This model's schematic flow is shown in Figure 1 and driven by a system of differential equations with delay (DDEs) (2.1)-(2.4) with assumption that an individual can not co-infected directly, an individual can not recovered once infected by permanent disease, and one can get susceptible after getting recovered due to loss of immunity,

$$\frac{dS}{dt} = \Lambda - \beta_1 S I_1 - \beta_2 S I_2 - \beta_3 S I_{12} + \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_{12}(t-\tau) - \mu S, \tag{2.1}$$

$$\frac{dI_1}{dt} = \beta_1 S I_1 + (1-p)\beta_3 S I_{12} - (1-q)\beta_4 I_1 I_2 - \gamma_1 I_1 - \mu I_1, \tag{2.2}$$

$$\frac{dI_2}{dt} = \beta_2 S I_2 + p\beta_3 S I_{12} - q\beta_4 I_1 I_2 - \gamma_2 I_2 - \mu I_2, \tag{2.3}$$

$$\frac{dI_{12}}{dt} = \beta_4 I_1 I_2 - \gamma_3 I_{12}(t-\tau) - (\mu+\delta) I_{12}, \tag{2.4}$$

with the initial conditions, let $C([-\tau,0],R^4)$ be a Banach space of continuous functions χ defined as $\chi : [-\tau,0] \to R^4$, with norm $\| \chi \| = \sup_{\tau<\nu<0} \{| \chi_1(\nu) |, | \chi_2(\nu) |, | \chi_3(\nu) |, | \chi_4(\nu) |\}$, where $\chi = (\chi_1,\chi_2,\chi_3,\chi_4)$. The given initial conditions of system are $S(\nu) = \chi_1(\nu)$, $I_1(\nu) = \chi_2(\nu)$, $I_2(\nu) = \chi_3(\nu)$, $I_{12}(\nu) = \chi_4(\nu)$, $\nu \in [-\tau,0]$, where the initial function $\chi = (\chi_1,\chi_2,\chi_3,\chi_4)$ belongs to Banach space C. The initial condition, with biological feasibility, we choose as

$$\chi_i(\nu) \geqslant 0, \text{ where } \quad \nu \in [-\tau,0], \text{ for } \quad i = 1,2,3,4.$$

By the fundamental theorem of functional differential equations [12], delayed system possesses the unique solution with above initial conditions. All system parameters are positive, and these parameters are defined as: Λ is susceptible recruitment rate, $\frac{1}{\mu}$ is an average life span, β_1 is transmission coefficient from susceptible to infection class I_1, β_2 is transmission coefficient from susceptible to infected class I_2, β_3 is transmission coefficient from susceptible to infected class I_1 & I_2 by co-infected I_{12} interaction, β_4 is transmission coefficient from I_1 & I_2 to co-infected class I_{12}, γ_1 is the rate of recovering from the infected class I_1, γ_2 is recovery rate from infected class I_2, γ_3 is recovery rate from co-infected class I_{12}, δ death rate due to co-infection, τ delay in recovery from co-infected class, and p, q are constants.

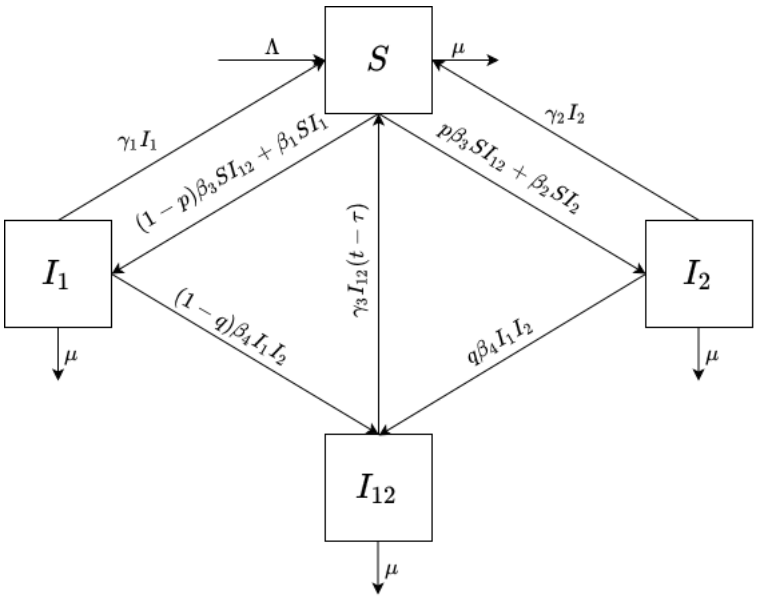

Figure 1: Schematic flow of proposed $S - I_1 - I_2 - I_{12} - S$ model with two delays.

3. Boundedness and positivity

This part has some lemmas for boundedness and positivity of the system solution (2.1)-(2.4).

Lemma 3.1. *Non-negative initial conditions system* (2.1)-(2.4) *possesses non-negative solution for all* $t \geqslant 0$.

