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On a system of (p, q)-analogues of the natural transform for solving (p, q)-differential equations

Sansumpan Jirakulchaiwong[a], Kamsing Nonlaopon[a,*], Jessada Tariboon[b], Sortiris K. Ntouyas[c], Shrideh Al-Omari[d]

[a]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand.

[b]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.

[c]Department of Mathematics, University of Ioannina, Ioannina 45110, Greece.

[d]Faculty of Engineering Technology, Al-Balqa Applied University, Amman 11134, Jordan.

Abstract

In this work, we apply the concept of (p, q)-calculus or post quantum calculus to establish the definitions of (p, q)-analogues of the natural transform of the first and second kind, which is a symmetric relation between (p, q)-analogues of the natural, Laplace, and Sumudu transforms. Moreover, as a result of the convolution theorem, some properties and some functions present in the table of (p, q)-analogues of the natural transform are discussed. Also, we apply them to solve higher order (p, q)-IVP with constants and coefficients, and to show the performance and effectiveness of the proposed transform.

Keywords: (p, q)-natural transforms, (p, q)-derivative, (p, q)-integral, (p, q)-calculus, (p, q)-difference equations, (p, q)-convolution theorem.

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1. Introduction

Integral transform techniques have been used for a long time as a significantly powerful tool to solve some problems in various fields, such as applied mathematics, physics, and engineering. They have also played a crucial part in a variety of theories and applications. Among numerous techniques, two of the most frequently used techniques are Laplace and Sumudu transforms, first introduced in 1780 and 1993, respectively. Following such introductions, in 2008, Khan [34] established a new integral transform, called the natural transform, and applied it to the unsteady flow over a plane wall.

In 2011, Silambarasan and Belgacem [44] derived certain electric field solutions of the natural transform to Maxwell's equation in conducting media and showed that the natural transform converges to Laplace and Sumudu transforms. In 2012, they studied some properties and applied this transform to Maxwell's equations describing transient electromagnetic [16]. In 2013, Al-Omari [8] applied the natural

*Corresponding author
Email address: nkamsi@kku.ac.th (Kamsing Nonlaopon)

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transform to some ordinary differential equations and some spaces of Boehmians. Recently, Al-Omari and Araci [12] investigated the definition and the properties of the generalized natural transform on sets of generalized functions.

Quantum calculus or q-calculus, is also known as calculus without limits. Euler was the first mathematician to study quantum calculus in the early eighteenth, which Gauss and Ramanujan later developed. In 1910, Jackson [27, 28] studied q-calculus in a symmetrical and introduced q-derivative and q-integral, which are known as Jackson derivative and Jackson integral. Quantum calculus has been used in the applications of diverse areas such as mathematics, applied mathematics, and physics; for instance, Fock [24] studied the symmetry operators of the hydrogen atoms using the q-difference equations. Many researchers have generalized and developed the q-calculus as found in [4, 5, 15, 21, 23, 30, 37, 45, 46] and their references. Also, the fundamental explanation of the q-calculus aspects can be found in the book by Kac and Cheung [32].

Around a decade ago, the topic of q-integral transform has piqued many researchers' interest, leading to various investigation forms. In 2011, Ganie and Jain [25] presented the q-Laplace transform of two variables and sought for application to figure out a generalized form of diffusion, wave, and space-time telegraphic equations. In 2013, Albayrak et al. [6] investigated q-analogues of Sumudu transform and derived specific properties. In 2014, Chung et al. [20] studied the q-analogues of the Laplace transform and pinpointed some distinct properties of the q-Laplcae transform to further the investigation. In 2018, Al-Omari [9] proposed the q-analogues of the natural transform on many functions of a special kind with the first kind and the second kind, and some of their respective properties. In 2020, he proposed the q-analogues and properties of the Laplace-type integral operator in the quantum calculus [10]. Recently, he presented the generalized q-theory of the q-Mellin transform and its specific properties in a set of q-generalized functions [11]; see [7, 13, 48] for more details.

The post quantum calculus or (p, q)-calculus is a generalized form of q-calculus. It was first studied in 1991 by Chakrabarti and Jagannathan [18]. It is pertinent that the direct substitution of q by q/p in q-calculus cannot provide valid quantum calculus; however, if $p = 1$ in (p, q)-calculus, it will reduce to q-calculus. In 2013, Sadjang [41] studied the concept of the (p, q)-derivative, the (p, q)-integration, (p, q)-Taylor formulas, and the fundamental theorem of (p, q)-calculus. The studies and developments of the (p, q)-calculus have been conducted many times as found in [1–3, 14, 17, 19, 22, 26, 33, 36, 38, 39] and their references.

A slew of extensive research about (p, q)-integral transforms can also be seen later. In 2017, Sadjang [40] studied the properties of (p, q)-analogues of the Laplace transform and attempted to apply those properties to solve certain (p, q)-difference equations. In 2019, Sadjang [42] studied the (p, q)-analogues of the Sumudu transform and gave some properties to solve (p, q)-difference equations. In 2020, Tassaddiq et al. [47] proposed (p, q)-analogues of Laplace and (p, q)-analogues of Sumudu transforms with (p, q)-Aleph function. Recently, Jirakulchaiwong et al. [31] established (p, q)-analogues of Laplace-type integral transforms and use some obtained properties to apply some (p, q)-differential equations.

We are fascinated by such inspirational literature as mentioned above, and therefore, propose to extend the q-analogues of the natural transform to (p, q)-analogues of the natural transform while giving a result of convolution theorem, some properties, and some functions that cover (p, q)-analogues of Laplace and Sumudu transform just by letting $u = 1$ and $v = 1$, respectively, and to apply those properties to solve higher order (p, q)-IVP with constants and coefficients.

The outline of this paper is as follows. Section 2 consists of some basic knowledge and notations that use in the following sections. Section 3 comprises of some properties of the (p, q)-analogues of the natural transform. Section 4 demonstrates the application to (p, q)-differential equations. Section 5 is about discussion, and the last section includes is the conclusion.

2. Preliminaries

For the convenience, we give some usual notations and definitions used in the (p,q)-calculus, which can be found in [36, 39–41]. Throughout this paper, let $0 < q < p \leqslant 1$ be constants.

The (p,q)-analogue or (p,q)-number of $n \in \mathbb{N}$ is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{2.1}$$

If $p = 1$ in (2.1), then (2.1) is the q-analogue of n or q-number.

The (p,q)-factorial is defined by

$$[n]_{p,q}! = \begin{cases} \prod_{j=1}^{n} [j]_{p,q} = [n]_{p,q}[n-1]_{p,q} \cdots [1]_{p,q}, & \text{for } n \geqslant 1, \\ 1, & \text{for } n = 0. \end{cases} \tag{2.2}$$

If $p = 1$ in (2.2), then (2.2) is q-factorial. The (p,q)-binomial coefficients are defined by

$$\binom{n}{j}_{p,q} = \frac{[n]_{p,q}!}{[j]_{p,q}![n-j]_{p,q}!} = \binom{n}{n-j}_{p,q} \tag{2.3}$$

for $0 \leqslant j \leqslant n$. If $p = 1$ in (2.3), then (2.3) reduces to the q-binomial coefficients. The (p,q)-analogue of derivative of a function $f : [0,\infty) \to \mathbb{R}$ is defined by

$$D_{p,q} f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \quad t \neq 0. \tag{2.4}$$

If $p = 1$ in (2.4), then $D_{p,q} f(t) = D_q f(t)$, which is the q-derivative of the function f, see [32]. Also, if $q \to 1$ in (2.4), then we get the classical derivative.

