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Generalized relative Hilali conjecture

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Abstract

Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected elliptic CW-complexes. Motivated by the famous Hilali conjecture, Yamaguchi and Yokura [12] proposed a relative version of Hilali conjecture which speculates that

$$\dim \ker\,(\pi_*(p)\otimes \mathbb{Q}) \leqslant \dim \ker\, H_*(p;\mathbb{Q})+1.$$

Our purpose in this paper is to generalize the relative Hilali conjecture. Therefore, we suggest

$$\dim \ker\,(\pi_*\,(j)\otimes \mathbb{Q})+\dim \ker\,(\pi_*\,(p)\otimes \mathbb{Q}) \leqslant \dim \ker\, H_*\,(j;\mathbb{Q})+\dim \ker\, H_*\,(p;\mathbb{Q})+1.$$

It includes the Hilali conjecture and Yamaguchi-Yokura conjecture as special cases. Furthermore, we prove this conjecture for non trivial cases.

Keywords: Rational homotopy theory, Sullivan models, elliptic space, relative Hilali conjecture, rational cohomology.

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1. Introduction

In this paper, all spaces are simply connected CW-complex and are of finite type over $\mathbb{Q}$, i.e., have finite dimensional rational cohomology in each degree.

A space X is said to be rationally elliptic if the dimensions of rational cohomology and homotopy groups are both finite (cf. [2, 3]), i.e., $\dim\,(\pi_*(X)\otimes\mathbb{Q})<\infty$ and $\dim H^*(X;\mathbb{Q})<\infty$, where

$$\pi_*(X)\otimes\mathbb{Q}=\sum_{i\geqslant 2}\pi_i(X)\otimes\mathbb{Q} \text{ and } H^*(X;\mathbb{Q})=\sum_{i\geqslant 0}H^i(X;\mathbb{Q}).$$

The computation of rational cohomology and homotopy groups is a very active research subject. At present time there are a very small number of know methods to determine these groups. In his doctoral thesis [5], inspired by the problem of classification of elliptic spaces, Hilali proposed the following.

Conjecture 1.1. Let X be a simply connected elliptic CW-complex, then

$$\dim\,(\pi_*(X)\otimes\mathbb{Q})\leqslant \dim H_*(X;\mathbb{Q}). \tag{H}$$

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The verification of the conjecture (H) has been receiving a growing attention and has become a popular subject of study with a lot of progresses [4, 6, 7, 9, 14, 15].

The main tool we shall use is the Sullivan minimal model. We assume that the reader is familiar with the basics of rational homotopy theory. Our references for this material are [2, 3]. In particular, We merely recall some tools and aspects which play a larger role in the paper.

Definition 1.2. A commutative differential graded algebra (cdga) is a graded algebra $A = \oplus_{i \geqslant 0} A^i$ with a differential $d : A^i \to A^{i+1}$ such that $d^2 = 0$, $xy = (-1)^{ij} yx$, and $d(xy) = d(x)y + (-1)^i x d(y)$ for all $x \in A^i$ and $y \in A^j$. A cdga (A, d) is called simply connected if $H^0(A, d) = \mathbb{Q}$ and $H^1(A, d) = 0$. A morphism $f : (A, d) \to (B, d)$ of cdga's is called a quasi-isomorphism if $H^*(f)$ is an isomorphism.

A commutative graded algebra A is free if it is of the form

$$\Lambda V = S(V^{even}) \otimes E(V^{odd}),$$

where $V^{even} = \oplus_{i \geqslant 1} V^{2i}$ and $V^{odd} = \oplus_{i \geqslant 0} V^{2i+1}$. A Sullivan algebra is a cgda $(\Lambda V, d)$, where $V = \oplus_{i \geqslant 1} V^i$ admits a homogeneous basis $\{x_i\}_{i \in I}$ indexed by a well ordered set I such $dx_i \in \Lambda(\{x_i\})_{i<j}$. A Sullivan algebra is called minimal if $dV \subset \Lambda^{\geqslant 2} V$. If there is a quasi-isomorphism $f : (\Lambda V, d) \to (A, d)$, where $(\Lambda V, d)$ is a Sullivan minimal algebra, then we say that $(\Lambda V, d)$ is a Sullivan minimal algebra of (A, d).