Proof. Let (S, I_1, I_2, I_{12}) be the solution with nonnegative initial population of the proposed system. For $t \in [0,\tau]$, the equation (2.1) forms the relation:

$$\frac{d(S)}{dt} \geqslant -(\mu + \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_{12}) S,$$

which evidences that

$$S(t) \geqslant S(0)e^{-\int \mu+\beta_1 I_1(t)+\beta_2 I_2(t)+\beta_3 I_{12}(t)\ dt} > 0,$$

also for $t \in [0,\tau]$, the equation (2.2) state that

$$\frac{dI_1}{dt} \geqslant -((1-q)\beta_4 I_2 + \gamma_1 + \mu)I_1,$$

which results

$$I_1 \geqslant I_1(0)e^{-\int (1-q)\beta_4 I_2(t)+\gamma_1+\mu\ dt} > 0,$$

Also for $t \in [0,\tau]$, the equation (2.3) state that

$$\frac{dI_2}{dt} \geqslant -(q\beta_4 I_1 + \gamma_2 + \mu)I_2,$$

which results

$$I_2 \geqslant I_1(0)e^{-\int q\beta_4 I_1(t)+\gamma_2+\mu\ dt} > 0,$$

finally for $t \in [0,\tau]$, the equation (2.4) state that

$$\frac{dI_{12}}{dt} \geqslant -(\gamma_3 \frac{I_{12}(t-\tau)}{I_{12}} + \mu + \delta)I_{12},$$

which results

$$I_{12} \geqslant I_{12}(0)\ e^{-\int \gamma_3 \frac{I_{12}(t-\tau)}{I_{12}}+\mu+\delta\ dt} > 0.$$

Similarly for the interval $[0,\tau], [\tau,2\tau]\cdots[(n-1)\tau, n\tau]\cdots$, where $n \in \mathbb{N}$, it can prove that $S(t)$, $I_1(t)$, $I_2(t)$, and $I_{12}(t)$ all are non-negative. For this reason, the population remains positive for the system (2.1)-(2.4), i.e., $S(t)$, $I_1(t)$, $I_2(t)$, $I_{12}(t)$, $\geqslant 0$ for all $t \geqslant 0$. □

Lemma 3.2. *If the initial population is positive then the proposed system solution* (2.1)-(2.4) *is bounded uniformly in* Ω*, where,*

$$\Omega = \left\{(S, I_1, I_2, I_{12}) : 0 \leqslant S + I_1 + I_2 + I_{12} \leqslant \frac{\Lambda}{\mu}\right\}.$$

Proof. Assuming that $P(t) = S(t) + I_1(t) + I_2(t) + I_{12}(t)$ is total population at any instant 't', now differentiating $P(t)$ with respect to 't', we have as follows from the system (2.1)-(2.4):

$$\frac{dP(t)}{dt} = \Lambda - \mu(S(t) + I_1(t) + I_2(t) + I_{12}(t)) - \delta I_{12} \leqslant \Lambda - \mu P \quad \Longrightarrow \quad P(t) \leqslant P(0)e^{-\mu t} + \frac{\Lambda}{\mu},$$

as $t \to \infty$, $P(t) \to \frac{\Lambda}{\mu}$. Clearly the system (2.1)-(2.4) is bounded above for its each population. Since initially all population are positive hence system is bounded below by zero. Therefore the bounded, feasible biological region is given by $\Omega = \left\{(S, I_1, I_2, I_{12}) : 0 \leqslant S + I_1 + I_2 + I_{12} \leqslant \frac{\Lambda}{\mu}\right\}$. □

4. System dynamical study

Under this portion, we explore all possible feasible equilibria and the basic reproduction number/ratio of the system (2.1)-(2.4). As the system is positively invariant under the feasible region Ω, therefore, we only consider solutions, which follow initial conditions within the region Ω and also the uniqueness and existence conditions with result of continuation.

4.1. Possible equilibrium points

The system (2.1)-(2.4) possesses several equilibria: when both disease are absent (disease free equilibrium), and when a disease (one or both simultaneously) is present. In the latter case an equilibrium is called endemic.

(i) Disease/Infection-free equilibrium (DFE) $E^0(S^0, 0, 0, 0,)$, always exists, where $S^0 = \frac{\Lambda}{\mu}$.
(ii) Infection (I_2)-free equilibrium (IFE) $E^1(S^1, I_1^1, 0, 0,)$, exists, if C1 holds, where $S^1 = \frac{\mu+\gamma_1}{\beta_1}$, $I_1^1 = \frac{-\mu^2+\Lambda\beta_1-\mu\gamma_1}{\mu\beta_1}$, and C1 : $\frac{\beta_1 S^0}{\gamma_1+\mu} > 1$.
(iii) Infection (I_1)-free equilibrium (IFE) $E^2(S^2, 0, I_1^2, 0,)$, exists, if C2 holds, where $S^2 = \frac{\mu+\gamma_2}{\beta_2}$, $I_1^2 = \frac{-\mu^2+\Lambda\beta_2-\mu\gamma_2}{\mu\beta_2}$, and C2 : $\frac{\beta_2 S^0}{\gamma_2+\mu} > 1$.
(iv) Endemic Equilibrium (EE) E^*, when both infections are present the expression for endemic equilibrium cannot be obtained analytically. Thus we e will analyze this through numerical simulation.

