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Riccati technique for oscillation of half-linear/Emden-Fowler neutral dynamic equations

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Abstract

By using the Riccati technique, which reduces the higher order dynamic equations to a Riccati dynamic inequality, we will establish some new sufficient conditions for the oscillation of half-linear/Emden-Fowler neutral dynamic equation of the form

$$(r(\rho)((\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho)))^{\Delta})^{\gamma})^{\Delta}+q(\rho)\mathbf{x}^{\alpha}(\delta(\rho))+\nu(\rho)\mathbf{x}^{\beta}(\eta(\rho))=0,$$

on a time scale $\mathcal{T}$, where γ, α, and β are quotients of odd positive integers. An example with particular equation is constructed in consistent to the above equation and oscillation criteria are established for its solution.

Keywords: Oscillation, nonoscillation, half-linear/Emden-Fowler neutral dynamic equation, time scales.

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1. Introduction and background

The oscillation theory provides significant insights into the dynamics of solutions of problems modeled with equations in various areas of engineering and science. In recent years, the study of the oscillation theory of fractional order difference equation has become remarkably constructive, advancing rapidly and being the focus of research for many scientists; for instance, the reader can refer to [14, 20] for models where oscillation and/or delay actions may be formulated by means of cross-diffusion terms.

In the recent two decades, many authors have studied the oscillation of the second order nonlinear neutral delay dynamic equations on time scales and established several sufficient conditions for different

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types of equations. For completeness, we review some relevant works. In [23], Saker investigated the oscillation of second order neutral delay dynamic equations of Emden-Fowler type of the form

$$\left[r(\rho)\left(\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho))\right)^{\Delta}\right]^{\Delta}+q(\rho)\left|\mathbf{x}(\delta(\rho))\right|^{\gamma}\operatorname{sign}(\mathbf{x}(\delta(\rho)))=0,$$

on a time scale $\mathcal{T}$, where, $\gamma > 1$, r, p, q, τ, and δ are real-valued functions defined on $\mathcal{T}$ with $\tau(\rho) \leqslant \rho$, $\delta(\rho) \leqslant \rho$ for all $\rho \in \mathcal{T}$ and $\lim_{\rho\to\infty}\tau(\rho) = \lim_{\rho\to\infty}\delta(\rho) = \infty$, $\int_{\rho_0}^{\infty}\frac{1}{r(\rho)}^{\frac{1}{\gamma}}\Delta\rho = \infty$, $r^{\Delta}(\rho) \geqslant 0$ and $0 \leqslant p(\rho) < 1$. Further in [24] and under similar assumptions, the same author studied the oscillation of the neutral delay dynamic equation of the form

$$\left[r(\rho)\left([x(\rho)+p(\rho)x(\tau(\rho))]^{\Delta}\right)^{\gamma}\right]^{\Delta}+q(\rho)x^{\gamma}(\delta(\rho)))=0,$$

where $\gamma > 0$ is a quotient of odd positive integers. The results represented further improvements for those given for superlinear and sublinear neutral dynamic equations.

Thandapani and Piramanantham [28] considered the oscillation of second order nonlinear neutral dynamic equations on time scales of the form

$$\left[r(\rho)\left((x(\rho)+p(\rho)x(\rho-\tau))^{\Delta}\right)^{\gamma}\right]^{\Delta}+q(\rho)x^{\beta}(\rho-\delta)=0,\ \rho\in\mathcal{T},$$

where $\mathcal{T}$ is a time scale. They obtained their results under the conditions $\gamma \geqslant 1$ and $\beta > 0$ which are quotients of odd positive integers, τ, δ are fixed nonnegative constants, r, p, q are real valued positive *rd*-continuous functions defined on $\mathcal{T}$ such that $0 \leqslant p < 1$. The results are proved for the cases

$$\int_{\rho_0}^{\infty}\frac{1}{a(\rho)}^{\frac{1}{\gamma}}\Delta\rho=\infty \quad \text{and} \quad \int_{\rho_0}^{\infty}\frac{1}{a(\rho)}^{\frac{1}{\gamma}}\Delta\rho<\infty. \tag{1.1}$$

In [27], Sun et al. worked on the oscillation of a second order quasiliniear neutral delay dynamic equation on a time scales of the form

$$\left[r(\rho)\left(\left[\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho))\right]^{\Delta}\right)^{\gamma}\right]^{\Delta}+q_1(\rho)\mathbf{x}^{\alpha}(\tau_1(\rho))+q_2(\rho)\mathbf{x}^{\beta}(\tau_2(\rho))=0,$$

on a time scale $\mathcal{T}$, where α, β, γ are quotients of odd positive integers, r, p, q_1, q_2 are rd-continuous functions on $\mathcal{T}$ and r, q_1, q_2 are positive, $-1 < -p_0 \leqslant p(\rho) < 1$, $p_0 > 0$, the delay functions $\tau_i : \mathcal{T} \to \mathcal{T}$ satisfy $\tau_i(\rho) \leqslant \rho$ for $\rho \in \mathcal{T}$ and $\tau_i(\rho) \to \infty$ as $\rho \to \infty$, for $i = 1, 2$ and there exists a function $\tau : \mathcal{T} \to \mathcal{T}$ satisfying $\tau(\rho) \leqslant \tau_1(\rho)$, $\tau(\rho) \leqslant \tau_2(\rho)$, $\tau(\rho) \to \infty$ as $\rho \to \infty$. The main results are established by the help of Riccati transformation and under the cases (1.1).

On the other direction, the authors in [15] established some oscillation theorems for second order neutral delay dynamic equation on time scales of the form

$$\left[r(\rho)\left(\left[\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho))\right]^{\Delta}\right)^{\gamma}\right)^{\Delta}+q_1(\rho)\mathbf{x}^{a}(\tau_1(\rho))+q_2(\rho)\mathbf{x}^{\beta}(\tau_2(\rho))=0,$$

where γ, a, β are ratios of odd positive integers, r, p, q_1, q_2 are rd-continuous functions on $\mathcal{T}$ and r, q_1, q_2 are positive, and the delay functions τ_1 and τ_2 satisfy the same conditions in [21]. The main theorems are proved by comparison technique and under the case $\int_{\rho_0}^{\infty}\frac{1}{a(\rho)}^{\frac{1}{\gamma}}\Delta\rho = \infty$.