The (p,q)-derivatives of higher order are given by

$$\left(D^0_{p,q} f\right)(t) = f(t) \quad \text{and} \quad \left(D^k_{p,q} f\right)(t) = D_{p,q}\left(D^{k-1}_{p,q} f\right)(t), \quad k \in \mathbb{N}.$$

Example 2.1. Define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(t) = t^2 + 5t + c$ and $t \neq 0$, where c is a constant, then

$$\begin{aligned} D_{p,q}(t^2 + 5t + c) &= \frac{(p^2t^2 + 5pt + c) - (q^2t^2 + 5qt + c)}{(p-q)t} \\ &= \frac{(p^2 - q^2)t^2 + 5(p-q)t}{(p-q)t} = (p+q)t + 5. \end{aligned} \tag{2.5}$$

The (p,q)-derivatives of the product and quotient rules of functions f and g are as follows:

$$D_{p,q}(f(t)g(t)) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t), \tag{2.6}$$

$$D_{p,q}\left(\frac{f(t)}{g(t)}\right) = \frac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(t)}{g(pt)g(qt)}, \quad g(t) \neq 0. \tag{2.7}$$

If $p = 1$ in (2.6) and (2.7), then (2.6) and (2.7) reduce to q-derivative of the product and quotient rules of functions f and g, respectively.

The (p,q)-integral of f and (p,q)-integral of f on $[0,\infty)$ are defined by

$$\int f(t) d_{p,q} t = (p-q)t \sum_{j=0}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}} t\right), \tag{2.8}$$

$$\int_0^{\infty} f(t) d_{p,q} t = (p-q) \sum_{j=-\infty}^{\infty} \frac{q^j}{p^{j+1}} f\left(\frac{q^j}{p^{j+1}}\right). \tag{2.9}$$

If $p = 1$ in (2.9), then (2.9) reduces to the q-integral of the function f, see [32]. Also, if $q \to 1$ in (2.9), then we get the classical integral.

The (p, q)-integral in an interval $[a, b]$ is given by

$$\int_a^b D_{p,q} f(t) d_{p,q} t = f(b) - f(a). \tag{2.10}$$

The (p, q)-integration by parts is given by

$$\int_a^b f(pt)(D_{p,q} g(t)) d_{p,q} t = f(b)g(b) - f(a)g(a) - \int_a^b g(qt) D_{p,q} f(t) d_{p,q} t. \tag{2.11}$$

Note that $b = \infty$ is allowed. If $p = 1$ in (2.11), then (2.11) reduces to the q-integration by parts.

The two types of (p, q)-exponential functions are defined by

$$e_{p,q}(z) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}}{[n]_{p,q}!} z^n, \tag{2.12}$$

$$E_{p,q}(z) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{[n]_{p,q}!} z^n. \tag{2.13}$$

If $p = 1$ in (2.12) and (2.13), then we have the q-exponential function, see [32]. Also, if $q \to 1$, then (2.12) and (2.13) reduce to the classical exponential function.

Moreover, the derivative of the (p, q)-exponential functions is given by

$$\begin{aligned} D_{p,q} e_{p,q}(nt) &= n e_{p,q}(npt), \\ D_{p,q} E_{p,q}(nt) &= n E_{p,q}(nqt). \end{aligned} \tag{2.14}$$

Due to (2.12) and (2.13), the (p, q)-analogues of the trigonometric functions cosine and sine are as follows:

$$\cos_{p,q}(z) = \frac{e_{p,q}(iz) + e_{p,q}(-iz)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$

$$\mathrm{Cos}_{p,q}(z) = \frac{E_{p,q}(iz) + E_{p,q}(-iz)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n}{2}}}{[2n]_{p,q}!} z^{2n},$$

$$\sin_{p,q}(z) = \frac{e_{p,q}(iz) - e_{p,q}(-iz)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n p^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1},$$

$$\mathrm{Sin}_{p,q}(z) = \frac{E_{p,q}(iz) - E_{p,q}(-iz)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{2n+1}{2}}}{[2n+1]_{p,q}!} z^{2n+1}.$$

The (p, q)-integral of f on $[0, \infty)$ for $a \in \mathbb{R} \backslash \{0\}$ can be written as below:

$$\int_0^\infty f(at) d_{p,q} t = \frac{1}{a} \int_0^\infty f(t) d_{p,q} t. \tag{2.15}$$

Example 2.2. Define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(t) = e_{p,q}(-t)$, then by (2.14) and (2.10), we obtain

$$\int_0^\infty e_{p,q}(-5t) d_{p,q} t = -\frac{p}{5} \int_0^\infty D_{p,q} e_{p,q}\left(-\frac{5t}{p}\right) d_{p,q} t = -\frac{p}{5}(0 - 1) = \frac{p}{5},$$

and

$$\frac{1}{5} \int_0^\infty e_{p,q}(-t) d_{p,q} t = -\frac{p}{5} \int_0^\infty D_{p,q} e_{p,q}\left(-\frac{t}{p}\right) d_{p,q} t = -\frac{p}{5}(0 - 1) = \frac{p}{5}.$$

Therefore,

$$\int_0^\infty e_{p,q}(-5t) d_{p,q} t = \frac{1}{5} \int_0^\infty e_{p,q}(-t) d_{p,q} t.$$

The (p,q)-gamma function of the first kind is defined by

$$\Gamma_{p,q}(n) = p^{\frac{n(n-1)}{2}} \int_0^\infty t^{n-1} E_{p,q}(-qt) d_{p,q}t. \tag{2.16}$$

Using (2.16) and (2.11), we have

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!. \tag{2.17}$$

The (p,q)-gamma function of the second kind is defined by

$$\gamma_{p,q}(n) = q^{\frac{n(n-1)}{2}} \int_0^\infty t^{n-1} e_{p,q}(-pt) d_{p,q}t. \tag{2.18}$$

Using (2.18) and (2.11), we have

$$\gamma_{p,q}(n+1) = [n]_{p,q}!.$$

The (p,q)-beta function is defined by

$$B_{p,q}(s,t) = \int_0^1 x^{s-1}(1-qx)_{p,q}^{t-1} d_{p,q}x. \tag{2.19}$$

The relation between the (p,q)-gamma function and the (p,q)-beta function is

$$B_{p,q}(s,t) = p^{\frac{(t-1)(2s+t-2)}{2}} \frac{\Gamma_{p,q}(s)\Gamma_{p,q}(t)}{\Gamma_{p,q}(s+t)}, \tag{2.20}$$

see [36] for more details.

3. Some properties of the (p,q)-natural transform

The natural transform of a function $f(t)$ for $t, u, v, M, j_k > 0$, and $k \in \mathbb{N}$, was proposed by Khan [34], which is defined over the set A,

$$A = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < Me^{|t|/j_k}, t \in (-1)^k \times [0,\infty), k = 1,2 \right\},$$

which is given as

$$N(f(t); u, v) = \frac{1}{u} \int_0^\infty f(t) \exp\left(-\frac{vt}{u}\right) dt,$$

where the constants $u, v > 0$. The natural transform can be converted to Laplace and Sumudu transforms just by letting $u = 1$ and $v = 1$, respectively. Hence, the relation between the natural, Laplace, and Sumudu transforms is a symmetric relation.

Next, Al-Omari [9] extended to the q-natural transform which is defined over the sets B and C, respectively,

$$B = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < ME_q\left(\frac{|t|}{j_k}\right), t \in (-1)^k \times [0,\infty), k = 1,2 \right\},$$
$$C = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < Me_q\left(\frac{|t|}{j_k}\right), t \in (-1)^k \times [0,\infty), k = 1,2 \right\},$$

by the following formulas:

$$N_q(f(t); u, v) = \frac{1}{u} \int_0^\infty f(t) E_q\left(-\frac{qvt}{u}\right) d_qt \quad \text{and} \quad N^q(f(t); u, v) = \frac{1}{u} \int_0^\infty f(t) e_q\left(-\frac{vt}{u}\right) d_qt,$$

where the constants $u, v > 0$.