4.2. Basic reproduction ratio/number

The basic reproduction ratio R_0, which mathematically characterizes the spread of infection is calculated below as defined in [8, 35].

$$F = \begin{pmatrix} \beta_1 S^0 & 0 & (1-p)\beta_3 S^0 \\ 0 & \beta_2 S^0 & p\beta_3 S^0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma_1+\mu & 0 & 0 \\ 0 & \gamma_2+\mu & 0 \\ 0 & 0 & \gamma_3+\mu+\delta \end{pmatrix}.$$

So R_0 is given by the matrix FV^{-1} spectral radius, i.e.,

$$R_0 = \max\left\{\frac{\beta_1 S^0}{\gamma_1+\mu} = R_{10}, \frac{\beta_2 S^0}{\gamma_2+\mu} = R_{20}\right\}.$$

Therefore it is clear from existing conditions C1 and C2 that the steady states E^1 and E^2 exist only when $R_{10} > 1$ and $R_{20} > 1$, respectively.

4.3. Stability criterion for disease-free equilibrium E^0

The local stability dynamic of the disease free equilibrium E^0 is derived as calculated in given lemmas of [17, 31, 32]. For a equation of the form (first degree transcendental)

$$\lambda + r_0 + q_0 e^{-\lambda\tau} = 0 \tag{4.1}$$

for the given cases:

(A1) $r_0 + q_0 > 0$;
(A2) $(r_0 + q_0)(r_0 - q_0) > 0$;
(A3) $(r_0 + q_0)(r_0 - q_0) < 0$.

Lemma 4.1. *For equation of the form (4.1):*

(i) *the equation (4.1) shows all the roots with negative real parts $\forall\ \tau \geqslant 0$, if (A1)-(A2) satisfy;*
(ii) *if cases (A1)-(A2) satisfy, then the equation (4.1) possesses all negative roots or having nagative real parts;*
(iii) *when $\tau = \tau_j^+$, then all the roots of equation (4.1) except purely imaginary $\pm i\omega_+$ have negative real parts, if (A1) and (A3) satisfy.*

Now, the stability condition for equation with second degree transcendental polynomial

$$\lambda^2 + p_1\lambda + r_1 + (s_1\lambda + q_1)e^{-\lambda\tau} = 0, \tag{4.2}$$

as

(B1) $p_1 + s_1 > 0$;
(B2) $q_1 + r_1 > 0$;
(B3) either $s_1^2 - p_1^2 + 2r_1 < 0$ and $r_1^2 - q_1^2 > 0$ or $(s_1^2 - p_1^2 + 2r_1)^2 < 4(r_1^2 - q_1^2)$;
(B4) either $r_1^2 - q_1^2 < 0$ or $s_1^2 - p_1^2 + 2r_1 > 0$ and $(s_1^2 - p_1^2 + 2r_1)^2 = 4(r_1^2 - q_1^2)$;
(B5) either $r_1^2 - q_1^2 > 0$ or $s_1^2 - p_1^2 + 2r_1 > 0$ and $(s_1^2 - p_1^2 + 2r_1)^2 > 4(r_1^2 - q_1^2)$.

Lemma 4.2. *For the equation (4.2):*

(i) *the equation (4.2) shows all the roots with negative real parts $\forall\ \tau \geqslant 0$, if (B1)-(B3) hold;*
(ii) *when $\tau = \tau_j^+$, then all the solutions of (4.2) except purely imaginary $\pm i\omega_+$ have negative real parts, if (B1),(B2), and (B4) hold;*
(iii) *all the solutions of equation (4.2) except purely imaginary roots $\pm i\omega_+ (\pm i\omega_-)$ have negative real parts for $\tau = \tau_j^+$ (respectively $\tau = \tau_j^-$), if (B1), (B2), and (B5) hold.*

Theorem 4.3. *For the system of equations (2.1)-(2.4), we have defined the stability condition of disease free equilibrium E^0 as:*

(i) *the infection free equilibrium E^0 is locally asymptotically stable for $\tau_1 = 0$, if $R_0 < 1$ holds;*
(ii) *the infection free equilibrium E^0 is locally asymptotically stable for all $\tau_1 > 0$, if $\mu > \gamma_3 - \delta$ and $R_0 < 1$ hold.*

Proof. The variational matrix at DFE E^0 is

$$\begin{pmatrix} -\mu & -S^0\beta_1 + \gamma_1 & -S^0\beta_2 + \gamma_2 & -S^0\beta_2 + \gamma_3 e^{-\lambda\tau} \\ 0 & -\mu + S^0\beta_1 - \gamma_1 & 0 & (1-p)S^0\beta_3 \\ 0 & 0 & -\mu + S^0\beta_1 - \gamma_1 & 0 \\ 0 & 0 & 0 & -\delta - \mu - e^{-\lambda\tau}\gamma_0 \end{pmatrix}$$

and its characteristics equation is given by

$$(\lambda + \mu)(\lambda + \mu - S^0\beta_1 + \gamma_1)(\lambda + \mu - S^0\beta_2 + \gamma_2)(\delta + \mu + \lambda + \gamma_3 e^{-\lambda\tau}) = 0, \tag{4.3}$$

So it is clear from characteristic equation (4.3) that three roots $-\mu$, $-(\gamma_1 + \mu - S^0\beta_1)$, and $-(\gamma_2 + \mu - S^0\beta_2)$ are negative if $R_0 < 1$ and remaining roots are given by equations

$$\delta + \mu + \lambda + \gamma_3 e^{-\lambda\tau} = 0. \tag{4.4}$$

Case (i). If $\tau = 0$, then the equation (4.4) becomes $(\delta + \mu + \lambda + \gamma_3) = 0$, which gives negative root. So therefore the DFE E^0 will be locally asymptotically stable if $R_0 < 1$.