In the recent paper [26], Sethi considered the second order sublinear neutral delay dynamic equation of the form

$$\left[r(\rho)\left(\left[\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho))\right]^{\Delta}\right)^{\gamma}\right]^{\Delta}+q_1(\rho)\mathbf{x}^{\gamma}(\tau_1(\rho))+q_2(\rho)\mathbf{x}^{\gamma}(\tau_2(\rho))=0,$$

where $0 < \gamma \leqslant 1$ is a quotient of odd positive integers, p, q_1, q_2 are rd-continuous functions on $\mathcal{T}$ and the delay functions satisfy the usual assumptions. Both cases in (1.1) are considered and the results are proved by the aid of Riccati transformation.

In this paper, we are concerned with a certain class of the following half-linear/Emden-Fowler neutral delay equation

$$\left[r(\rho)\left((\mathbf{x}(\rho)+p(\rho)\mathbf{x}(\tau(\rho)))^{\Delta}\right)^{\gamma}\right]^{\Delta}+q_1(\rho)\mathbf{x}^{\alpha}(\delta(\rho))+q_2(\rho)\mathbf{x}^{\beta}(\eta(\rho))=0, \text{ for } \rho\in[\rho_0,\infty)_{\mathcal{T}}, \tag{1.2}$$

where γ, α, β are quotients of odd positive integers, $r \in C_{rd}([\rho_0,\infty)_{\mathcal{T}},(0,\infty))$ and p, $q_1, q_2 \in C_{rd}([\rho_0,\infty)_{\mathcal{T}},\mathbb{R}^+)$ with $0 \leqslant p(\rho) < 1$, and τ, δ, $\eta \in C_{rd}([\rho_0,\infty)_{\mathcal{T}},\mathbb{R}^+)$ and $\tau(\rho) \leqslant \rho$, $\delta(\rho) \leqslant \rho$, $\eta(\rho) \leqslant \rho$ with $\lim_{\rho\to\infty}\tau(\rho) = \lim\limits_{\rho\to\infty}\delta(\rho) = \lim_{\rho\to\infty}\eta(\rho) = \infty$. By a *solution* of (1.2), we mean a nontrivial real-valued function $\mathbf{x}(\rho) \in C^1_{rd}([T_{\mathbf{x}},\infty),\mathbb{R})$, $T_{\mathbf{x}} \geqslant \rho_0$, which has the properties that $r(\mathbf{x}^{\Delta})^{\gamma})^{\Delta} \in C^1_{rd}([T_{\mathbf{x}},\infty),\mathbb{R})$ such that $\mathbf{x}(\rho)$ satisfies (1.2) for all $[T_{\mathbf{x}},\infty)_{\mathcal{T}}$. The half-linear/Emden-Fowler equations have numerous applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous medium; for more details see for instance the papers [5–7].

The objective of this paper is to establish new sufficient conditions for the oscillation of equation (1.2) by employing the Riccati technique and applying some basic lemmas. Reported results are obtained under the condition

(H_0) $\int_{\rho_0}^{\infty} \frac{1}{a(\rho)}^{\frac{1}{\gamma}}\Delta\rho = \infty.$

We say that a solution $\mathbf{x}$ of (1.2) has a generalized zero at ρ if $\mathbf{x}(\rho) = 0$ and has a generalized zero in $(\rho,\sigma(\rho))$ in case $\mathbf{x}(\rho)\mathbf{x}^{\sigma}(\rho) < 0$ and the graininess function $\mu(\rho) := \sigma(\rho)-\rho > 0$. To investigate the oscillation properties of (1.2) it is proper to use some notions such as *conjugacy* and *disconjugacy* of the equation (1.2). Equation (1.2) is *disconjugate* on the interval $[\rho_0, b]_{\mathcal{T}}$, if there is no nontrivial solution of (1.2) with two (or more) generalized zeros in $[\rho_0, b]_{\mathcal{T}}$. Equation (1.2) is said to be *nonoscillatory* on $[\rho_0,\infty]_{\mathcal{T}}$ if there exists $c \in [\rho_0,\infty]_{\mathcal{T}}$ such that this equation is *disconjugate* on $[c, d]_{\mathcal{T}}$ for every $d > c$. Otherwise, (1.2) is said to be *oscillatory* on $[\rho_0,\infty]_{\mathcal{T}}$. A solution $\mathbf{x}(\rho)$ of (1.2) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is *oscillatory*. We say that (1.2) is right disfocal (left disfocal) on $[a, b]_{\mathcal{T}}$ if the solutions of (1.2) such that $\mathbf{x}^{\Delta}(a) = 0$ ($\mathbf{x}^{\Delta}(b) = 0$) have no generalized zeros in $[a, b]_{\mathcal{T}}$.

2. Main results

This section is devoted to the main oscillation results for equation (1.2) under the hypothesis (H_0). Throughout the paper, we use the notation

$$\mathbf{z}(\rho) = \mathbf{x}(\rho) + p(\rho)\mathbf{x}(\tau(\rho)). \tag{2.1}$$

Lemma 2.1 ([2]). *Assume that* (H_0) *holds and* $r(\rho) \in C^1_{rd}([(a,\infty),\mathbb{R}^+)$ *such that* $r^{\Delta}(\rho) \geqslant 0$. *Let* $\mathbf{x}(\rho)$ *be an eventually positive real valued function such that* $(r(\rho)(\mathbf{x}^{\Delta}(\rho))^{\gamma})^{\Delta} \leqslant 0$, *for* $\rho \geqslant \rho_1 > \rho_0$. *Then* $\mathbf{x}^{\Delta}(\rho) > 0$ *and* $\mathbf{x}^{\Delta\Delta}(\rho) < 0$ *for* $\rho \geqslant \rho_1 > \rho_0$.

Lemma 2.2. *Assume that Lemma 2.1 holds and let* $\tau(\rho)$ *be a positive continuous function such that* $\tau(\rho) \leqslant \rho$ *and* $\lim\limits_{\rho\to\infty}\tau(\rho) = \infty$. *Then there exists* $\rho_l > \rho_1$ *such that for each* $l \in (0, 1)$,

$$\frac{\mathbf{x}(\tau(\rho))}{\mathbf{x}(\delta(\rho))} \geqslant l\frac{\tau(\rho)}{\delta(\rho)}.$$

Proof. Indeed, for $\rho \geqslant \rho_1$,

$$u(\delta(\rho)) - u(\tau(\rho)) = \int_{\tau(\rho)}^{\delta(\rho)} u^{\Delta}(s)\Delta s \leqslant (\delta(\rho)-\tau(\rho))u^{\Delta}(\tau(\rho)),$$

which implies that

$$\frac{u(\delta(\rho))}{u(\tau(\rho))} \leqslant 1 + (\delta(\rho) - \tau(\rho))\frac{u^{\Delta}(\tau(\rho))}{u(\tau(\rho))}.$$