Now, we present the (p, q)-natural transform which covers (p, q)-Laplace and (p, q)-Sumudu transforms. We introduce $N^1_{p,q}$ and $N^2_{p,q}$ which are called $N_{p,q}$-transform of the first kind and second kind, respectively. Let

$$D = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < ME_{p,q}\left(\frac{|t|}{j_k}\right), t \in (-1)^k \times [0, \infty), k = 1, 2 \right\},$$

and

$$E = \left\{ f(t) \mid \exists M, j_1, j_2 > 0, |f(t)| < Me_{p,q}\left(\frac{|t|}{j_k}\right), t \in (-1)^k \times [0, \infty), k = 1, 2 \right\}.$$

Next, we define $N^1_{p,q}$ and $N^2_{p,q}$ in Definition 3.1.

Definition 3.1. The $N^1_{p,q}$ and $N^2_{p,q}$ of the function $f(t)$ of exponential order are defined over the set of functions D and E, respectively, as follows:

$$N^1_{p,q}(f(t); u, v) = \frac{1}{u}\int_0^\infty f(t)E_{p,q}\left(-\frac{qvt}{u}\right) d_{p,q}t, \tag{3.1}$$

and

$$N^2_{p,q}(f(t); u, v) = \frac{1}{u}\int_0^\infty f(t)e_{p,q}\left(-\frac{pvt}{u}\right) d_{p,q}t,$$

where the constants $u, v > 0$.

Now, we only show the properties of $N^1_{p,q}$ but $N^2_{p,q}$ omitted because $N^2_{p,q}$ can be showed in the same way.

Theorem 3.2 (Linearity)**.** *If $f, g \in D$, then the following formula holds:*

$$N^1_{p,q}(\alpha f(t) + \beta g(t); u, v) = \alpha N^1_{p,q}(f(t); u, v) + \beta N^1_{p,q}(g(t); u, v),$$

where α and β are constants.

Proof. Use the Definition 3.1, we have

$$\begin{aligned} N^1_{p,q}(\alpha f(t) + \beta g(t); u, v) &= \frac{1}{u}\int_0^\infty (\alpha f(t) + \beta g(t))E_{p,q}\left(-\frac{qvt}{u}\right) d_{p,q}t \\ &= \frac{\alpha}{u}\int_0^\infty f(t)E_{p,q}\left(-\frac{qvt}{u}\right) d_{p,q}t + \frac{\beta}{u}\int_0^\infty g(t)E_{p,q}\left(-\frac{qvt}{u}\right) d_{p,q}t \\ &= \alpha N^1_{p,q}(f(t); u, v) + \beta N^1_{p,q}(g(t); u, v). \end{aligned}$$

This completes the proof. □

Theorem 3.3 (Scaling)**.** *If $f \in D$, then the following formula holds:*

$$N^1_{p,q}(f(\xi t); u, v) = N^1_{p,q}(f(t); \xi u, v),$$

where ξ is a non-zero constant.

Proof. Using (3.1), we have

$$N^1_{p,q}(f(\xi t); u, v) = \frac{1}{u}\int_0^\infty f(\xi t)E_{p,q}\left(-\frac{qvt}{u}\right) d_{p,q}t.$$

Next, using (2.15), we get

$$N^1_{p,q}(f(\xi t); u, v) = \frac{1}{\xi u}\int_0^\infty f(t)E_{p,q}\left(-\frac{qvt}{\xi u}\right) d_{p,q}t = N^1_{p,q}(f(t); \xi u, v).$$

Therefore, the proof is completed. □

In Table 1, we provide the (p,q)-natural transforms of the first kind and second kind of some basic functions.

Table 1: Table of the (p,q)-natural transforms.

No.	Function $f(t)$	First kind $N^1_{p,q}(f(t);u,v)$	Second kind $N^2_{p,q}(f(t);u,v)$
1	1	$\frac{1}{v}$	$\frac{1}{v}$
2	t	$\frac{u}{pv^2}$	$\frac{u}{qv^2}$
3	t^α	$\frac{\Gamma_{p,q}(n+1)u^n}{v^{n+1}p^{\binom{n+1}{2}}}$, $\alpha > -1$	$\frac{\gamma_{p,q}(n+1)u^n}{v^{n+1}q^{\binom{n+1}{2}}}$, $\alpha > -1$
4	t^n	$\frac{[n]_{p,q}!u^n}{v^{n+1}p^{\binom{n+1}{2}}}$, $n \in \mathbb{N}$	$\frac{[n]_{p,q}!u^n}{v^{n+1}q^{\binom{n+1}{2}}}$, $n \in \mathbb{N}$
5	$e_{p,q}(at)$	$\frac{p}{vp-au}$, $u < \left\lvert\frac{vp}{a}\right\rvert$	$\frac{1}{v}\sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}\left(\frac{au}{v}\right)^n}{q^{\binom{n+1}{2}}}$
6	$E_{p,q}(at)$	$\frac{1}{v}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}\left(\frac{au}{v}\right)^n}{p^{\binom{n+1}{2}}}$	$\frac{q}{vq-au}$, $u < \left\lvert\frac{vq}{a}\right\rvert$
7	$\sin_{p,q}(at)$	$\frac{apu}{v^2p^2+a^2u^2}$	$\frac{au}{v^2}\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n+1}{2}}\left(\frac{au}{v}\right)^{2n}}{q^{\binom{2n+2}{2}}}$
8	$\cos_{p,q}(at)$	$\frac{p^2v}{v^2p^2+a^2u^2}$	$\frac{1}{v}\sum_{n=0}^{\infty}\frac{(-1)^n p^{\binom{2n}{2}}\left(\frac{au}{v}\right)^{2n}}{q^{\binom{2n+1}{2}}}$
9	$\sinh_{p,q}(at)$	$\frac{apu}{v^2p^2-a^2u^2}$, $u < \left\lvert\frac{vp}{a}\right\rvert$	$\frac{au}{v^2}\sum_{n=0}^{\infty}\frac{p^{\binom{2n+1}{2}}\left(\frac{au}{v}\right)^{2n}}{q^{\binom{2n+2}{2}}}$
10	$\cosh_{p,q}(at)$	$\frac{p^2v}{v^2p^2-a^2u^2}$, $u < \left\lvert\frac{vp}{a}\right\rvert$	$\frac{1}{v}\sum_{n=0}^{\infty}\frac{p^{\binom{2n}{2}}\left(\frac{au}{v}\right)^{2n}}{q^{\binom{2n+1}{2}}}$
11	$H(t-a)$	$\frac{1}{v}E_{p,q}\left(-\frac{av}{u}\right)$	$\frac{1}{v}e_{p,q}\left(-\frac{av}{u}\right)$
12	$\delta(t-a)$	$\frac{1}{u}E_{p,q}\left(-\frac{aqv}{u}\right)$	$\frac{1}{u}e_{p,q}\left(-\frac{aqv}{u}\right)$

Theorem 3.4 (Transforms of derivatives). *If* $f \in D$ *and* $D^n_{p,q}$ *has the* $N^1_{p,q}$ *of type one for each* $n \in \mathbb{N}$, *then the transforms of the first, second, and n-th derivatives of* f *can be written in the following forms:*

(i)

$$N^1_{p,q}(D_{p,q}f(t);u,v) = -\frac{f(0)}{u} + \frac{vN^1_{p,q}(f(t);pu,v)}{u};$$

(ii)

$$N^1_{p,q}(D^2_{p,q}f(t);u,v) = -\frac{f'(0)}{u} - \frac{vf(0)}{pu^2} + \frac{v^2N^1_{p,q}(f(t);p^2u,v)}{pu^2};$$

(iii)

$$N^1_{p,q}(D^n_{p,q}f(t);u,v) = \begin{cases} -\frac{f(0)}{u} + \frac{vN^1_{p,q}(f(t);pu,v)}{u}, & \text{for } n = 1, \\ \frac{v^nN^1_{p,q}(f(t);up^n,v)}{u^np^{\binom{n}{2}}} - \sum_{k=0}^{n-3}\frac{v^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2}+1}} & \\ -\sum_{k=n-2}^{n-1}\frac{v^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p^{n-k-1}}, & \text{for } n = 2,3,\ldots. \end{cases}$$

Proof.