Case (ii). If $\tau > 0$, then on comparing the equation (4.4) with the equation (4.1), we will get $r_0 = \mu + \delta$, $q_0 = \gamma_3$. Clearly both (A1) and (A2) hold simultaneously when $\mu > \gamma_3 - \delta$, which gives all negative roots for 4.4. So therefore the DFE E^0 will be locally asymptotically stable if $R_0 < 1$ and $\mu > \gamma_3 - \delta$ for all $\tau_1 > 0$. Otherwise E^0 will be quasi locally asymptotically stable if $R_0 < 1$ and $\mu < \gamma_3 - \delta$ for all $\tau_1 > 0$. □

4.4. Stability criterion for infection I_2-free equilibrium E^1

The variational matrix at equilibrium point E^1 is given as:

$$\begin{pmatrix} -\mu - I_1^1\beta_1 & -S^1\beta_1 + \gamma_1 & -S^1\beta_2 + \gamma_2 & -S^1\beta_3 + e^{-\lambda\tau_1}\gamma_3 \\ I_1^1\beta_1 & -\mu + S^1\beta_1 - \gamma_1 & -(1-q)I_1^1\beta_4 & (1-p)S^1\beta_3 \\ 0 & 0 & -\mu + S^1\beta_2 - qI_1^1\beta_4 - \gamma_2 & pS^1\beta_3 \\ 0 & 0 & I_1^1\beta_4 & -\delta - \mu - e^{-\lambda\tau_1}\gamma_3 \end{pmatrix}$$

and characteristic equation is

$$(\lambda + \mu)\left(\lambda + \mu + (I_1^1 - S^1)\beta_1 + \gamma_1\right)\left(\left(\lambda^2 + P_1\lambda + R_1\right) + \left(S_1\lambda + Q_1\right)e^{-\lambda\tau}\right) = 0, \tag{4.5}$$

where

$$P_1 = \left(-S^1\beta_2 + I_1^1\beta_4 q + 2\mu + \gamma_2 + \delta\right), \qquad R_1 = \left(-S^1\beta_2 + \mu + \gamma_2 + I_1^1\beta_4 q\right)(\delta + \mu) - I_1^1\beta_4 p S^1\beta_3,$$
$$S_1 = \gamma_3, \qquad Q_1 = \gamma_3\left(-S^1\beta_2 + \mu + \gamma_2 + I_1^1\beta_4 q\right),$$

then the roots of equation (4.5) are $-\mu, -(\mu + (I_1^1 - S^1)\beta_1 + \gamma_1)$ and the roots given by equation

$$\left(\lambda^2 + P_1\lambda + R_1\right) + (S_1\lambda + Q_1)\, e^{-\lambda\tau} = 0. \tag{4.6}$$

Case (i). If $\tau = 0$, then by Routh Hurwitz Criterion all roots of equation (4.6) will be negative if condition C3: $P_1 + S_1 > 0$ & $R_1 + Q_1 > 0$ holds. So therefore equilibrium point E^1 will be stable if conditions $R_{10} > 1$ and C3 hold.

Case (ii). If $\tau > 0$, then by Lemma 4.2 all roots of equation (4.6) will be negative if conditions (B1)-(B3) hold. So therefore equilibrium point E^1 will be stable if condition $R_{10} > 1$ together with (B1)-(B3) hold. Further, if conditions (B1), (B2), and (B3) hold, then by Lemma 4.2 equation (4.6) will have set of imaginary roots. Put $\lambda = iw$ in (4.6), then we have

$$(iw)^2 + P_1(iw) + R_1 + (S_1(iw) + Q_1)\, e^{-(iw)\tau} = 0,$$

on comparing real and imaginary parts on both sides, we get

$$-w^2 + R_1 + S_1 w \sin w\tau + Q_1 \cos w\tau = 0, \tag{4.7}$$
$$P_1 w + S_1 w \cos w\tau - Q_1 \sin w\tau = 0, \tag{4.8}$$

on simplifying equations (4.7) and (4.8), we get

$$\sin w\tau = \frac{S_1 w^3 + (P_1 Q_1 - R_1 S_1)w}{S_1^2 w^2 + Q_1^2},$$
$$\cos w\tau = \frac{(Q_1 - P_1 S_1)W^2 - Q_1 R_1}{S_1^2 w^2 + Q_1^2}, \tag{4.9}$$
$$w^4 + (P_1^2 - 2R_1 - S_1^2)w^2 + (R_1^2 - Q_1^2) = 0. \tag{4.10}$$

Since $R_1^2 - Q_1^2$ holds, hence by Descart's rule of sign, there exists at least one positive root of equation (4.10). Let this positive root be w_0, hence by equation (4.9), we have

$$\tau_j = \frac{1}{w_0}\left[\cos^{-1}\left(\frac{(Q_1 - P_1 S)w_0^2 - Q_1 R_1}{S_1^2 w_0^2 + Q_1^2}\right) + 2j\pi\right], \quad \text{where } j = 0, 1, 2, \ldots.$$