On the other hand, it follows that

$$u(\tau(\rho)) - u(\rho_1) = \int_{\rho_1}^{\tau(\rho)} u^{\Delta}(s)\Delta s \geqslant (u(\rho) - \rho_1)u^{\Delta}(\tau(\rho)).$$

That is $\forall\ l \in (0,1)$, $\exists$ a $\rho_l > \rho_1$ such that

$$l(\tau(\rho)) \leqslant \frac{u(\tau(\rho))}{u^{\Delta}(\tau(\rho))},\ \rho \geqslant \rho_l.$$

Consequently,

$$\frac{u(\delta(\rho))}{u(\tau(\rho))} \leqslant 1 + (\delta(\rho) - \tau(\rho))\frac{u^{\Delta}(\tau(\rho))}{u(\tau(\rho))} \leqslant \frac{\delta(\rho)}{l\tau(\rho)}.$$

The proof is complete. □

For convenience, we use the following notations:

$$a_1(\rho) := \int_{\rho}^{\infty} [q_1(s)(1 - p(\delta(s)))] \left(\frac{l\delta(s)}{\sigma(s)}\right)^{\alpha} \Delta s + \int_{\rho}^{\infty} [q_2(s)(1 - p(\delta(s)))] \left(\frac{l\delta(s)}{\delta(s)}\right)^{\alpha} \Delta s,$$

and

$$A_1(\rho, K_1) := \left[a_1(\rho) + K_1 \int_{\rho}^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}},\ \text{for}\ \rho \in [\rho_0, \infty)_{\mathcal{T}},$$

where $K_1 > 0$ is an arbitrary constant.

Theorem 2.3. *Assume that* (H_0) *holds and let* $0 \leqslant p(\rho) \leqslant a < 1$, $r^{\Delta}(\rho) > 0$ *and* $\gamma < \alpha < \beta$, $\eta(\rho) \geqslant \delta(\rho)$ *and* $\delta^{\Delta}(\rho) \geqslant 1$ *for* $\rho \in [\rho_0, \infty)_{\mathcal{T}}$. *If*

$(H_{1\alpha})$ $\limsup_{\rho\to\infty} a_1(\rho) < \infty$;

$(H_{1\beta})$ $\int_{\rho_0}^{\infty} (\frac{1}{r(s)})^{\frac{1}{\gamma}} A_1^{\sigma}(s, K_1)\Delta s = \infty$,

then every solution of (1.2) *oscillates on* $[\rho_0, \infty)_{\mathcal{T}}$.

Proof. Suppose the contrary that $\mathbf{x}(\rho)$ is a nonoscillatory solution of (1.2). Without loss of generality, we may assume that $\mathbf{x}(\rho) > 0$ for $\rho \geqslant \rho_0$. Hence there exists $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ such that $\mathbf{x}(\rho) > 0$, $\mathbf{x}(\tau(\rho)) > 0$, $\mathbf{x}(\delta(\rho)) > 0$ and $\mathbf{x}(\eta(\rho)) > 0$ for $\rho \geqslant \rho_1$. Using (2.1), we see that (1.2) becomes

$$(r(\rho)(\mathbf{z}^{\Delta}(\rho))^{\gamma})^{\Delta} = -q(\rho)\mathbf{x}^{\alpha}(\delta(\rho)) - \nu(\rho)\mathbf{x}^{\beta}(\eta(\rho)) \leqslant 0,\ \text{for}\ \rho \geqslant \rho_2. \tag{2.2}$$

So $r(\rho)(\mathbf{z}^{\Delta}(\rho))^{\gamma}$ is nonincreasing on $[\rho_1, \infty)_{\mathcal{T}}$, that is, either $\mathbf{z}^{\Delta}(\rho) > 0$ or $\mathbf{z}^{\Delta}(\rho) < 0$. By Lemma 2.1, it follows that $\mathbf{z}^{\Delta}(\rho) > 0$ for $\rho \geqslant \rho_2$. Hence there exists $\rho_3 > \rho_2$ such that

$$\begin{aligned} \mathbf{z}(\rho) - p(\rho)\mathbf{z}(\tau(\rho)) &= \mathbf{x}(\rho) + p(\rho)\mathbf{x}(\tau(\rho)) - p(\rho)\mathbf{x}(\tau(\rho)) - p(\rho)p(\tau(\rho))p(\tau(\tau(\rho))) \\ &= \mathbf{x}(\rho) - p(\rho)p(\tau(\rho))p(\tau(\tau(\rho))) \leqslant \mathbf{x}(\rho), \end{aligned}$$

which implies that

$$\mathbf{x}(\rho) \geqslant (1 - p(\rho))\mathbf{z}(\rho),\ \text{for}\ \ \rho \in [\rho_3, \infty)_{\mathcal{T}}.$$

Therefore (1.2) can be written as

$$(r(\rho)(\mathbf{z}^{\Delta}(\rho))^{\gamma})^{\Delta}+q(\rho)(1-p(\delta(\rho)))^{a}\mathbf{z}^{a}(\delta(\rho))+\nu(\rho)(1-p(\eta(\rho)))^{a}\mathbf{z}^{a}(\eta(\rho))\leqslant 0,$$

where $\gamma<a<\beta$. Define Riccati transformation

$$\mathbf{w}(\rho)=r(\rho)\frac{(\mathbf{z}^{\Delta}(\rho))^{\gamma}}{\mathbf{z}^{a}(\rho)},\quad \text{for}\quad \rho\in[\rho_3,\infty)_{\mathcal{T}}. \tag{2.3}$$

By using the product and quotient rules, we see that

$$\mathbf{w}^{\Delta}(\rho)=\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}-\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\sigma}(\mathbf{z}^{a})^{\Delta}}{\mathbf{z}^{a}(\mathbf{z}^{\sigma})^{a}},\quad \text{for } \rho\in[\rho_3,\infty)_{\mathcal{T}}. \tag{2.4}$$

Now, since $\eta(\rho)>\delta(\rho)$ and due to (2.3) and (2.4), we have

$$\mathbf{w}^{\Delta}(\rho)\leqslant -q(1-p^{\delta})^{a}-\nu(1-p^{\delta})^{a}\frac{(\mathbf{z}^{\delta})^{a}}{(\mathbf{z}^{\sigma})^{a}}-\frac{\mathbf{w}^{\sigma}(\mathbf{z}^{a})^{\Delta}}{\mathbf{z}^{a}},\ \text{for } \rho\in[\rho_3,\infty)_{\mathcal{T}}.$$