(i). Using Definition 3.1 and (2.11) and (2.15), we have

$$N^1_{p,q}(D_{p,q}f(t);u,v) = \frac{1}{u}\int_0^\infty D_{p,q}f(t)E_{p,q}\left(-\frac{qvt}{u}\right)d_{p,q}t$$

$$= \frac{1}{u}\left[\lim_{a\to\infty}\left[f(t)E_{p,q}\left(-\frac{\nu t}{u}\right)\right]_0^a - \int_0^\infty f(pt)D_{p,q}E_{p,q}\left(-\frac{\nu t}{u}\right)d_{p,q}t\right]$$
$$= -\frac{f(0)}{u} + \frac{\nu}{u^2 p}\int_0^\infty f(t)E_{p,q}\left(-\frac{q\nu t}{pu}\right)d_{p,q}t = -\frac{f(0)}{u} + \frac{\nu}{u}N^1_{p,q}(f(t);pu,\nu).$$

Applying the result of (i), we get

$$N^1_{p,q}(D^2_{p,q}f(t);u,\nu) = \frac{1}{u}\int_0^\infty D^2_{p,q}f(t)E_{p,q}\left(-\frac{q\nu t}{u}\right)d_{p,q}t$$
$$= -\frac{f'(0)}{u} + \frac{\nu}{u}N^1_{p,q}(f'(t);pu,\nu) = -\frac{f'(0)}{u} - \frac{\nu f(0)}{pu^2} + \frac{\nu^2}{pu^2}N^1_{p,q}(f(t);p^2u,\nu),$$

which is (ii).

(iii). If $n = 1$, it is not difficult to see that

$$N^1_{p,q}(D^n_{p,q}f(t);u,\nu) = \frac{\nu^n N^1_{p,q}(f(t);up^n,\nu)}{u^n p^{-n} p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-1}\frac{\nu^{n-k-1}u^{-n+k}(D^k_{p,q}f(0))}{p^{n-k-1}}. \tag{3.2}$$

If $n > 1$, we apply (3.2) by changing p^{n-k-1} to $p^{\frac{(n-k-2)(n-k+1)}{2}+1}$ and putting $n = 2$. We can write

$$\begin{aligned}
&N^1_{p,q}(D^n_{p,q}f(t);u,\nu)\\
&= -\frac{(D^{n-1}_{p,q}f)(0)}{u} - \frac{\nu(D^{n-2}_{p,q}f)(0)}{pu^2} + \frac{\nu^2 N^1_{p,q}(D^{n-2}_{p,q}f(t);p^2u,\nu)}{pu^2}\\
&= -\frac{(D^{n-1}_{p,q}f)(0)}{u} - \frac{\nu(D^{n-2}_{p,q}f)(0)}{pu^2} + \frac{\nu^2}{pu^2}\left[\frac{\nu^{n-2}N^1_{p,q}(f(t);up^n,\nu)}{u^n p^{-n}p^{2n-1}p^{\binom{n-1}{2}}}\right.\\
&\quad\left. - \sum_{k=0}^{n-3}\frac{\nu^{n-k-3}(p^2u)^{-n+k+2}(D^k_{p,q}f)(0)}{p^{\frac{(n-k-4)(n-k-1)}{2}+1}}\right]\\
&= -\frac{(D^{n-1}_{p,q}f)(0)}{u} - \frac{\nu(D^{n-2}_{p,q}f)(0)}{pu^2} + \frac{\nu^n N^1_{p,q}(f(t);up^n,\nu)}{u^n p^{-n}p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-3}\frac{\nu^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p(p^{2n-2k-4})(p^{\frac{(n-k-4)(n-k-1)}{2}+1})}\\
&= -\frac{(D^{n-1}_{p,q}f)(0)}{u} - \frac{\nu(D^{n-2}_{p,q}f)(0)}{pu^2} + \frac{\nu^n N^1_{p,q}(f(t);up^n,\nu)}{u^n p^{-n}p^{\binom{n+1}{2}}} - \sum_{k=0}^{n-3}\frac{\nu^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p^{\frac{(n-k-2)(n-k-1)}{2}+1}}\\
&= \frac{\nu^n N^1_{p,q}(f(t);up^n,\nu)}{u^n p^{\binom{n}{2}}} - \sum_{k=0}^{n-3}\frac{\nu^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2}+1}} - \sum_{k=n-2}^{n-1}\frac{\nu^{n-k-1}u^{-n+k}(D^k_{p,q}f)(0)}{p^{n-k-1}}.
\end{aligned}$$

Therefore, the proof is completed. □

Theorem 3.5 (Transforms of integrals). *If* $f \in D$, *then the following Formulae hold:*

(i)
$$N^1_{p,q}\left(\int_0^t f(x)d_{p,q}x;pu,\nu\right) = \frac{u}{\nu}N^1_{p,q}(f(t);u,\nu);$$

(ii)
$$N^1_{p,q}\left(\int_0^t\int_0^x f(\tau)d_{p,q}\tau d_{p,q}x;p^2u,\nu\right) = \frac{pu^2}{\nu^2}N^1_{p,q}(f(t);u,\nu);$$

(iii)

$$N^1_{p,q}\left(\underbrace{\int_0^t\left(\int_0^{x_2}\cdots\left(\int_0^{x_n}f(x_1)d_{p,q}x_1\right)\cdots d_{p,q}x_{n-1}\right)d_{p,q}x_n}_{n\ times};p^n u,v\right)=\frac{p^{\binom{n}{2}}u^n}{v^n}N^1_{p,q}(f(t);u,v).$$

Proof. Using (2.11) and (2.15) to prove (i)-(iii), we have

$$N^1_{p,q}\left(\int_0^t f(x)d_{p,q}x;u\right)=\frac{1}{u}\int_0^\infty E_{p,q}\left(-\frac{qvt}{u}\right)\int_0^t f(x)d_{p,q}xd_{p,q}t.$$

We set $\Phi(t)=\int_0^t f(x)d_{p,q}x$ and applying the (p,q)-integration by part, we get

$$\int_0^\infty \Phi(pt)D_{p,q}E_{p,q}\left(-\frac{vt}{u}\right)d_{p,q}t=\left[\Phi(t)E_{p,q}\left(-\frac{vt}{u}\right)\right]_{t=0}^\infty-\int_0^\infty E_{p,q}\left(-\frac{qvt}{u}\right)D_{p,q}\Phi(t)d_{p,q}t.$$

Next, we obtain

$$\frac{v}{u}\int_0^\infty \Phi(pt)E_{p,q}\left(-\frac{qvt}{u}\right)d_{p,q}t=\int_0^\infty E_{p,q}\left(-\frac{qvt}{u}\right)f(t)d_{p,q}t.$$

Consequently,

$$N^1_{p,q}\left(\int_0^t f(x)d_{p,q}x;pu,v\right)=\frac{u}{v}N^1_{p,q}(f(t);u,v).$$