Now if transversality condition $Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right] \neq 0$ holds at τ_0^+, then hopf bifurcation will necessary occur at τ_0^+. Hence by differentiating λ with respect to τ in equation (4.6), we get

$$\frac{d\lambda}{d\tau} = \frac{\lambda(S_1\lambda + Q_1)e^{-\lambda\tau}}{2\lambda + P_1 + S_1 e^{-\lambda\tau} - (S_1\lambda + Q_1)\tau e^{-\lambda\tau}}.$$

Thus, at $\tau = \tau_0^+$ and $\lambda = iw_0$, we get

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{Q_1 J - S_1 w_0 K}{w_0(Q_1^2 + S_1^2 w_0^2)}, \tag{4.11}$$

here $J = P_1 \sin w_0\tau_0 + 2w_0 \cos w_0\tau_0$ and $K = S_1 + P_1 \cos w_0\tau_0 - 2w_0 \sin w_0\tau_0$. On simplifying equation (4.11), we have

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} \neq 0 \quad \text{if} \quad Q_1 J \neq S_1 w_0 K.$$

4.5. Stability criterion for infection I_1-free equilibrium E^2

The variational matrix at equilibrium point E_2 is given as:

$$\begin{pmatrix} -\mu - I_2^2\beta_2 & -S^2\beta_1 + \gamma_1 & -S^2\beta_2 + \gamma_2 & -S^2\beta_3 + e^{-\lambda\tau}\gamma_3 \\ 0 & -\mu + S^2\beta_1 - (1-q)I_2^2\beta_4 - \gamma_1 & 0 & (1-p)S^2\beta_3 \\ I_2^2\beta_2 & -qI_2^2\beta_4 & -\mu + S^2\beta_2 - \gamma_2 & pS^2\beta_3 \\ 0 & I_2^2\beta_4 & 0 & -\delta - \mu - e^{-\lambda\tau}\gamma_3 \end{pmatrix}$$

and characteristic equation is

$$(\lambda+\mu)\left(\lambda+\mu+\left(-S^2+I_2^2\right)\beta_2+\gamma_2\right)\left(\left(\lambda^2+P_2\lambda+R_2\right)+\left(S_2\lambda+Q_2\right)e^{-\lambda\tau}\right)=0, \tag{4.12}$$

where

$$\begin{aligned} P_2 &= \left(-S^2\beta_1+2\mu+\gamma_1+\delta+I_2^2\beta_4(1-q)\right), \\ R_2 &= \left(-S^2\beta_1+2\mu+\gamma_1+\delta+I_2^2\beta_4(1-q)\right)(\delta+\mu)-I_2^2\beta_4(1-p)S^2\beta_3, \\ S_2 &= \gamma_3, \qquad Q_2 = \gamma_3\left(-S^2\beta_1+\mu+\gamma_1+I_2^2\beta_4(1-q)\right), \end{aligned}$$

then the roots of equation (4.12) are $-\mu, -(\mu+(I_2^2-S^2)\beta_2+\gamma_2)$ and the roots given by equation

$$\left(\lambda^2+P_2\lambda+R_2\right)+\left(S_2\lambda+Q_2\right)e^{-\lambda\tau}=0. \tag{4.13}$$

Case(i): if $\tau = 0$, by Routh Hurwitz Criterion all roots will be negative if condition C4: $P_2+S_2 > 0$ & $R_2+Q_2>0$ hold. So therefore equilibrium point E_2 will be stable if conditions $R_{20}>1$ and C4 hold.
Case(ii): if $\tau>0$, by Lemma 4.2 all roots of equation (4.13) will be negative if conditions (B1)-(B3) hold. So therefore equilibrium point E_2 will be stable if conditions $R_{20}>1$ together with (B1)-(B3) hold. Further, if conditions (B1), (B2), and (B3) hold, then by Lemma 4.2 equation (4.13) will have set of imaginary roots. Put $\lambda = iw$ in 4.13, then we have

$$(iw)^2+P_2(iw)+R_2+(S_2(iw)+Q_2)e^{-(iw)\tau}=0,$$

on comparing real and imaginary parts on both sides, we get

$$-w^2+R_2+S_2w\sin w\tau+Q_2\cos w\tau=0, \tag{4.14}$$
$$P_2w+S_2w\cos w\tau-Q_2\sin w\tau=0, \tag{4.15}$$

on simplifying equations (4.14) and (4.15), we get

$$\begin{aligned} \sin w\tau &= \frac{S_2w^3+(P_2Q_2-R_2S_2)w}{S_2^2w^2+Q_2^2}, \\ \cos w\tau &= \frac{(Q_2-P_2S_2)W^2-Q_2R_2}{S_2^2w^2+Q_2^2}, \end{aligned} \tag{4.16}$$

and

$$w^4+(P_2^2-2R_2-S_2^2)w^2+(R_2^2-Q_2^2)=0. \tag{4.17}$$

Since $R_2^2-Q_2^2$ holds hence by Descart's rule of sign, there exist at least one positive root of equation (4.17). Let this positive root be w_0, hence by equation (4.16), we have

$$\tau_k=\frac{1}{w_0}\left[\cos^{-1}\left(\frac{(Q_2-P_2S)w_0^2-Q_2R_2}{S_2^2w_0^2+Q_2^2}\right)+2k\pi\right], \text{ where } k=0,1,2,\ldots.$$