To a simply connected topological space X of finite type, Sullivan associates in a functorial way a cdga $A_{PL}(X)$ of piecewise linear forms on X such that $H^*(A_{PL}(X)) \cong H^*(X; \mathbb{Q})$ [11]. A Sullivan minimal model of X is a Sullivan minimal model of $A_{PL}(X)$. Moreover, the rational homotopy type of X is completely determined by its Sullivan minimal model $(\Lambda V, d)$. In particular, there are isomorphisms

$$H^*(X; \mathbb{Q}) \cong H^*(\Lambda V, d) \text{ as commutative graded algebras,}$$
$$\mathrm{Hom}(\pi_*(X) \otimes \mathbb{Q}, \mathbb{Q}) \cong V \text{ as graded vector spaces.}$$

Sullivan minimal models behave nicely with respect to fibrations. Let $\xi : F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected spaces. The relative Sullivan model for ξ is a short exact sequence:

$$(\Lambda W, d_W) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \xrightarrow{P} (\Lambda V, d_V)$$

of cdga, in which $J : (\Lambda W, d_W) \to (\Lambda W \otimes \Lambda V, D)$ is the inclusion and $P : (\Lambda W \otimes \Lambda V, D \to (\Lambda V, d_V)$ is the projection onto the quotient of $(\Lambda W \otimes \Lambda V, D)$ by the ideal generated by $\Lambda^+ W$. The differential D satisfies: $D(w) = d_W(w)$ for $w \in W$ and $D(v) - d_V(v) \in \Lambda^+ W.(\Lambda W \otimes \Lambda V)$ for $v \in V$. Here $(\Lambda W, d_W)$ and $(\Lambda V, d_V)$ are the Sullivan minimal models for B and F, respectively (Proposition 15.5 of [2]). The cdga $(\Lambda W \otimes \Lambda V, D)$ is a Sullivan model for the total space E but is not in general minimal.

We now turn our attention to the relative Hilali conjecture. At present, a little is known about rational cohomology of function spaces. However, as we mentioned earlier, Yamaguchi and Yokura generalized the conjecture (H) as follows.

Conjecture 1.3. Let $F \to E \xrightarrow{p} B$ be a fibration of simply connected elliptic CW-complexes, then

$$\dim \ker (\pi_*(p) \otimes \mathbb{Q}) \leqslant \dim \ker H_*(p; \mathbb{Q}) + 1. \tag{RH}$$

Here, we recall that

$$\ker (\pi_*(p) \otimes \mathbb{Q}) = \oplus_{i \geqslant 2} \ker (\pi_i(p) \otimes \mathbb{Q} : \pi_i(E) \otimes \mathbb{Q} \to \pi_i(B) \otimes \mathbb{Q}), \text{ and}$$
$$\ker H_*(p; \mathbb{Q}) = \oplus_{i \geqslant 0} \ker (H_i(p; \mathbb{Q}) : H_i(E; \mathbb{Q}) \to H_i(B; \mathbb{Q})).$$

In [12], the authors proved that this conjecture can be reformulated algebraically as follows.

Conjecture 1.4. If $(\Lambda W, d_W) \xrightarrow{J} (\Lambda W \otimes \Lambda V, D) \xrightarrow{P} (\Lambda V, d_V)$ is the relative Sullivan model for a fibration of simply connected CW-complexes, then

$$\dim \text{ coker } H^*(J, D_1) \leqslant \dim \text{ coker } H^*(J; \mathbb{Q}) + 1. \tag{RH}$$

This is an interesting and challenging problem in rational homotopy theory, and positive answer to this conjecture would have some interesting results to prove the conjecture (H). Until now the conjecture (RH) is affirmed for certain fibrations [1, 12, 13, 16], but in general it still remains open question.

In the following, we assume that all spaces are **rationally simply connected elliptic CW-complex**, unless otherwise stated.

2. Generalized relative Hilali conjecture

Inspired by the works of Yamaguchi and Yokura (see [13] and [12]), in this section we propose a relative version of the conjecture (RH), which could be called the generalized relative Hilali conjecture. In particular, this conjecture includes the conjectures (H) and (RH) as special cases. At the end of this section, we present some results concerning this conjecture. Without loss of generality, we suggest the following.