By using the chain rule [8], we get that

$$(\mathbf{z}^{a}(\rho))^{\Delta}=a\int_0^1[(1-h)\mathbf{z}(\rho)+hz(\sigma(\rho))]^{a-1}dhz^{\Delta}(\rho)\geqslant\begin{cases}a(\mathbf{z}(\rho))^{a-1}\mathbf{z}^{\Delta}(\rho),\ a>1,\\ a(\mathbf{z}(\sigma(\rho)))^{a-1}\mathbf{z}^{\Delta}(\rho),\ 0<a\leqslant 1.\end{cases}$$

Since $\mathbf{z}(\rho)$ is a nondecreasing function on $[\rho_3,\infty)_{\mathcal{T}}$, then for $\rho\geqslant\rho_3$,

$$\frac{(\mathbf{z}^{a}(\rho))^{\Delta}}{\mathbf{z}^{a}(\rho)}\geqslant\begin{cases}a\frac{\mathbf{z}^{\Delta}(\rho)}{\mathbf{z}(\rho)}, & \text{for } a>1,\\ a\frac{(\mathbf{z}(\sigma(\rho)))^{a-1}}{\mathbf{z}^{a}(\rho)}\mathbf{z}^{\Delta}(\rho), & \text{for } 0<a\leqslant 1.\end{cases}$$

Using the fact that $\rho\leqslant\sigma(\rho)$, we have

$$\frac{(\mathbf{z}^{a})^{\Delta}}{\mathbf{z}^{a}}\geqslant a\frac{\mathbf{z}^{\Delta}}{\mathbf{z}^{\sigma}},\ a>0\quad\text{on}\quad[\rho_3,\infty)_{\mathcal{T}}.$$

Therefore (2.4) yields that

$$\mathbf{w}^{\Delta}\leqslant -q(1-p^{\delta})^{a}-\nu(1-p^{\delta})^{a}\frac{(\mathbf{z}^{\sigma})^{a}}{(\mathbf{z}^{\delta})^{a}}-a\mathbf{w}^{\sigma}\frac{\mathbf{z}^{\Delta}}{\mathbf{z}^{\sigma}},\ \rho\geqslant\rho_3. \tag{2.5}$$

Now, since $\left(r^{\frac{1}{\gamma}}\mathbf{z}^{\Delta}\right)$ is nonincreasing on $[\rho_3,\infty)_{\mathcal{T}}$, then for $\rho\leqslant\sigma(\rho)$, we have that

$$\mathbf{z}^{\Delta}\geqslant r^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{\frac{1}{\gamma}}(\mathbf{z}^{\sigma})^{\frac{a}{\gamma}},\ \rho\geqslant\rho_3. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$\mathbf{w}^{\Delta}\leqslant -q(1-p^{\delta})^{a}\frac{(\mathbf{z}^{\delta})^{a}}{(\mathbf{z}^{\sigma})^{a}}-\nu(1-p^{\delta})^{a}\frac{(\mathbf{z}^{\delta})^{a}}{(\mathbf{z}^{\sigma})^{a}}-ar^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}}(\mathbf{z}^{\sigma})\frac{a}{\gamma}-1,\ \rho\geqslant\rho_3.$$

Since $\mathbf{z}(\rho)$ is nondecreasing on $[\rho_3,\infty)_{\mathcal{T}}$, then there exist $\rho_4>\rho_3$ and $C>0$ such that

$$(\mathbf{z}(\sigma(\rho)))^{\frac{a}{\gamma}-1}\geqslant(\mathbf{z}(\rho))^{\frac{a}{\gamma}-1}\geqslant C,\quad\text{for}\quad\rho\geqslant\rho_4.$$

By using Lemma 2.2, it follows from the last inequality that

$$\mathbf{w}^{\Delta}(\rho)\leqslant -q(1-p(\delta(\rho)))^{a}\left(\frac{l\delta(\rho)}{\sigma(\rho)}\right)^{a}-\nu(1-p(\delta(\rho)))^{a}\left(\frac{l\delta(\rho)}{\sigma(\rho)}\right)^{a}-aCr^{-\frac{1}{\gamma}}(\rho)(\mathbf{w}^{\sigma}(\rho))^{1+\frac{1}{\gamma}},\ \rho\geqslant\rho_l>\rho_4.$$

Integrating the above inequality from ρ to u $(\rho < u)$ for $\rho, u \in [\rho_4, \infty)_{\mathcal{T}}$, we obtain

$$-\mathbf{w}(\rho) \leqslant \mathbf{w}(u) - \mathbf{w}(\rho) \leqslant -\int_{\rho}^{u} \left[q(1-p^{\delta})^{\alpha} \left(\frac{l\delta(\rho)}{\sigma(\rho)} \right)^{\alpha} + \nu(1-p^{\delta})^{\alpha} \left(\frac{l\delta(\rho)}{\sigma(\rho)} \right)^{\alpha} + \alpha C r^{-\frac{1}{\gamma}}(\rho)(\mathbf{w}^{\sigma}(\rho))^{1+\frac{1}{\gamma}} \right] \Delta s,$$

that is,

$$\mathbf{w}(\rho) \geqslant a_1(\rho) + K_1 \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)\mathbf{w}(\sigma(s))^{1+\frac{1}{\gamma}} \Delta s, \ \rho \geqslant \rho_1,$$

where $K_{1\alpha} = C\alpha$. Indeed, $\mathbf{w}(\rho) > a_1(\rho)$ implies that

$$\mathbf{w}(\rho) \geqslant a_1(\rho) + K_1 \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(\sigma(s)))^{1+\frac{1}{\gamma}} \Delta s = A_1^{\gamma}(\rho, K_1).$$

Since $\rho \leqslant \sigma(\rho)$, we see

$$r(\mathbf{z}^{\Delta})^{\gamma} \geqslant (r(\mathbf{z}^{\Delta})^{\gamma})^{\sigma},$$

which implies that

$$\frac{r(\mathbf{z}^{\Delta})^{\gamma}}{(\mathbf{z}^{\sigma})^{\alpha}} \geqslant \frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\sigma}}{(\mathbf{z}^{\sigma})^{\alpha}} = \mathbf{w}^{\sigma} \geqslant (A_1^{\gamma}(\rho, k_1))^{\sigma},$$

that is,

$$(\mathbf{z}^{\sigma})^{\delta}\mathbf{z}^{\Delta} \geqslant r^{-\frac{1}{\gamma}}(A_1^{\sigma}(\rho, k_1)), \ \rho \in [\rho_5, \infty]_{\mathcal{T}},$$

where $\delta = (\frac{\alpha}{\gamma}) > 1$. Using the chain rule, we have

$$(\mathbf{z}^{1-\delta}(\rho))^{\Delta} = (1-\delta)\int_0^1 [(1-h)\mathbf{z}(\rho) + hz(\sigma(\rho))]^{\delta} dh z^{\Delta}(\rho) \leqslant (1-\delta)(\mathbf{z}(\sigma(\rho)))^{-\delta}\mathbf{z}^{\Delta}(\rho),$$

that is,

$$\frac{(\mathbf{z}^{1-\delta}(\sigma(\rho)))^{\Delta}}{1-\delta} \geqslant \mathbf{z}(\sigma(\rho))^{-\delta}\mathbf{z}^{\Delta}(\sigma(\rho)).$$