Now if transversality condition $Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right] \neq 0$ holds at τ_0^+, then hopf bifurcation will necessary occur at τ_0^+. Hence by differentiating λ with respect to τ in equation (4.13), we get

$$\frac{d\lambda}{d\tau} = \frac{\lambda(S_2\lambda + Q_2)e^{-\lambda\tau}}{2\lambda + P_2 + S_2e^{-\lambda\tau} - (S_2\lambda + Q_2)\tau e^{-\lambda\tau}},$$

thus at $\tau = \tau_0^+$ and $\lambda = iw_0$, we get

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{Q_2L - S_1w_0M}{w_0(Q_2^2 + S_2^2w_0^2)}, \tag{4.18}$$

here $L = P_2 \sin w_0\tau_0 + 2w_0 \cos w_0\tau_0$ and $M = S_2 + P_2 \cos w_0\tau_0 - 2w_0 \sin w_0\tau_0$. On simplifying equation (4.18), we have

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1} \neq 0 \quad \text{if} \quad Q_2L \neq S_2w_0M.$$

4.6. Stability criterion of endemic equilibrium E^*

Endemic equilibrium explored numerically due to complexity in analytical expression.

5. Numerical simulation

The numerical simulations would be used to demonstrate previously defined effects for different sets of parameter values.

(a) When $\tau = 0$: for set of values of parameters $\Lambda = 0.10; \mu = 0.06; \beta_1 = 0.11; \beta_2 = 0.11; \beta_3 = 0.07; \beta_4 = 0.25; \gamma_1 = 0.10; \gamma_2 = 0.10; \gamma_3 = 0.012; \delta = 0.0005; p = 0.1; q = 0.3$, we obtain basic reproduction number $R_0 = 0.92 < 1$. The system (2.1)-(2.4) has an DFE E^0 (1.428, 0, 0, 0). Also, the local stability conditions $R_0 < 1$ and $\mu > \gamma_3 - \delta$ ae well satisfied. Therefore the DFE E^0 is locally asymptotically stable and presence of the delays does not affect the stability even choosing any positive value of delay parameters (see Figure 2).

(b) For set of values of parameters $\Lambda = 0.10; \mu = 0.06; \beta_1 = 0.33; \beta_2 = 0.10; \beta_3 = 0.07; \beta_4 = 0.25; \gamma_1 = 0.10; \gamma_2 = 0.11; \gamma_3 = 0.012; \delta = 0.0005; p = 0.1; q = 0.3$, we obtain basic reproduction number $R_{20} = 0.98 < 1$ and $R_{10} = 3.43$. The system (2.1)-(2.4) has an I_2-free equilibrium E^0 (0.48, 1.18, 0, 0). Also, the local stability conditions $R_{20} < 1$ and C5 are satisfied. Therefore the DFE E^0 is locally asymptotically stable and presence of the delays does not affect the stability even choosing any positive value of delay parameters (see Figure 3).

(c) For set of values of parameters $\Lambda = 0.10; \mu = 0.06; \beta_1 = 0.10; \beta_2 = 0.33; \beta_3 = 0.07; \beta_4 = 0.25; \gamma_1 = 0.11; \gamma_2 = 0.10; \gamma_3 = 0.012; \delta = 0.0005; p = 0.1; q = 0.3$, we obtain basic reproduction number $R_{10} = 0.98 < 1$ and $R_{20} = 3.43$. The system (2.1)-(2.4) has an I_1-free equilibrium E^0 (0.48, 0, 1.18, 0). Also, the local stability conditions $R_{10} < 1$ and C6 are satisfied. Therefore the DFE E^0 is locally asymptotically stable and presence of the delays does not affect the stability even choosing any positive value of delay parameters (see Figure 4).

(d) For set of values of parameters $\Lambda = 0.20; \mu = 0.06; \beta_1 = 0.15; \beta_2 = 0.43; \beta_3 = 0.70; \beta_4 = 0.30; \gamma_1 = 0.10; \gamma_2 = 0.10; \gamma_3 = 0.12; \delta = 0.0005; p = 0.3; q = 0.3$, we obtain basic reproduction number $R_0 = 8.95 > 1$. The system (2.1)-(2.4) has an Endemic equilibrium E^* (0.34, 0.40, 1.54, 1.03). Therefore the system along EE E^* is locally asymptotically stable if delay parameter has less value than critical threshold $\tau = 31.21$ and start bifurcating beyond the critical threshold and hopf bifurcation occurs. This phenomena is shown in the Figures 5 and 6.