Conjecture 2.1. Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration of simply connected elliptic CW-complexes, then

$$\dim \ker \pi_* (j) + \dim \ker \pi_* (p) \leqslant \dim \ker H_* (j; \mathbb{Q}) + \dim \ker H_* (p; \mathbb{Q}) + 1. \tag{GRH}$$

In special cases of the preceding that is of interest to us is the one in which:

- B is contractible i.e., B is a one-point space, then the conjecture (GRH) is just the conjecture (H). Indeed, since in this case we have $F \simeq E$. Hence, as is easily checked we find that

  $$\ker \pi_*(j) \cong \ker H_*(j; \mathbb{Q}) \cong 0.$$

  Furthermore, we have $\ker \pi_*(p) \cong \pi_*(E)$ and $\ker H_*(p) \oplus \mathbb{Q} \cong H_*(E; \mathbb{Q})$.
- $\dim \ker \pi_*(j) = \dim \ker H_*(j)$, of course we immediately retrieve the conjecture (RH).

Thus the above conjecture is a generalization of the conjectures (H) and (RH).

We go now to establish an upper bound of the left hand side of the conjecture (GRH). It is well known that $\dim \ker \pi_*(j) \leqslant \dim \pi_*(F)$ and then collecting together with result in [12] imply the following.

Remark 2.2. As same notation in conjecture (GRH), we get $\dim \ker \pi_* (p) + \dim \ker \pi_* (j) \leqslant 2 \dim \pi_* (F)$.

In the next, we identify the conjecture (GRH) in terms of Sullivan minimal models. Let us denote by $H^*(P, D_1) : H^*(W \oplus V, D_1) \to V$, where $D_1 = D/D^2$ is the linear part of D. By using ([3, Proposition 2.65]), we form the following commutative diagram

$$\begin{array}{ccc} \text{Hom}(\pi_* (E), \mathbb{Q}) & \xrightarrow{\text{Hom}(\pi_*(j), \mathbb{Q})} & \text{Hom}(\pi_* (F), \mathbb{Q}) \\ \| & & \| \\ H^*(W \oplus V, D_1) & \xrightarrow{H^*(P, D_1)} & V \end{array}$$

So, we notice that $\text{Hom}(\ker \pi_* (j), \mathbb{Q}) \cong \text{coker } H^* (P, D_1)$. Hence, as all is finite dimensional we have that

$$\dim \ker \pi_* (j) = \dim \text{ coker } H^* (P, D_1). \tag{2.1}$$

On the other hand, we have the following commutative diagram

$$\begin{array}{ccc} H^*(E,\mathbb{Q}) & \xrightarrow{H^*(f;\mathbb{Q})} & H^*(F,\mathbb{Q}) \\ \| & & \| \\ H^*(\Lambda W\otimes\Lambda V,D) & \xrightarrow{H^*(P;\mathbb{Q})} & H^*(\Lambda V,d_V) \end{array}$$

We argue exactly as in the above to deduce that

$$\dim\ker H^*(p;\mathbb{Q}) = \dim\operatorname{coker} H^*(P;\mathbb{Q}). \tag{2.2}$$

Combining (2.1) with (2.2) and the results in [12], the conjecture (GRH) can be rewritten algebraically as follows.

Conjecture 2.3. If $(\Lambda W, d_W) \xrightarrow{J} (\Lambda W\otimes\Lambda V, D) \xrightarrow{P} (\Lambda V, d_V)$ is the relative Sullivan model for a fibration of simply connected CW-complexes, then

$$\dim\operatorname{coker} H^*(J,D_1) + \dim\operatorname{coker} H^*(P,D_1) \leqslant \dim\operatorname{coker} H^*(J;\mathbb{Q}) + \dim\operatorname{coker} H^*(P;\mathbb{Q}) + 1. \tag{GRH}$$

Furthermore, the identification of this conjecture allows us to conclude several results of interest. A very difficulty in proving this conjecture globally is in the differential D of the Sullivan model of E. Before proving the conjecture (GRH) in some cases, we illustrate some examples so that is valid.

Example 2.4. Consider the following fibration

$$\xi : \mathbb{C}P^{n-1} \xrightarrow{j} \mathbb{C}P^{2n-1} \xrightarrow{p} \mathbb{S}^{2n} \text{ for } n \geqslant 2.$$

First, we observe that

$\pi_i\left(\mathbb{S}^{2n}\right) \cong \mathbb{Q}$ for $i = 2n, 4n-1$, $\pi_i\left(\mathbb{C}P^{2n-1}\right) \cong \mathbb{Q}$ for $i = 2, 4n-1$, $\pi_i\left(\mathbb{C}P^{n-1}\right) \cong \mathbb{Q}$ for $i = 2$ and $i = 2n-1$.