Hence

$$\frac{(\mathbf{z}^{1-\delta}(\rho))^{\Delta}}{1-\delta} \geqslant (\mathbf{z}(\sigma(\rho)))^{-\delta}\mathbf{z}^{\Delta}(\rho),$$

and then due to (2.6), we see that

$$\frac{(\mathbf{z}^{1-\delta}(\rho))^{\Delta}}{1-\delta} \geqslant r^{-\frac{1}{\gamma}}(\rho)(A_1^{\sigma}(\rho, k_1)), \ \rho \in [\rho_5, \infty)_{\mathcal{T}}.$$

Integrating the above inequality from ρ_5 to ρ, we get

$$\int_{\rho_5}^{\rho} r(s)^{-\frac{1}{\gamma}} (A_1^{\sigma}(s, K_1))^{\frac{1}{\gamma}} \Delta s < \infty,$$

which contradicts $(H_1\beta)$. □

Theorem 2.4. *Let* $0 \leqslant p(\rho) \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ *for* $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ *and* $\gamma = \alpha = \beta$, $\eta(\rho) \geqslant \sigma(\rho)$ *and assume that* (H_0) *and* $(H_{1\alpha})$ *hold. Furthermore, assume that*

(H_2) $\limsup_{\rho\to\infty} \left(\int_{\rho_0}^{\rho} r^{-\frac{1}{\gamma}}(s)A_1(s, K_1)\Delta s \right) > 1.$

Then every solution of (1.2) *oscillates.*

Proof. Proceeding as in the proof of Theorem 2.3, we get

$$\mathbf{w}(\rho) \geqslant A_1^{\gamma}(\rho, K_1) \text{ for } \rho \in [\rho_4, \infty)_{\mathbb{T}}.$$

Using the fact that $r^{\frac{1}{\gamma}}\mathbf{z}^{\Delta}$ is nonincreasing on $[\rho_4, \infty)_{\mathcal{T}}$, we get

$$\mathbf{z}(\rho) = \mathbf{z}(\rho_4) + \int_{\rho_4}^{\rho} \mathbf{z}^{\Delta}(s)\Delta s = \mathbf{z}(\rho_4) + \int_{\rho_4}^{\rho} r^{-\frac{1}{\gamma}}(s)\Big(r(s)^{-\frac{1}{\gamma}}\mathbf{z}^{\Delta}(s)\Big)\Delta s \geqslant r^{\frac{1}{\gamma}}(\rho)\mathbf{z}^{\Delta}(\rho)r^{-\frac{1}{\gamma}}(s)\Delta s,$$

that is,

$$\frac{r(\rho)^{\frac{1}{\gamma}}\mathbf{z}^{\Delta}(\rho)}{\mathbf{z}(\rho)} \leqslant \Big(\int_{\rho_4}^{\rho} r(s)^{-\frac{1}{\gamma}}\Delta s\Big)^{-1}, \ \rho \geqslant \rho_4. \tag{2.7}$$

Consequently,

$$A_1(\rho, K_1) \leqslant \mathbf{w}^{\frac{1}{\gamma}}(\rho) = \frac{r(\rho)^{\frac{1}{\gamma}}\mathbf{z}'(\rho)}{\mathbf{z}(\rho} \leqslant \Big(\int_{\rho_2}^{\rho} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1},$$

which implies that

$$\Big(\int_{\rho_4}^{\rho} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)A_1(\rho, K_1) \leqslant 1,$$

which contradicts (H_2). Hence the theorem is proved. □

Theorem 2.5. *Let* $0 \leqslant p(\rho) \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ *for* $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ *and* $\gamma > \alpha > \beta$, $\eta(\rho) \geqslant \sigma(\rho)$ *and assume that* (H_0) *and* $(H_{1\beta})$ *hold. Furthermore, assume that*

(H_3) $\limsup_{\rho\to\infty}(a_1(\rho))^{\frac{(\gamma-\alpha)}{\alpha\gamma}}\left(\int_{\rho_0}^{\rho} r^{-\frac{1}{\gamma}}(s)\Delta s\right)\left[a_1(\rho) + K_1\int_{\rho}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.2) *oscillates.*

Proof. Following similar steps as in the proof of Theorem 2.3, we obtain (2.2) and (2.3) and hence $\mathbf{w}(\rho) > a_1(\rho)$, for $\rho \in [\rho_4, \infty)$. Consequently, it follows from (2.3) that

$$r^{\frac{1}{\gamma}}\mathbf{z}^{\Delta} > \mathbf{z}^{\frac{\alpha}{\gamma}}a_1^{\frac{1}{\gamma}}, \quad \text{for} \quad \rho \geqslant \rho_4.$$

We deduce from $(r\mathbf{z}^{\Delta})^{\gamma})^{\Delta} \leqslant 0$ that there exists a constant $C > 0$ and $\rho_5 > \rho_4$ such that $r^{\frac{1}{\gamma}}\mathbf{z}^{\Delta} \leqslant C$, for $\rho \geqslant \rho_5$, that is $C \geqslant r^{\frac{1}{\gamma}}\mathbf{z}^{\Delta} > \mathbf{z}^{\frac{\alpha}{\gamma}}a_1^{\frac{1}{\gamma}}$ and hence

$$\mathbf{z}(\rho) \leqslant C^{\frac{\gamma}{\alpha}}a_1(\rho)^{-\frac{1}{\alpha}}, \text{ for } \rho \in [\rho_5, \infty)_{\mathcal{T}}, \tag{2.8}$$

which implies that

$$(\mathbf{z}^{\sigma})^{\frac{(\alpha-\gamma)}{\gamma}} \geqslant C^{\frac{(\alpha-\gamma)}{\alpha}}(a_1^{\sigma})^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \text{ for } \rho \in [\rho_5, \infty)_{\mathcal{T}}. \tag{2.9}$$