Let $\Phi(x)=\int_0^x f(\tau)d_{p,q}\tau$, then we get

$$\begin{aligned}N^1_{p,q}\left(\int_0^t\int_0^x f(\tau)d_{p,q}\tau d_{p,q}x;p^2u,v\right)&=N^1_{p,q}\left(\int_0^t\Phi(x)d_{p,q}x;p^2u,v\right)\\&=\frac{pu}{v}N^1_{p,q}(\Phi(t);pu,v)\\&=\frac{pu}{v}N^1_{p,q}\left(\int_0^t f(\tau)d_{p,q}\tau;pu,v\right)=\frac{pu^2}{v^2}N^1_{p,q}(f(t);u,v).\end{aligned}$$

Next, we set $\Phi_1(x)=\int_0^x\int_0^{x_1}f(\tau_1)d_{p,q}\tau_1d_{p,q}x_1$, then we obtain

$$N^1_{p,q}\left(\int_0^t\int_0^x\int_0^{x_1}f(\tau_1)d_{p,q}\tau_1d_{p,q}x_1d_{p,q}x;p^3u,v\right)=\frac{p^3u^3}{v^3}N^1_{p,q}(f(t);u,v).$$

Similarly, after continuing this process, we obtain the sequence

$$N^1_{p,q}\left(\underbrace{\int_0^t\left(\int_0^{x_n}\cdots\left(\int_0^{x_2}f(x_1)d_{p,q}x_1\right)\cdots d_{p,q}x_{n-1}\right)d_{p,q}x_n}_{n\text{ times}};p^n u,v\right)=\frac{p^{\binom{n}{2}}u^n}{v^n}N^1_{p,q}(f(t);u,v).$$

Therefore, the proof is completed. □

Theorem 3.6 (Convolution theorem)**.** *If $f_1,f_2\in D$ are defined by $f_1(t)=t^\delta$ and $f_2(t)=t^{\zeta-1}$ for $\delta\geqslant 0$ and $\zeta\geqslant 1$, then*

$$N^1_{p,q}((f_1*f_2)_{p,q};u,v)=up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}(t^\delta;u,v)N^1_{p,q}(t^{\zeta-1};u,v),$$

where

$$(f_1*f_2)_{p,q}(t)=\int_0^t f_1(\eta)f_2(t-q\eta)d_{p,q}\eta. \tag{3.3}$$

Proof. Using (3.3), we get

$$(f_1 * f_2)_{p,q}(t) = \int_0^t \eta^{\delta}(t-q\eta)^{\zeta-1} d_{p,q}\eta. \tag{3.4}$$

Then, we change the variables in (3.4) by $\eta = rt$ and use (2.19), which results in the following form:

$$(f_1 * f_2)_{p,q}(t) = t\int_0^1 r^{\delta}t^{\delta}(t-qrt)_{p,q}^{\zeta-1} d_{p,q}r = t^{\delta+\zeta}\int_0^1 r^{\delta}(1-qr)_{p,q}^{\zeta-1} d_{p,q}r = t^{\delta+\zeta}B_{p,q}(\delta+1,\zeta). \tag{3.5}$$

Thus, using (2.17) and (2.20) in (3.5), we get

$$\begin{aligned}
\frac{N^1_{p,q}((f*g)_{p,q};u,v)}{up^{(\zeta^2-3\beta-2\delta)/2}} &= \frac{B_{p,q}(\delta+1,\zeta)\int_0^{\infty} E_{p,q}\left(-\frac{qvt}{u}\right)t^{\delta+\zeta}d_{p,q}t}{u^2p^{(\zeta^2-3\zeta-2\delta)/2}} \\
&= \frac{p^{(\zeta-1)(2\delta+\zeta)/2}\Gamma_{p,q}(\delta+1)\Gamma_{p,q}(\zeta)u^{\delta+\zeta}[\delta+\zeta]_{p,q}!}{up^{(\zeta^2-3\zeta-2\delta)/2}p^{\binom{\delta+\zeta+1}{2}}\Gamma_{p,q}(\delta+\zeta+1)v^{\delta+\zeta+1}} \\
&= \frac{[\delta]_{p,q}![\zeta-1]_{p,q}!u^{\delta+\zeta-1}}{p^{\binom{\delta+1}{2}}p^{\binom{\beta}{2}}} = N^1_{p,q}(t^{\delta};u,v)N^1_{p,q}(t^{\zeta-1};u,v).
\end{aligned}$$

Hence, we obtain

$$N^1_{p,q}((f_1*f_2)_{p,q};u,v) = up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}(t^{\delta};u,v)N^1_{p,q}(t^{\zeta-1};u,v).$$

Therefore, the proof is completed. □

Theorem 3.7. *If $f_1, f_2 \in D$ are defined by $f_1(t) = \sum_{i=1}^{\infty} a_i t^{\delta_i}$ and $f_2(t) = t^{\zeta-1}$ for $\delta \geqslant 0$ and $\zeta \geqslant 1$, then we have*

$$N^1_{p,q}((f_1*f_2)_{p,q};u,v) = up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}(f_1;u,v)N^1_{p,q}(f_2;u,v).$$

Proof. By using Theorem 3.6, we obtain

$$\begin{aligned}
N^1_{p,q}((f_1*f_2)_{p,q};u,v) &= \sum_{i=1}^{\infty} a_i up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}((t^{\delta_i}*t^{\zeta-1})_{p,q};u,v) \\
&= \sum_{i=1}^{\infty} a_i up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}(t^{\delta};u,v)N^1_{p,q}(t^{\zeta-1};u,v) \\
&= up^{(\zeta^2-3\zeta-2\delta)/2}N^1_{p,q}(f_1;u,v)N^1_{p,q}(f_2;u,v).
\end{aligned}$$

Therefore, the proof is completed. □

4. Solving (p,q)-IVP by the (p,q)-natural transform

In this section, we apply the (p,q)-natural transform to solve high order (p,q)-IVP with constant and variable coefficients. Also, we apply it to obtain solutions.

4.1. *Solving (p,q)-IVP with constant coefficient*

Consider the (p,q)-IVP for $a_i \in \mathbb{R}, i = 1,2,\dots,n$ and $n \in \mathbb{N}$ of the form

$$D^n_{p,q}y(t) + a_1D^{n-1}_{p,q}y(pt) + \cdots + a_ny(p^nt) = g(t) \tag{4.1}$$

with the initial conditions

$$y(0) = y_0, D_{p,q}y(0) = y_1, \dots, D^{n-1}_{p,q}y(0) = y_{n-1}. \tag{4.2}$$

Now, we apply $N^1_{p,q}$ to both sides of (4.1) and use initial conditions (4.2). We divide it into two cases in the process including following.

Case 1 ($n = 1$): Using Theorems 3.4 (i) and 3.3, we get

$$\begin{aligned} N^1_{p,q}(D_{p,q}y(t) + a_1y(pt); u, v) &= N^1_{p,q}(g(t); u, v), \\ \frac{y(0)}{u} + \frac{v}{u}N^1_{p,q}(y(t); pu, v) + a_1N^1_{p,q}(y(pt); u, v) &= N^1_{p,q}(g(t); u, v), \\ \frac{y_0}{u} + \frac{v}{u}N^1_{p,q}(y(t); pu, v) + a_1N^1_{p,q}(y(t); pu, v) &= N^1_{p,q}(g(t); u, v), \\ -\frac{y_0}{u} + \left(\frac{v}{u} + a_1\right) N^1_{p,q}(y(t); pu, v) &= N^1_{p,q}(g(t); u, v). \end{aligned}$$

Then, we have

$$N^1_{p,q}(y(t); u, v) = \frac{\frac{u}{p}N^1_{p,q}(g(t); \frac{u}{p}, v)}{v + \frac{a_1u}{p}} + \frac{y_0}{v + \frac{a_1u}{p}}. \tag{4.3}$$

Finally, by applying the inverse transform $N^1_{p,q}$ on both sides of (4.3), we obtain the exact solution which is in the form:

$$y(t) = (N^1_{p,q})^{-1}\left[\frac{\frac{u}{p}N^1_{p,q}(g(t); \frac{u}{p}, v)}{v + \frac{a_1u}{p}}\right] + (N^1_{p,q})^{-1}\left[\frac{y_0}{v + \frac{a_1u}{p}}\right]. \tag{4.4}$$

Example 4.1. The (p, q)-Cauchy problem is in the following form:

$$D_{p,q}y(t) + a_1y(pt) = 0$$

with the initial condition $y(0) = 1$.