It follows from the homotopy long exact sequence that

$$\dim\ker\pi_*(p) = \dim\ker\pi_*(j;\mathbb{Q}) = 1.$$

Otherwise, from degree reasons we find that

$$\dim\ker H_*(p;\mathbb{Q}) = 2n-1 \text{ and } \dim\ker H_*(j;\mathbb{Q}) = 0.$$

By summation, it follows that

$$\dim\ker\pi_*(j) + \dim\ker\pi_*(p) \leqslant \dim\ker H_*(j;\mathbb{Q}) + \dim\ker H_*(p;\mathbb{Q}) + 1.$$

We have shown that ξ satisfies the conjecture (GRH).

Example 2.5. Let a fibration

$$\xi : \mathbb{S}^3 \xrightarrow{j} \mathbb{S}^2 \times \mathbb{S}^{2n+1} \xrightarrow{p} \mathbb{C}P^n,$$

which is modelled by

$$(\Lambda(x,y), d) \xrightarrow{J} (\Lambda(x,y,z), D) \xrightarrow{P} (\Lambda(z), 0)$$

with $|x| = 2, |z| = 3$, and $|y| = 2n+1$. The differential is given by $dx = Dx = 0$, $dy = Dy = x^{n+1}$, and $Dz = x^2$. Without going into details, a simple check reveals that

$$\dim\operatorname{coker} H^*(J,D_1) = 1 \text{ and } \dim\operatorname{coker} H^*(P,D_1) = 0.$$

On the other hand, a direct computation shows that

$$\dim\operatorname{coker} H^*(J;\mathbb{Q}) = 2 \text{ and } \dim\operatorname{coker} H^*(P;\mathbb{Q}) = 1.$$

Thus ξ satisfies the conjecture (GRH).

Now, we go on to establish the conjecture (GRH) for fibration whose base space is a homogeneous space G/H, where H is a closed subgroup of a compact connected Lie group G. This is particularly interesting, because these spaces constitute a nice class of highly important geometric examples.

Theorem 2.6. *Conjecture* (GRH) *holds for a principal fibration* $H \overset{j}{\to} G \overset{p}{\to} G/H$ *with* rank $G =$ rank H.

Proof. We recall our aim is to prove that

$$\dim \ker \pi_* (j) + \dim \ker \pi_* (p) \leqslant \dim \ker H_* (j; \mathbb{Q}) + \dim \ker H_* (p; \mathbb{Q}) + 1.$$

So, from Remark (2.2) it is sufficient to show that

$$2 \dim \pi_* (H) \leqslant \dim \ker H_* (j; \mathbb{Q}) + \dim \ker H_* (p; \mathbb{Q}) + 1.$$

First, we consider G a compact connected Lie group such that

$$\pi_* (G) \cong \mathbb{Q} \langle y_1, \dots, y_q \rangle ,$$

where the degree of y_i is odd for $1 \leqslant i \leqslant q$. Using the fact rank $G =$ rank H, one deduces that the rational homology of G/H is evenly graded, i.e., any cycle of odd degree is exact (see [3]). On the other hand, if we consider the map induced on homology of p and from the general considerations above, we get

$$H_* (p; \mathbb{Q}) : H_* (G; \mathbb{Q}) \cong \Lambda (y_1, \dots, y_q) \to H_* (G/H; \mathbb{Q}) \cong H_{\text{even}} (G/H; \mathbb{Q}) .$$

Now according to rank of G we consider two different cases.

Case 1. Suppose that $\dim \pi_* (G) > 3$, i.e., $q > 3$. Without loss of generality, we set

$$\Gamma = \mathbb{Q} \langle y_1, \dots, y_q \rangle \oplus \mathbb{Q} \langle y_{i_0} y_{i_1} y_{i_2} \text{ for } i_0, i_1, i_2 \in \{1, \dots, q\} \text{ and } i_0 \neq i_1 \neq i_2 \rangle .$$

For each element α in Γ, we have the degree of α is odd and the class of α is non null in $H_* (G; \mathbb{Q})$. Furthermore, since the homology of G/H is concentrated in even degree, we deduce that the class of α is non null in $\ker H_* (p; \mathbb{Q})$. This proves that

$$\dim \ker H_* (p; \mathbb{Q}) \geqslant \dim \Gamma \geqslant 2q \geqslant 2 \dim \pi_* (H) .$$