Due to (2.5), (2.6), and using Lemma 2.2, we have that

$$\mathbf{w}^{\Delta}(\rho) \leqslant -q(1-p(\delta(\rho)))^{\alpha}\Big(\frac{l\delta(\rho)}{\sigma(\rho)}\Big)^{\alpha} - \nu(1-p(\delta(\rho)))^{\alpha}\Big(\frac{l\delta(\rho)}{\sigma(\rho)}\Big)^{\alpha} - \alpha C r^{-\frac{1}{\gamma}}(\rho)(\mathbf{w}^{\sigma}(\rho))^{1+\frac{1}{\gamma}}(\mathbf{z}^{\sigma}(\rho))^{\frac{(\alpha-\gamma)}{\alpha}}.$$

Integrating the last inequality as in the proof of Theorem 2.3 and using (2.8), we obtain for $\rho \geqslant \rho_1 \geqslant \rho_5$ that

$$\mathbf{w}(\rho) \geqslant a_1(\rho) + K_3\int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(s))^{1+\frac{1}{\gamma}}\Delta s, \text{ for } \rho \in [\rho_l, \infty)_{\mathcal{T}},$$

where $K_1 = aC^{\frac{(a-\gamma)}{\gamma}}$. Substituting (2.9) into (2.3), it is easy to verify that

$$(\mathbf{z}(\rho))^{\frac{(a-\gamma)}{\gamma}} \frac{r^{\frac{1}{\gamma}}(\rho)\mathbf{z}^{\Delta}(\rho)}{\mathbf{z}(\rho)} \geqslant \left[a_1(\rho) + K_1 \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}}. \tag{2.10}$$

Using (2.7) and (2.9) in (2.10), we can find

$$C^{\frac{a-\gamma}{a}} a_1(\rho)^{\frac{(\gamma-a)}{a\gamma}} \left(\int_{\rho_2}^{\rho} r^{-\frac{1}{\gamma}}(s)\Delta s\right)^{-1} \geqslant \left[a_1(\rho) + K_1 \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}}, \text{ for } \rho \in [\rho_1, \infty)_{\mathcal{T}}.$$

Therefore, for $\rho \geqslant \rho_1$ we have

$$(a_1(\rho))^{\frac{(\gamma-a)}{a\gamma}} \left(\int_{\rho_2}^{\rho} r^{-\frac{1}{\gamma}}(s)\Delta s\right) \left[a_1(\rho) + K_1 \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}} \leqslant C^{\frac{a-\gamma}{a}},$$

which contradicts (H_3). □

Theorem 2.6. *Let $0 \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ for $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ and $\gamma < \beta < a$, $\eta(\rho) \geqslant \sigma(\rho)$. If (H_0) and $(H_{1\beta})$ hold, then every solution of (1.2) oscillates.*

The proof of the theorem follows from Theorem 2.3. Hence the details are omitted.

Theorem 2.7. *Let $0 \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ for $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ and $a > \gamma > \beta$, $\eta(\rho) \geqslant \sigma(\rho)$. If (H_0) and $(H_{1\beta})$ hold, then every solution of (1.2) oscillates.*

The proof of the theorem follows from Theorem 2.3.

Theorem 2.8. *Let $0 \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ for $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ and $a < \beta < \gamma$, $\eta(\rho) \geqslant \sigma(\rho)$. If (H_0), (H_{1a}), and (H_2) hold, then every solution of (1.2) oscillates.*

The proof of the theorem follows from Theorem 2.3 and Theorem 2.5.

Theorem 2.9. *Let $0 \leqslant p(\rho) \leqslant 1$, $r^{\Delta}(\rho) \geqslant 0$ for $\rho \in [\rho_0, \infty)_{\mathcal{T}}$ and $a < \gamma < \beta$, $\eta(\rho) \geqslant \sigma(\rho)$. If (H_0), (H_{1a}), and (H_2) hold, then every solution of (1.2) oscillates.*

The proof of the theorem follows from Theorems 2.3 and 2.5.

In the following theorems we will denote

$$a_2(\rho) = \int_{\rho}^{\infty} \left[\lambda Q(s) \left(\frac{l\delta(\rho)}{\delta(\rho)}\right)^{a} + \mu V(s) \left(\frac{l\delta(\rho)}{\sigma(\rho)}\right)^{a}\right] \Delta s, \ \rho \in [\rho_0, \infty)_{\mathcal{T}},$$

and

$$A_2(\rho, K_2) = \left[\frac{\lambda a_2(\tau^{-1}(\rho))}{1 + a^{a}} + \frac{\mu a_2(\tau^{-1}(\rho))}{1 + a^{a}} + K_2 \int_{\tau^{-1}(\rho)}^{\infty} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} ((a_2(\tau^{-\delta}(s)))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}},$$

where K_2 is an arbitrary positive constant and $a > 0$ $\lambda, \mu > 0$ are positive constants, and

$$Q(\rho) = \min\{q(\rho), q(\tau(\rho))\} \text{ and } V(\rho) = \min\{v(\rho), v(\tau(\rho))\}.$$

From the definitions of τ, δ, η, we see that $\tau^{-1}, \delta^{-1}, \eta^{-1} : \mathbb{T} \to \mathbb{T}$ and $\tau^{-1}, \delta^{-1}, \eta^{-1}$ are rd-continuous functions and $\tau^{-1}(\rho) \geqslant \rho, \delta^{-1}(\rho) \geqslant \rho$ and $\eta^{-1}(\rho) \geqslant \rho$.

Theorem 2.10. *Let $1 \leqslant p(\rho) \leqslant p < \infty$, $r^{\Delta}(\rho) \geqslant 0$ $\tau(\delta(\rho)) = \delta(\tau(\rho))$, $\tau(\eta(\rho)) = \eta(\tau(\rho))$ and $\gamma < a < \beta$, $\eta(\rho) \geqslant \delta(\rho)$ with (H_0) and the following conditions hold:*

(H_4) $\limsup_{\rho\to\infty} a_2(\rho) < \infty$;

(H_5) $\int_{\rho_0}^{\infty}(\frac{1}{r(s)})^{\frac{1}{\gamma}}A_2^{\sigma}(s,K_2)\Delta s=\infty,$

then every solution of (1.2) *oscillates.*

Proof. Let $\mathbf{x}(\rho)$ be a nonoscillatory solution of (1.2) such that $\mathbf{x}(\rho)>0$ for $\rho \geqslant \rho_0$. Proceeding as in the proof of Theorem 2.3, we get (2.2) for $\rho \in [\rho_2,\infty)$, that is either $\mathbf{z}^{\Delta}(\rho)>0$ or $\mathbf{z}^{\Delta}(\rho)<0$. By Lemma 2.1, it follows that $\mathbf{z}^{\Delta}(\rho)>0$. From (1.2), it is easy to see for $\rho \geqslant \rho_1$, that

$$\begin{aligned}(r(\rho)(\mathbf{z}^{\Delta}(\rho))^{\gamma})^{\Delta}&+p^{\beta}(r(\tau(\rho))(\mathbf{z}^{\Delta}(\tau(\rho))^{\gamma})^{\Delta}+q(\rho)\mathbf{x}^{a}(\delta(\rho)))\\&+p^{\beta}q(\tau(\rho))\mathbf{x}^{a}(\delta(\tau(\rho))+\nu(\rho)\mathbf{x}^{\beta}(\eta(\rho)))+p^{\beta}\nu(\tau(\rho))\mathbf{x}^{\beta}(\eta(\tau(\rho))=0.\end{aligned}\tag{2.11}$$