By using (4.4), the previous equation of (4.4) equals to zero. Since $g(t) = 0$, we obtain

$$y(t) = (N^1_{p,q})^{-1}\left[\frac{p}{vp + a_1u}\right].$$

Hence, the solution is as follows:

$$y(t) = e_{p,q}(-a_1t). \tag{4.5}$$

In addition, if $p = 1$ and $q \to 1$, then equation (4.5) reduces to $y(t) = \exp(-a_1t)$, which is a solution of $y'(t) + a_1y(t) = 0$ with the initial condition $y(0) = 1$, which appeared in [40].

Example 4.2. Consider the first order (p, q)-differential equation of the form

$$D_{p,q}y(t) + a_1y(pt) = t$$

with the initial condition $y(0) = 1$.

By using (4.4), we get

$$\begin{aligned} y(t) &= (N^1_{p,q})^{-1}\left[\frac{\frac{u}{p}N^1_{p,q}(t; \frac{u}{p}, v)}{v + \frac{a_1u}{p}}\right] + (N^1_{p,q})^{-1}\left[\frac{1}{v + \frac{a_1u}{p}}\right] \\ &= (N^1_{p,q})^{-1}\left[\frac{1}{a_1^2(bu + pv)} - \frac{1}{a_1^2pv} + \frac{u}{a_1p^2v^2}\right] + (N^1_{p,q})^{-1}\left[\frac{p}{vp + a_1u}\right] \\ &= \frac{t}{a_1p} - \frac{1}{a_1^2p} + \left(1 + \frac{1}{a_1^2}\right) e_{p,q}(-a_1t). \end{aligned} \tag{4.6}$$

In addition, if $p = 1$ and $q \to 1$, then (4.6) reduces to

$$y(t) = \frac{t}{a_1} - \frac{1}{a_1^2} + \left(1 + \frac{1}{a_1^2}\right) \exp(-a_1t),$$

which is a solution of $y'(t) + a_1y(t) = t$ with the initial condition $y(0) = 1$.

Case 2 ($n > 1$): Using Theorems 3.4 (iii) and 3.3, we have

$$\begin{aligned}&\mathbb{N}^1_{p,q}(D^n_{p,q}y(t) + a_1 D^{n-1}_{p,q}y(pt) + a_2 D^{n-2}_{p,q}y(p^2t) + \cdots + a_{n-1}D_{p,q}y(p^{n-1}t) + a_n y(p^n t); u, v)\\ &\quad = \mathbb{N}^1_{p,q}(g(t); u, v).\end{aligned}$$

Then, we have

$$\begin{aligned}&\left[\frac{v^n}{u^n p^{\binom{n}{2}}}\mathbb{N}^1_{p,q}(y(t); up^n, v) - \sum_{k=0}^{n-3}\frac{v^{n-k-1}u^{-n+k}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2}+1}} - \sum_{k=n-2}^{n-1}\frac{v^{n-k-1}u^{-n+k}(D^n_{p,q}y)(0)}{p^{n-k-1}}\right]\\
&+ a_1\left[\frac{v^{n-1}\mathbb{N}^1_{p,q}(y(pt); up^{n-1}, v)}{u^{n-1}p^{\binom{n-1}{2}}} - \sum_{k=0}^{n-4}\frac{v^{n-k-2}u^{-n+k+1}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-3)(n-k)}{2}+1}}\right.\\
&\left.-\sum_{k=n-3}^{n-2}\frac{v^{n-k-2}u^{-n+k+1}(D^n_{p,q}y)(0)}{p^{n-k-2}}\right] + a_2\left[\frac{v^{n-2}\mathbb{N}^1_{p,q}(y(p^2t); up^{n-2}, v)}{u^{n-2}p^{\binom{n-2}{2}}}\right.\\
&\left.-\sum_{k=0}^{n-5}\frac{v^{n-k-3}u^{-n+k+2}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-4)(n-k-1)}{2}+1}} - \sum_{k=n-5}^{n-4}\frac{v^{n-k-3}u^{-n+k+2}(D^n_{p,q}y)(0)}{p^{n-k-3}}\right]\\
&+\cdots + a_{n-1}\left[-\frac{y_0}{u} + \frac{v\mathbb{N}^1_{p,q}(y(p^{n-1}t); pu, v)}{u}\right] + a_n\mathbb{N}^1_{p,q}(y(p^n t); u, v) = \mathbb{N}^1_{p,q}(g(t); u, v).\end{aligned} \tag{4.7}$$

After simplifying (4.7), we get

$$\begin{aligned}\mathbb{N}^1_{p,q}(y(t); u, v) = &\frac{\mathbb{N}^1_{p,q}(g(t); \frac{u}{p^n}, v)}{\frac{v^n}{u^n p^{\binom{n}{2}-n^2}} + \frac{a_1 v^{n-1}}{u^{n-1}p^{\binom{n-1}{2}-n^2+n}} + \frac{a_2 v^{n-2}}{u^{n-2}p^{\binom{n-2}{2}-n^2+2n}} + \cdots + \frac{a_{n-1}v}{up^{-n}} + a_n}\\
&- \frac{G(\frac{u}{p^n}, v)}{\frac{v^n}{u^n p^{\binom{n}{2}-n^2}} + \frac{a_1 v^{n-1}}{u^{n-1}p^{\binom{n-1}{2}-n^2+n}} + \frac{a_2 v^{n-2}}{u^{n-2}p^{\binom{n-2}{2}-n^2+2n}} + \cdots + \frac{a_{n-1}v}{up^{-n}} + a_n},\end{aligned}$$

where

$$\begin{aligned}G(u, v) = &-\sum_{k=0}^{n-3}\frac{v^{n-k-1}u^{-n+k}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-2)(n-k+1)}{2}+1}} - \sum_{k=n-2}^{n-1}\frac{v^{n-k-1}u^{-n+k}(D^k_{p,q}y)(0)}{p^{n-k-1}}\\
&- a_1\sum_{k=0}^{n-4}\frac{v^{n-k-2}u^{-n+k+1}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-3)(n-k)}{2}+1}} - a_1\sum_{k=n-3}^{n-2}\frac{v^{n-k-2}u^{-n+k+1}(D^k_{p,q}y)(0)}{p^{n-k-2}}\\
&- a_2\sum_{k=0}^{n-5}\frac{v^{n-k-3}u^{-n+k+2}(D^k_{p,q}y)(0)}{p^{\frac{(n-k-4)(n-k-1)}{2}+1}} - a_2\sum_{k=n-5}^{n-4}\frac{v^{n-k-3}u^{-n+k+2}(D^k_{p,q}y)(0)}{p^{n-k-3}} - \cdots - \frac{a_{n-1}y(0)}{u}.\end{aligned}$$