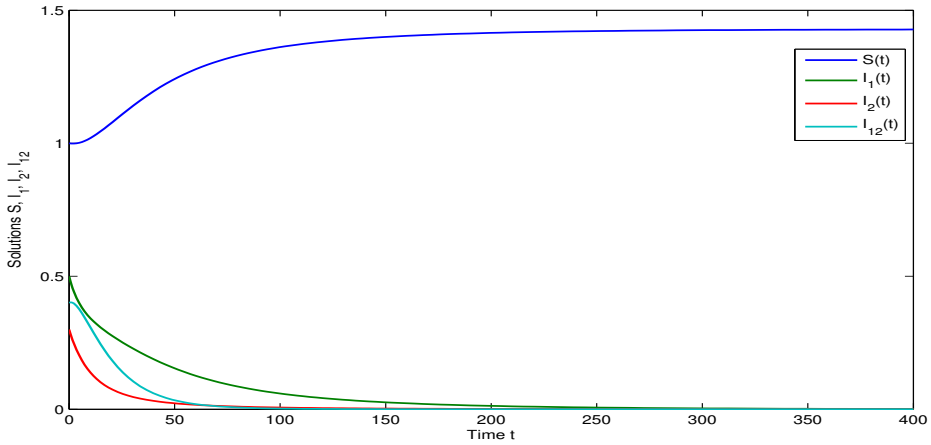

Figure 2: Population densities when $R_0 < 1$ and $\tau > 0$.

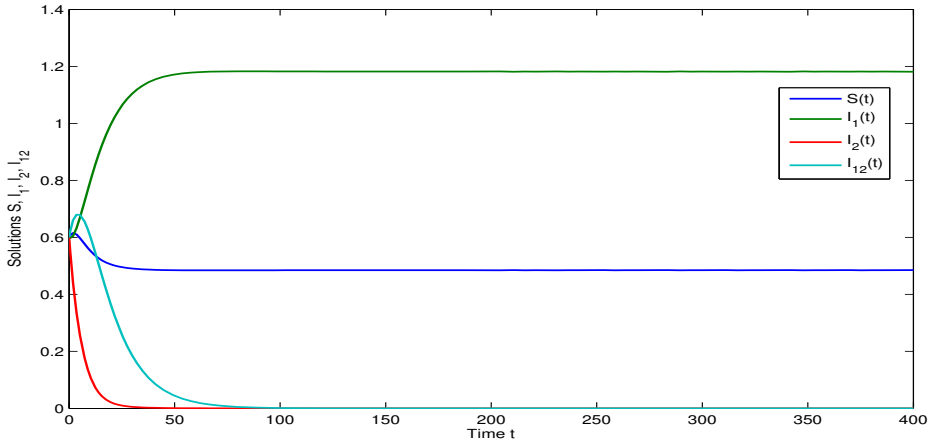

Figure 3: Population densities when $R_{10} > 1$ and $\tau > 0$.

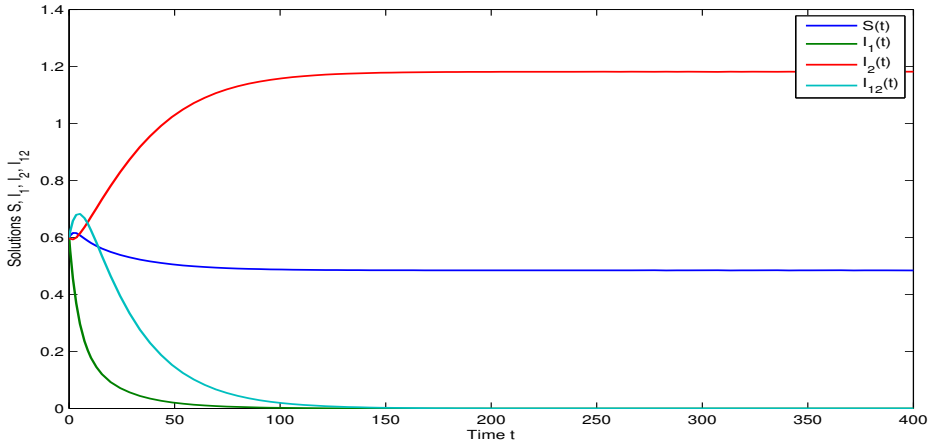

Figure 4: Population densities when $R_{20} > 1$ and $\tau > 0$.