Assuming that there exists $\lambda>0$ such that $u^{\gamma}(\mathbf{x})+u^{\gamma}(y)\geqslant \lambda u^{\gamma}(\mathbf{x}+y)$, $\mathbf{x}$, $y\in\mathbb{R}^{+}$, and there exists $\mu>0$ such that $u^{\gamma}(\mathbf{x})+u^{\gamma}(y)\geqslant \mu u^{\gamma}(\mathbf{x}+y)$, $\mathbf{x}$, $y\in\mathbb{R}^{+}$, we obtain (note that $\gamma<a<\beta$) that

$$(r(\rho)(\mathbf{z}^{\Delta}(\rho))^{\gamma})^{\Delta}+p^{a}(r(\tau(\rho))(\mathbf{z}^{\Delta}(\tau(\rho))^{\gamma})^{\Delta}+\lambda Q(\rho)\mathbf{z}^{a}(\delta(\rho))+\mu V(\rho)\mathbf{z}^{a}(\eta(\rho))\leqslant 0,$$

for $\rho\in[\rho_2,\infty)_{\mathcal{T}}$, where $\mathbf{z}(\rho)\leqslant \mathbf{x}(\rho)+px(\tau(\rho))$. Define $\mathbf{w}(\rho)$ as in (2.3), upon using the fact that

$$\omega^{\Delta}(\rho)=\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\delta})^{a}}-\frac{r(\mathbf{z}^{\Delta})^{\gamma})^{\delta}(\mathbf{z}^{a})^{\Delta}}{\mathbf{z}^{a}(\mathbf{z}^{\delta})^{a}}$$

and

$$\frac{(\mathbf{z}^{a})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}\geqslant a\frac{(\mathbf{z}^{\Delta})}{\mathbf{z}^{\sigma}},\ a>0 \text{ for } \rho\in[\rho_3,\infty)_{\mathcal{T}}.$$

Since $\mathbf{z}(\rho)$ is nondecreasing, then using (2.11) in (2.10) we obtain

$$\omega^{\Delta}\leqslant\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}-a\mathbf{w}^{\sigma}\frac{\mathbf{z}^{\Delta}}{\mathbf{z}^{\rho}},\ \rho\geqslant\rho_3.$$

Due to (2.6) and $(\mathbf{z}(\sigma(\rho)))^{\frac{a}{\gamma}}\geqslant C$, there exists $\rho_4>\rho_3$ such that, for $\rho\in[\rho_4,\infty)_{\mathcal{T}}$,

$$\mathbf{w}^{\Delta}\leqslant\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}-aCr^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}}.\tag{2.12}$$

From (2.12), we find

$$\omega^{\Delta}+a^{a}\mathbf{w}^{\tau\Delta}\leqslant\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}-aCr^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}}+a^{a}\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\tau\Delta}}{(\mathbf{z}^{\sigma\Delta})^{a}}-aC(r^{\tau})^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma\tau})^{1+\frac{1}{\gamma}},$$

that is,

$$\omega^{\Delta}+a^{a}\mathbf{w}^{\tau\Delta}\leqslant\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\Delta}}{(\mathbf{z}^{\sigma})^{a}}+a^{a}\frac{(r(\mathbf{z}^{\Delta})^{\gamma})^{\tau\Delta}}{(\mathbf{z}^{\sigma\Delta})^{a}}-aC\left[r^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}}+a^{a}(r^{\tau})^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma\tau})^{1+\frac{1}{\gamma}}\right].$$

Applying the Lemma 2.2 to the above inequality we get

$$\omega^{\Delta}+a^{a}\mathbf{w}^{\tau\Delta}\leqslant-\lambda Q\left(\frac{l\delta}{\sigma}\right)^{a}-\mu V\left(\frac{l\delta}{\sigma}\right)^{a}-aC\left[r^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}}+a^{a}(r^{\tau})^{-\frac{1}{\gamma}}(\mathbf{w}^{\sigma\tau})^{1+\frac{1}{\gamma}}\right]$$

for $\rho\in[\rho_1,\infty)_{\mathcal{T}}$, that is

$$\omega^{\Delta}+a^{a}\mathbf{w}^{\tau\Delta}\leqslant-\lambda Q(\rho)\left(\frac{l\delta}{\sigma}\right)^{a}-\mu V(\rho)\left(\frac{l\delta}{\sigma}\right)^{a}-aCr^{-\frac{1}{\gamma}}(1+a^{a})(\mathbf{w}^{\sigma})^{1+\frac{1}{\gamma}},\tag{2.13}$$

where we used the fact that $r^{\Delta}(\rho) \geqslant 0$ and $\mathbf{w}(\rho)$ is a decreasing function due to (2.6) and (2.13) on $[\rho_1, \infty)_{\mathcal{T}}$. Integrating (2.13) from ρ to ν ($\rho < \nu$) for $\rho, \nu \in [\rho_1, \infty)_{\mathcal{T}}$, it is easy to verify that

$$\omega^{\Delta} + a^{a}\mathbf{w}^{\tau(\rho)} \geqslant \int_{\rho}^{\infty} \lambda Q(s) \left(\frac{l\delta}{\sigma}\right)^{a} \Delta s + \int_{\rho}^{\infty} \mu V(s) \left(\frac{l\delta}{\sigma}\right)^{a} \Delta s + aC(1+a^{a}) \int_{\rho}^{\infty} \left[r(s)^{-\frac{1}{\gamma}} \mathbf{w}(\sigma(s))^{1+\frac{1}{\gamma}}\right] \Delta s,$$

that is,

$$\omega^{\Delta} + a^{a}\mathbf{w}^{\tau(\rho)} = a_2(\rho) + aC(1+a^{a}) \int_{\rho}^{\infty} \left[r(s)^{-\frac{1}{\gamma}} \mathbf{w}(\sigma(s))^{1+\frac{1}{\gamma}}\right] \Delta s,$$

which implies that

$$(1+a^{a})\mathbf{w}(\tau(\rho)) \geqslant a_2(\rho) + aC(1+a^{a}) \int_{\rho}^{\infty} r^{-\frac{1}{\gamma}}(s) \mathbf{w}(\sigma(s))^{1+\frac{1}{\gamma}} \Delta s. \tag{2.14}$$