Next, taking inverse transform $\mathbb{N}^1_{p,q}$ on both sides of above equation, we obtain the exact solution which is in the form:

$$\begin{aligned}&\mathbb{N}^1_{p,q}(y(t); u, v)\\
&\quad = (\mathbb{N}^1_{p,q})^{-1}\left[\frac{\mathbb{N}^1_{p,q}(g(t); \frac{u}{p^n}, v)}{\frac{v^n}{u^n p^{\binom{n}{2}-n^2}} + \frac{a_1 v^{n-1}}{u^{n-1}p^{\binom{n-1}{2}-n^2+n}} + \frac{a_2 v^{n-2}}{u^{n-2}p^{\binom{n-2}{2}-n^2+2n}} + \cdots + \frac{a_{n-1}v}{up^{-n}} + a_n}\right]\\
&\qquad - (\mathbb{N}^1_{p,q})^{-1}\left[\frac{G(\frac{u}{p^n}, v)}{\frac{v^n}{u^n p^{\binom{n}{2}-n^2}} + \frac{a_1 v^{n-1}}{u^{n-1}p^{\binom{n-1}{2}-n^2+n}} + \frac{a_2 v^{n-2}}{u^{n-2}p^{\binom{n-2}{2}-n^2+2n}} + \cdots + \frac{a_{n-1}v}{up^{-n}} + a_n}\right].\end{aligned} \tag{4.8}$$

Example 4.3. Consider the second order (p, q)-differential equation of the form

$$D^2_{p,q}y(t) - y(p^2t) = t$$

with the initial conditions $y(0) = D_{p,q}y(0) = 1$. Using equation (4.8), we have

$$\begin{aligned}
y(t) &= (N^1_{p,q})^{-1}\left[\frac{N^1_{p,q}(t; \frac{u}{p^2}, v)}{\frac{v^2}{u^2p^{-3}} - 1}\right] - (N^1_{p,q})^{-1}\left[\frac{-\frac{y'(0)}{up^{-2}} - \frac{vy(0)}{u^2p^{-3}}}{\frac{v^2}{u^2p^{-3}} - 1}\right] \\
&= (N^1_{p,q})^{-1}\left[\frac{u^3}{p^3v^2(v^2p^3 - u^2)} + \frac{up^2}{v^2p^3 - u^2} + \frac{vp^3}{v^2p^3 - u^2}\right] \\
&= (N^1_{p,q})^{-1}\left[\frac{u}{p^3v^2 - u^2} - \frac{u}{p^3v^2} + \frac{up^2}{v^2p^3 - u^2} + \frac{vp^3}{v^2p^3 - u^2}\right] \\
&= (N^1_{p,q})^{-1}\left[\frac{(1+p^2)u}{p^3v^2 - u^2} - \frac{u}{p^3v^2} + \frac{vp^3}{v^2p^3 - u^2}\right] \\
&= -\frac{t}{p^2} + \cosh_{p,q}\left(\frac{t}{\sqrt{p}}\right) + \left(\frac{1+p^2}{p^2}\right)\sqrt{p}\sinh_{p,q}\left(\frac{t}{\sqrt{p}}\right).
\end{aligned} \tag{4.9}$$

In addition, if $p = 1$ and $q \to 1$, then (4.9) reduces to

$$y(t) = -t + \cosh(t) + 2\sinh(t),$$

which is a solution of $y''(t) - y(t) = t$ with the initial conditions $y(0) = y'(0) = 1$, which appeared in [35].

Example 4.4. Consider the second order (p, q)-differential equation of the form

$$D^2_{p,q}y(t) - 4D_{p,q}y(pt) - 12y(p^2t) = 3\exp(5t)$$

with the initial conditions $y(0) = D_{p,q}y(0) = 0$. Using (4.8), we have

$$y(t) = (N^1_{p,q})^{-1}\left[\frac{N^1_{p,q}(3\exp(5t); \frac{u}{p^2}, v)}{\frac{v^2}{u^2p^{-3}} - \frac{4v}{up^{-2}} - 12}\right] = (N^1_{p,q})^{-1}\left[\frac{3p^3u^2}{(p^3v^2 - 4p^2vu - 12u^2)(p^3v - 5u)}\right]. \tag{4.10}$$

In addition, if $p = 1$ and $q \to 1$, then (4.10) reduces to

$$\begin{aligned}
y(t) &= N^{-1}\left[\frac{3u^2}{(v - 6u)(v + 2u)(v - 5u)}\right] \\
&= N^{-1}\left[-\frac{3}{7(v - 5u)} + \frac{3}{56(2u + v)} + \frac{3}{8(v - 6u)}\right] = -\frac{3\exp(5t)}{7} + \frac{3\exp(-2t)}{56} + \frac{3\exp(6t)}{8},
\end{aligned}$$

which is a solution of $y''(t) - 4y'(t) - 12y(t) = 3\exp(5t)$ with the initial conditions $y(0) = y'(0) = 0$, see [34] for the inverse natural transform.

Example 4.5. Consider the third order (p, q)-differential equation of the form

$$D^3_{p,q}y(t) + 2D^2_{p,q}y(pt) + 2D_{p,q}y(p^2t) + 3y(p^3t) = \sin(t) + \cos(t)$$

with the initial conditions $y(0) = D^2_{p,q}y(0) = 0$ and $D_{p,q}y(0) = 1$. Using (4.8), we have

$$y(t) = (N^1_{p,q})^{-1}\left[\frac{N^1_{p,q}(\sin(t); \frac{u}{p^3}, v) + N^1_{p,q}(\cos(t); \frac{u}{p^3}, v)}{\frac{v^3}{u^3p^{-6}} + \frac{2v^2}{u^2p^{-5}} + \frac{2v}{up^{-3}} + 3}\right]$$

$$-\left[-\frac{y''(0)}{up^{-3}}-\frac{vy'(0)}{u^2p^{-5}}-\frac{v^2y(0)}{u^3p^{-6}}-\frac{2y'(0)}{up^{-3}}-\frac{2vy(0)}{u^2p^{-5}}-\frac{2y(0)}{up^{-3}}\right] \tag{4.11}$$
$$=(N^1_{p,q})^{-1}\left[\frac{\frac{up^4}{v^2p^8+u^2}+\frac{vp^8}{v^2p^8+u^2}+\frac{vp^5}{u^2}+\frac{2p^3}{u}}{\frac{v^3p^6}{u^3}+\frac{2v^2p^5}{u^2}+\frac{2vp^3}{u}+3}\right]$$
$$=(N^1_{p,q})^{-1}\left[\frac{p^3u(p^{10}v^3+2p^8uv^2+(p^2+\frac{1}{p})p^3u^2v+(2+p)u^3)}{(p^8v^2+u^2)(p^6v^3+2p^5uv^2+2p^3u^2v+3u^3)}\right].$$

In addition, if $p=1$ and $q\to 1$, then equation (4.11) reduces to

$$y(t)=N^{-1}\left[\frac{u}{v^2+u^2}\right]=\sin(t),$$

which is a solution of $y'''(t)+2y''(t)+2y'(t)+3y(t)=\sin(t)+\cos(t)$ with the initial conditions $y(0)=y''(0)=0$ and $y'(0)=1$, which appeared in [29].

4.2. Solving (p,q)-IVP with variable coefficients

Consider the following (p,q)-IVP problems of the form

$$a_1(t^n)D^n_{p,q}y\left(\frac{t}{p^nq^n}\right)+a_2(t^{n-1})D^{n-1}_{p,q}y\left(\frac{t}{p^{n-1}q^{n-1}}\right)+\cdots+a_n(t)y(t)=g(t)$$

with the initial conditions

$$y(0)=y_0, D_{p,q}y(0)=y_1,\ldots,D^{n-1}_{p,q}y(0)=y_{n-1},$$

where $a_i(t^n)=b_it^n, b_i\in\mathbb{R}, i=1,2,\ldots,n$ and $n\in\mathbb{N}$.