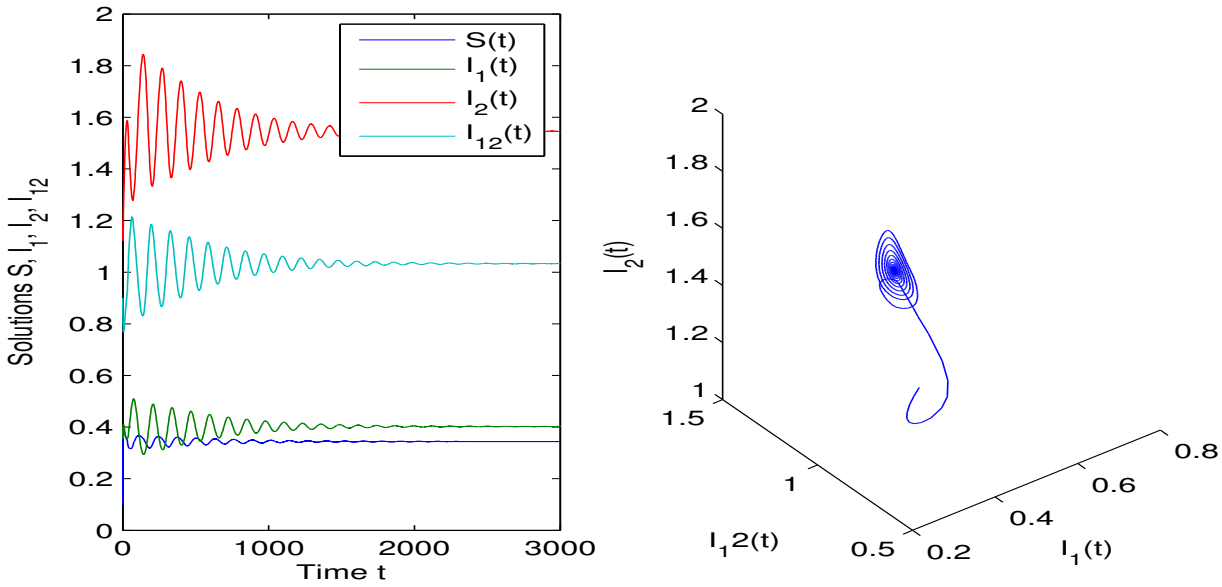

Figure 5: Population densities when $R_0 > 1$ and $\tau = 30.5 < 31.0$.

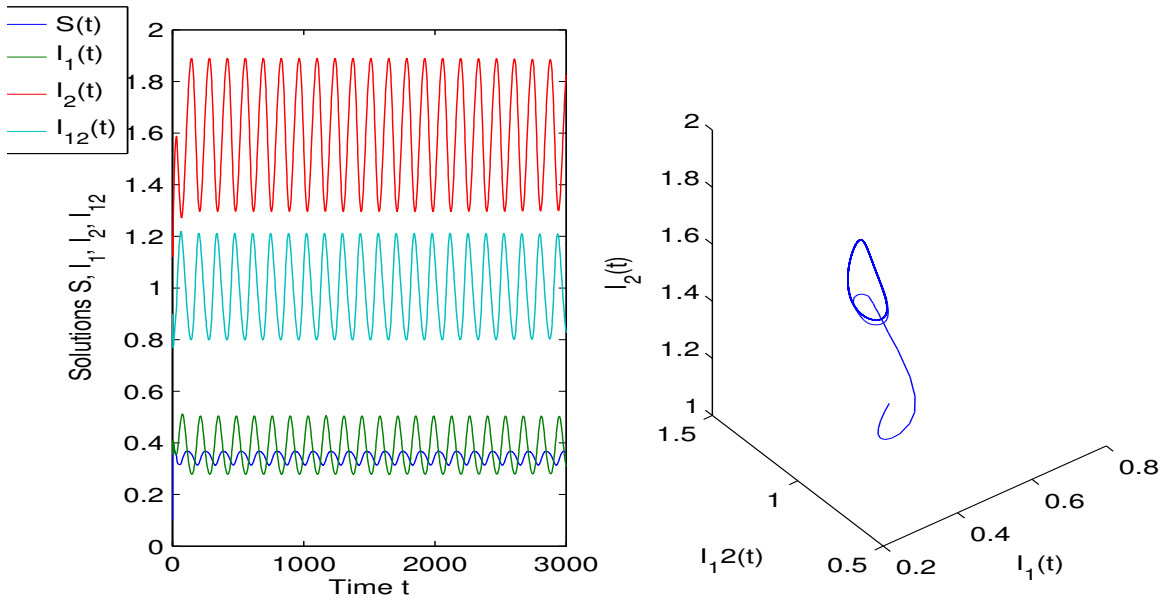

Figure 6: Population densities when $R_0 > 1$ and $\tau = 31.217 > 31.0$.

6. Conclusions

In this paper, we proposed a simple two short-term disease $S - I_1 - I_2 - I_{12} - S$ model with recovery delays featuring not simultaneous transmission of infection but recovery from duly infected individuals. In this model, there are four population classes susceptible to both infections S, susceptible to infection two but infectious of infection one I_1, susceptible to infection one but infectious of infection two I_2, and infectious of both infections I_{12}. We observed four biologically feasible states for the model: the disease-free, the infection-I_2-free, the infection-I_1-free, and the endemic steady state from the analysis. The basic reproduction ratio R_0 is determined for the system and observed that infection would die out if $R_0 < 1$

for the system without delay. But in the presence of delay, stability conditions do not remain sufficient and additional condition is required for stability which is defined in Theorem 4.3. The infection I_2-free equilibrium becomes stable if $R_{10} > 1$ with additional condition C3 for system without delay. The infection I_1-free equilibrium become stable if $R_{20} > 1$ with additional condition C4 for without delay system. Further, for both the boundary equilibria E^1, E^2 , Hopf bifurcation condition is derived for the system with delay but can not be satisfied for a practical set of values of parameters; hence does not show the bifurcation phenomenon in both the cases. Analytically endemic equilibrium expression is very lengthy, so the existence of the endemic state is verified numerically. Further, it is established that the coexistence state is stable only when $R_0 > 1$ and the delay parameter value is less than the critical threshold value $\tau < 31.217$ and becomes unstable beyond this value also shows hopf bifurcation at $\tau \geqslant 31.217$.

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