Case 2. Suppose that $\dim \pi_* (G) \leqslant 3$, i.e., in particular

$$\pi_* (G) \cong \mathbb{Q} \langle y_1, y_2, y_3 \rangle ,$$

where y_1, y_2, and y_3 are of odd degree. Thus, for degree reasons and considerations above, it is easy to see that $[y_1], [y_2]$, and $[y_3]$ are non null in $\ker H_* (p; \mathbb{Q})$. Further, a direct argument shows that there is $i_0, i_1 \in \{1, 2, 3\}$ and $i_0 \neq i_1$ such that $[y_{i_0} y_{i_1}]$ is non null in $\ker H_* (p; \mathbb{Q})$. Now, let $[\mu_{G/H}]$ denote the top class in $H_*(G/H; \mathbb{Q})$. From Theorem 32.6 in [2], it is easily seen that

$$|\mu_{G/H}| \leqslant \sum_{1 \leqslant i \leqslant 3} |y_i|.$$

By degree argument, we have immediately the class of $y_1 y_2 y_3$ is non null in $\ker H_*(p; \mathbb{Q})$. In summary, we have computed that

$$\dim \ker H_* (j; \mathbb{Q}) + \dim \ker H_* (p; \mathbb{Q}) + 1 \geqslant \dim \ker H_* (p; \mathbb{Q}) + 1 \geqslant 2 \dim \pi_* (H) ,$$

which completes the proof. □

Next, we present our second result concerning the conjecture (GRH). First, we recall that a fibration

$$\xi : F \xrightarrow{j} E \xrightarrow{p} B$$

is called totally non cohomologous to zero (abbreviated TNCZ) if the induced homomorphism $H^*(j;\mathbb{Q}) : H^*(E;\mathbb{Q}) \to H^*(F;\mathbb{Q})$ is surjective. It is equivalent to require that the Serre spectral sequence collapses at E_2-term. In this case, there is an isomorphism $H^*(E;\mathbb{Q}) \cong H^*(F;\mathbb{Q}) \otimes H^*(B;\mathbb{Q})$ of $H^*(B;\mathbb{Q})$-modules. Furthermore, we say that ξ is weakly trivial if $\pi_*(E) \cong \pi_*(F) \oplus \pi_*(B)$.

Now, we are in position to prove the following result:

Theorem 2.7. *Let $\xi : F \xrightarrow{j} E \xrightarrow{p} B$ be a weakly trivial TNCZ fibration. If F satisfies the conjecture (H), then ξ satisfies the conjecture (GRH).*

Proof. First since ξ is weakly trivial, we clearly have $\pi_*(j)$ is injective. In order to prove the conjecture (GRH), it is sufficient to show that

$$\dim \ker \pi_*(p) \leqslant \dim \ker H_*(j;\mathbb{Q}) + \dim \ker H_*(p;\mathbb{Q}) + 1.$$

On the other hand, under our second assumption (ξ is TNCZ), we have

$$\ker H_*(p;\mathbb{Q}) \cong H_*(F;\mathbb{Q}) \otimes H_*^+(B;\mathbb{Q}),$$

where $H_*^+(B;\mathbb{Q}) \cong \oplus_{i\geqslant 1} H_i(B;\mathbb{Q})$. It follows that

$$\begin{aligned} \dim \ker H_*(p;\mathbb{Q}) \geqslant \dim H_*(F;\mathbb{Q}) &\geqslant \dim \pi_*(F) \quad \text{(due to conjecture (H))} \\ &\geqslant \dim \ker \pi_*(p) \quad \text{(from Remark (2.2))}. \end{aligned}$$

□

If we restrict the fibre of ξ, we retrieve two immediate consequences of the Theorem above.

Corollary 2.8. *Let $\xi : F \xrightarrow{j} E \xrightarrow{p} B$ be a weakly trivial fibration with* $\operatorname{rank} \pi_{odd}(F) = \operatorname{rank} \pi_{even}(F) \leqslant 3$. *Then ξ satisfies the conjecture (GRH).*

Proof. It follows easily from [5, 8]. □

Now, due to [5, 10] we obtain the following.

Corollary 2.9. *The conjecture (GRH) holds for every weakly trivial fibration $G/H \xrightarrow{j} E \xrightarrow{p} B$ with rank $G =$ rank H.*

we give a further result concerning the conjecture (GRH).