Due to $(H_1\beta)$, (2.14) yields that

$$\mathbf{w}(\rho) \geqslant \frac{a_2(\tau^{-1}(\rho))}{(1+a^{a})} + aC \int_{\tau^{-1}(\rho)}^{\infty} r^{-\frac{1}{\gamma}}(s) \mathbf{w}(\sigma(s))^{1+\frac{1}{\gamma}} \Delta s.$$

Indeed

$$\mathbf{w}(\rho) \geqslant \frac{a_2(\tau^{-1}(\rho))}{(1+a^{a})}.$$

Hence the last inequality becomes

$$\begin{aligned} \mathbf{w}(\rho) &\geqslant \frac{a_2(\tau^{-1}(\rho))}{(1+a^{a})} + aC \int_{\tau^{-1}(\rho)}^{\infty} \left[r^{-\frac{1}{\gamma}}(s) \left(\frac{1}{1+a^{a}}\right)^{1+\frac{1}{\gamma}} a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right] \Delta s \\ &= \frac{a_2(\tau^{-1}(\rho))}{(1+a^{a})} + K_2 \int_{\tau^{-1}(\rho)}^{\infty} \left[r^{-\frac{1}{\gamma}}(s)(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right] \Delta s = A_2^{\gamma}(\rho, K_2), K_2 = aC\left(\frac{1}{1+a^{a}}\right)^{1+\frac{1}{\gamma}}. \end{aligned}$$

Proceeding as in the proof of Theorem 2.3, we obtain

$$\int_{\rho_4}^{\rho} r^{-\frac{1}{\gamma}}(s) A_2^{\sigma}(s, K_2) \Delta s < \infty,$$

a contradiction due to (H_5). Hence the theorem is complete. □

Theorem 2.11. *Let* $1 \leqslant p(\rho) \leqslant p < \infty$, $r^{\Delta}(\rho) \geqslant 0$ *for* $\rho \in [\rho_0, \infty)_{\mathcal{T}}$, $\tau(\delta(\rho)) = \delta(\tau(\rho))$, $\tau(\eta(\rho)) = \eta(\tau(\rho))$ *with* $\gamma = a = \beta$, $\eta(\rho) \geqslant \delta(\rho)$. *If* (H_0), (H_4), (H_5) *are satisfied and*

(H_6) $\limsup_{\rho\to\infty} \left(\int_{\rho_0}^{\rho} r^{-\frac{1}{\gamma}}(s) A_2(s, K_2) \Delta s\right) > 1$,

then every solution of (1.2) *oscillates.*

Theorem 2.12. *Let* $1 \leqslant p(\rho) \leqslant a < \infty$, $r^{\Delta}(\rho) \geqslant 0$ $\tau(\sigma(\rho)) = \sigma(\tau(\rho))$, $\tau(\eta(\rho)) = \eta(\tau(\rho))$, $\gamma > a > \beta$, $\eta(\rho) \geqslant \delta(\rho)$. *If* (H_0), $(H_{1\beta})$, (H_4), (H_6) *are satisfied and*

(H_7) $\limsup_{\rho\to\infty} (a_1(\rho))^{\frac{(\gamma-a)}{a\gamma}} \int_{\rho_0}^{\rho} r^{-\frac{1}{\gamma}}(s) \Delta s \left[a_1(\rho) + K_3 \int_{\rho}^{\infty} (\frac{1}{r(s)})^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}} = \infty$,

then every solution of (1.2) *oscillates.*

The proofs of the above two theorems follow as a consequence of the proofs of Theorems 2.4 and 2.10. Hence the details are omitted.

3. Examples

Two numerical examples are presented in this section. The theoretical results are verified and confirmed.

Example 3.1. Let $\mathcal{T} = \mathbb{R}$ and therefore $\sigma(t) = t$. Consider the equation

$$(\mathbf{x}(\rho) + (1 - \frac{1}{2\rho})\mathbf{x}(\delta(\rho)))'' + \frac{1}{\rho}\mathbf{x}^3(\frac{\rho}{2}) + \frac{1}{\rho}\mathbf{x}^3(\frac{\rho}{3}) = 0, \tag{3.1}$$

where $r(\rho) = 1$, $p(\rho) = 1 - \frac{1}{2\rho}$, $q_1(\rho) = q_2(\rho) = \frac{1}{\rho}$. Here, $R(t) = \int_{t_0}^{\infty} \left(\frac{1}{t^\gamma}\right)^{\frac{1}{\gamma}} dt = \infty$ and $a_1(\rho) = \frac{l^3}{8\rho} + \frac{l^3}{8\rho} = (\frac{l}{8\rho})^3$ so that $\lim_{t\to\infty} \sup a_1(\rho) < \infty$. Further, we have

$$A_1(\rho, K_1) = (\frac{l}{8\rho})^3 + k_1 \int_{\rho}^{\infty} (\frac{l}{8s})^3 ds = (\frac{l^3}{8}) + (\frac{l^6}{64})\frac{1}{\rho}.$$

Hence, all conditions of Theorem 2.3 are satisfied. Therefore, (3.1) is oscillatory.

Example 3.2. Let $\mathcal{T} = \mathbb{Z}$ and $\sigma(t) = t + 1$. Consider the equation

$$\Delta^2[\mathbf{x}(\rho) + (1 + e^{-\rho})\mathbf{x}(\tau(\rho))] + q_1(\rho)\mathbf{x}^3(\delta(\rho(p)) + q_2(\rho)\mathbf{x}^5(\eta(\rho(p)) = 0, \tag{3.2}$$

where $r(\rho) = 1$, $p(\rho) = 1 + e^{-\rho}$, $\gamma = 1$, $\alpha = 3$, $\beta = 5$, $q(\rho) = \rho - 2$, $\delta(\rho) = \rho - 3$, $\eta(\rho) = \rho - 2$, $q_1(\rho) = e^{-4t}(e^7 + e^{11})$, and $q_2(\rho) = 2e^{-6\rho+9}(1 + e + e^2) + 2e^{-7\rho+10}(1 + e^2)$. It is not difficult to check that (H_0), (H_5), and (H_6) hold true. Therefore, by Theorem 2.10, every solution of (3.2) oscillates. In particular $\mathbf{x}(\rho) = (-1)^\rho e^{-\rho}$ is such a solution.

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