Theorem 4.6 (Derivative of transforms)**.** *For $n\in\mathbb{N}$, the following formulas hold:*

(i)
$$N^1_{p,q}(tf(t);u,v)=-\frac{q}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)}\left(uN^1_{p,q}(f(t);qu,v)\right);$$

(ii)
$$N^1_{p,q}(t^2f(t);u,v)=\frac{q^3}{u}\frac{D^2_{p,q}}{D^2_{p,q}\left(\frac{v}{u}\right)}\left(uN^1_{p,q}(f(t);q^2u,v)\right);$$

(iii)
$$N^1_{p,q}(t^nf(t);u,v)=\frac{(-1)^nq^{\binom{n+1}{2}}}{u}\frac{D^n_{p,q}}{D^n_{p,q}\left(\frac{v}{u}\right)}\left(uN^1_{p,q}(f(t);q^nu,v)\right).$$

Proof. Using Definition (3.1) to prove (i), we have

$$N^1_{p,q}(f(t);qu,v)=\frac{1}{qu}\int_0^\infty f(t)E_{p,q}\left(-\frac{vt}{u}\right)d_{p,q}t. \tag{4.12}$$

Taking (p,q)-derivative to both sides with respect to $1/u$, we get

$$N^1_{p,q}(tf(t);u,v)=-\frac{q}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)}\left(uN^1_{p,q}(f(t);qu,v)\right).$$

From (4.12), taking the second (p,q)-derivative to both sides with respect to $1/u$ to prove (ii), we have

$$N^1_{p,q}(t^2f(t);u,v)=\frac{q^3}{u}\frac{D^2_{p,q}}{D^2_{p,q}\left(\frac{v}{u}\right)}\left(uN^1_{p,q}(f(t);q^2u,v)\right).$$

Following the same procedure, we can prove (iii). Therefore, the proof is completed. □

Theorem 4.7. *For* $n, m \in \mathbb{N}$, *the following formulas hold:*

(i)
$$N^1_{p,q}(t^n D_{p,q} f(t); u, v) = \frac{(-1)^n q^{\binom{n+1}{2}}}{u} \frac{D^n_{p,q}}{D^n_{p,q}\left(\frac{v}{u}\right)} \left(u N^1_{p,q}(D_{p,q} f(t); q^n u, v)\right);$$

(ii)
$$N^1_{p,q}(t^n D^2_{p,q} f(t); u, v) = \frac{(-1)^n q^{\binom{n+1}{2}}}{u} \frac{D^n_{p,q}}{D^n_{p,q}\left(\frac{v}{u}\right)} \left(u N^1_{p,q}(D^2_{p,q} f(t); q^n u, v)\right);$$

(iii)
$$N^1_{p,q}(t^n D^m_{p,q} f(t); u, v) = \frac{(-1)^n q^{\binom{n+1}{2}}}{u} \frac{D^n_{p,q}}{D^n_{p,q}\left(\frac{v}{u}\right)} \left(u N^1_{p,q}(D^m_{p,q} f(t); q^n u, v)\right).$$

Proof. The proof is similar to Theorem 4.6. □

Example 4.8. Consider the first order (p, q)-differential equation with variable coefficients of the form

$$b_1 t D_{p,q} y\left(\frac{t}{pq}\right) = b_2 t^2 \tag{4.13}$$

with the initial conditions $y(0) = 0$, where b_1 and b_2 are constants. Applying $N^1_{p,q}$ on both sides of (4.13), we obtain

$$b_1 N^1_{p,q}\left(t D_{p,q} y\left(\frac{t}{pq}\right); u, v\right) = b_2 N^1_{p,q}(t^2; u, v).$$

Using Theorems 3.4, 4.6, 4.7, and (2.15), we have

$$b_1\left[-\frac{q}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)} u N^1_{p,q}\left(D_{p,q} y\left(\frac{t}{pq}\right); qu, v\right)\right] = b_2 N^1_{p,q}(t^2; u, v),$$
$$b_1\left[-\frac{q}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)} u N^1_{p,q}\left(D_{p,q} y(t); \frac{u}{p}, v\right)\right] = b_2 N^1_{p,q}(t^2; u, v),$$
$$-\frac{b_1 q}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)} u\left[-\frac{y(0)}{up^{-1}} + \frac{v}{up^{-1}} N^1_{p,q}(y(t); u, v)\right] = b_2 N^1_{p,q}(t^2; u, v). \tag{4.14}$$

By substituting the initial conditions in (4.14), we have

$$-\frac{b_1 pq}{u}\frac{D_{p,q}}{D_{p,q}\left(\frac{v}{u}\right)}\left[v N^1_{p,q}(y(t); u, v)\right] = \frac{b_2(p+q)u^2}{v^3 p^3}. \tag{4.15}$$

Letting $u = 1$ in (4.15), which leads to (p, q)-Laplace transform of the first kind (see [40]), we get

$$-b_1 pq \frac{D_{p,q}}{D_{p,q} v}\left[v L_{p,q}(v)\right] = \frac{b_2(p+q)}{v^3 p^3}. \tag{4.16}$$

Now, using (2.8) in (4.16), we obtain

$$v L_{p,q}(v) = -\frac{b_2(p+q)}{b_1 p^4 q}\int v^{-3} d_{p,q} v = -\frac{b_2(p^2-q^2)v}{b_1 p^4 q}\sum_{j=0}^{\infty}\frac{q^j}{p^{j+1}} f\left(\frac{q^k}{p^{k+1}} v\right) = \frac{b_2 q}{b_1 p^2 v^2}.$$

We obtain the solution

$$y(t) = \frac{b_2 p t^2}{b_1(p+q)}. \tag{4.17}$$

In addition, if $b_1 = 1$, $b_2 = 1$, and $q \to 1$, then (4.17) reduces to $y(t) = \frac{t^2}{2}$, which is an exact solution of $t y'(t) = t^2$.

5. Discussion

We investigated the properties of the (p, q)-analogues of the natural transform that are necessary to solve the (p, q)-differential equations in Section 4, which was divided as follows.

(i) The solutions of (p, q)-IVP with constant coefficient in Subsection 4.1 consist of two cases: for $n = 1$ and $n > 1$. For $n = 1$, (4.4) was used in both Examples 4.1 and 4.2. In Example 4.1, if $p = 1$ and $q \to 1$, then the solution reduces to [40, Example 1]. For $n > 1$, (4.8) was used in Examples 4.3, 4.4, and 4.5 as follows: Example 4.3, if $p = 1$ and $q \to 1$, then the solution reduces to [35, Example 4], Example 4.5, the solution in the form of N^{-1} allows us to see that not every (p, q)-differential equation can be used to find the solution with direct properties, but if $p = 1$ and $q \to 1$, then the solution reduces to [29, Example 2].

(ii) The solution of (p, q)-IVP with variable coefficients in Subsection 4.2 consists of Example 4.8, which is used in Theorems 4.6 and 4.7. We obtained (4.15), and if $u = 1$ (if $v = 1$ in (4.15), then the equation format will become highly complexe and difficult to use), then the best choice is to apply the (p, q)-analogues of the Laplace transform. We get (4.17), but if $p = 1$ and $q \to 1$, then the solution reduces to classical solution.

(iii) The (p, q)-analogues of the natural transform is a generalization of the (p, q)-analogues of the Laplace and the Sumudu transforms.

6. Conclusion

In this work, we introduced the concept of natural transform by using (p, q)-calculus and established the definitions, the convolution theorem, and some properties of the (p, q)-analogues of the natural transform of the first and second kind. Then, the applications of the properties of (p, q)-analogues of the natural transform of the first kind to solve higher order (p, q)-IVP with constants and coefficients are conducted to show its performance and effectiveness of the proposed transform.

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