Theorem 2.10. *Let $\xi : F \simeq \prod_{i=1}^{r} S^{2n_i+1} \xrightarrow{j} E \xrightarrow{p} B \simeq \prod_{k=1}^{r} K(\mathbb{Q}, 2m_k)$ be a fibration, then the conjecture (GRH) is true.*

Proof. To prove our result we work with Sullivan minimal model framework. We make the following identification $W^n \cong \operatorname{Hom}(\pi_n(B), \mathbb{Q})$ and $V^n \cong \operatorname{Hom}(\pi_n(F), \mathbb{Q})$, where $(\Lambda W, d_W)$ and $(\Lambda V, d_V)$ are the Sullivan minimal models of B and F, respectively. Our hypothesis implies that the Sullivan minimal model $(\Lambda W, d_W)$ for B has trivial differential $d_W = 0$ with $W^{odd} = 0$. Write $W = \langle x_1, \dots, x_r \rangle$ and each $|x_k|$ is even for $1 \leqslant k \leqslant r$. In a similar manner as above, write

$$(\Lambda V, d_V) \cong (\Lambda(y_1, \dots, y_r), 0),$$

where $|y_i|$ is odd for $1 \leqslant i \leqslant r$ the Sullivan minimal model of F. So, the relative Sullivan model for ξ is of the form

$$(\Lambda(x_1, \dots, x_r), 0) \xrightarrow{J} (\Lambda(x_1, \dots, x_r, y_1, \dots, y_r), D) \xrightarrow{P} (\Lambda(y_1, \dots, y_r), 0),$$

where $D(x_k) = 0$ and $D(y_i) \in \Lambda^{\geqslant 2}(x_1, \ldots, x_r)$ by ellipticity argument. This proves that ξ is weakly trivial and further

$$\dim \operatorname{coker} H^*(P, D_1) = 0.$$

Hence, it remains to prove that $\dim \operatorname{coker} H^*(J, \mathbb{Q}) + \dim \operatorname{coker} H^*(P, \mathbb{Q}) + 1 \geqslant r$. A brute force calculation will display the result, but we opt to argue at a more general level so as to indicate some reason for which the conjecture is true. Indeed, this last part of our argument shows that $H^*(\Lambda W \otimes \Lambda V, D)$ is concentrated in even degree. Otherwise, by degree argument it follows, of course, that $[y_1], \ldots, [y_r]$ are non null in $\operatorname{coker} H^*(P, \mathbb{Q})$. Finally, we deduce that

$$\dim \operatorname{coker} H^*(P; \mathbb{Q}) \geqslant r \geqslant \dim \operatorname{coker} H^*(J, D_1) \quad \text{(from Remark 2.2)}.$$

Hence, the conjecture (GRH) follows. □

In the remainder of this section, we focus more on the relation between the conjectures (GRH) and (RH). Without loss of generality, it is easy to prove the following.

Proposition 2.11. *Let* $F \xrightarrow{j} E \xrightarrow{p} B$ *be a weakly trivial fibration, then the conjecture* (GRH) *is true whenever the conjecture* (RH) *is.*

Now, we consider the question of when the conjecture (GRH) implies the conjecture (RH).

Proposition 2.12. *Let* $F \xrightarrow{j} E \xrightarrow{p} B$ *be fibration with* $\dim \ker\ \pi_*(j) = \dim \ker H_*(j; \mathbb{Q})$. *Then the conjecture* (GRH) *holds if and only if the conjecture* (RH) *is.*

Notice that the result above does not require that the fibration be weakly trivial. It would be most interesting to prove the conjectures (GRH) and (RH) in other new cases. So, we offer the following as a specific question in this area which has been suggested by the referee.

Question 2.13. *Characterize fibrations* $F \xrightarrow{j} E \xrightarrow{p} B$ *for which* $\dim \ker \pi_*(j) = \dim \ker H_*(j; \mathbb{Q})$?

Of course if $F \xrightarrow{j} E \xrightarrow{p} B$ is trivial, i.e., $E \simeq F \times B$, a similar argument as in the proof of Theorem 2.7 shows that $\dim \ker \pi_*(j) = \dim \ker H_*(j; \mathbb{Q}) = 0$. Generally, it would be useful to have conditions under which the equality $\dim \ker \pi_*(j) = \dim \ker H_*(j; \mathbb{Q})$ holds. I hope to develop this line of investigation in future work.

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