



Stability of linear differential equation of higher order using Mahgoub transforms



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Abstract

In this paper, by applying Mahgoub transform, we show that the n^{th} order linear differential equation

$$x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_{\kappa} x^{(\kappa)}(v) = \psi(v)$$

has Hyers-Ulam stability, where a_{κ} 's are scalars and x is an n times continuously differentiable function of exponential order.

Keywords: Hyers-Ulam stability, linear differential equations, Mahgoub transform.

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1. Introduction

A simulating and famous talk presented by Ulam [32] in 1940, motivated the study of stability problems for various functional equations. He presented a many unsolved problems before a Mathematical Colloquium at the University of Wisconsin, one of his questions was that when is it true that a mapping that approximately satisfies a functional equation must be close to an exact solution of the equation? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [9] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In course of time, the theorem formulated by Hyers was generalized by Rassias [29], Aoki [4], and Bourgin [5] for additive mappings (see also [27, 33]).

The generalization of Ulam's question has been relatively recently proposed by replacing functional equations with differential equations. Let I be a sub-interval of \mathbb{R} , let \mathbb{K} denote either \mathbb{R} or \mathbb{C} , and let n be a positive integer. The differential equation $\psi(f, x, x', x'', \dots, x^{(n)}) = 0$ has the Hyers-Ulam stability if there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: If an n times continuously differentiable function $z : I \rightarrow \mathbb{K}$ satisfies the inequality $|\psi(f, z, z', z'', \dots, z^{(n)})|$ for all $t \in I$, then there

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exists a solution $y : I \rightarrow \mathbb{K}$ of the differential equation that satisfies the inequality $|z(t) - y(t)| \leq K\varepsilon$ for all $t \in I$.

Obłozza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [23, 24]). Then, in 1998, Alsina and Ger [3] continued the study of Obłozza's Hyers-Ulam stability of differential equations. Indeed, they proved in [3] the following theorem.

Theorem 1.1. *Let $I \neq \emptyset$ be an open sub-interval of \mathbb{R} . If a differentiable function $x : I \rightarrow \mathbb{R}$ satisfies the differential inequality $\|x'(t) - \lambda x(t)\| \leq \varepsilon$ for any $t \in I$ and for some $\varepsilon > 0$, then there exists a differentiable function $y : I \rightarrow \mathbb{R}$ satisfying $y'(t) = \lambda y(t)$ and $\|x(t) - y(t)\| \leq 3\varepsilon$ for any $t \in I$.*

This result of Alsina and Ger has been generalized by Takahashi et al. [31]. They proved that the Hyers-Ulam stability holds true for the Banach space valued differential equation $x'(t) = \lambda x(t)$. Indeed, the Hyers-Ulam stability has been proved for the first-order linear differential equations in more general settings (see [10, 11, 13, 18]).

In 2014, Alqifiary and Jung [2] proved the generalized Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t) = f(t)$$

by using the Laplace transform method, where α_k are scalars and $x(t)$ is an n times continuously differentiable function and of the exponential order (see also [30]).

In recent years, many authors are studying the Hyers-Ulam stability of differential equations, and a number of mathematicians are paying attention to the new results of the Hyers-Ulam stability of differential equations by applying different techniques (see [6–8, 15, 16, 25, 26]).

Note that, during these days most of the mathematicians are studied only the Hyers-Ulam stability of the higher order differential equations by various directions (see [17, 19, 20]).

In recent days, few authors have investigated the Ulam stability of the linear differential equations using various integral transform techniques, like, Fourier transform, Mahgoub transform and Aboodh transform in [12, 21, 22, 28].

Based on the above results, our main aim is to prove the Hyers-Ulam stability of the higher-order linear differential equation

$$x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_{\kappa} x^{(\kappa)}(v) = \psi(v) \quad (1.1)$$

by using the Mahgoub integral transform method.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminaries which are used throughout this paper.

Throughout this paper, \mathbb{F} denotes the real field \mathbb{R} or the complex field \mathbb{C} and a function $f : (0, \infty) \rightarrow \mathbb{F}$ of exponential order if there exists a constants $A, B \in \mathbb{R}$ such that $|f(t)| \leq Ae^{tB}$ for all $t > 0$.

Definition 2.1 ([1]). The Mahgoub integral transform of the function $f(t)$ is defined by

$$\mathcal{M}\{f(t)\} = u \int_0^{\infty} f(t) e^{-tu} dt = F(u), \quad (2.1)$$

for $t \geq 0$, $k_1 \leq u \leq k_2$, where \mathcal{M} is a Mahgoub integral transform operator.

The Mahgoub integral transform for the function $f(t)$ for $t > 0$ exists if $f(t)$ is piecewise continuous and of exponential order. These conditions are the only sufficient conditions for the existence of Mahgoub transform of the function $f(t)$.

Definition 2.2. There exists a unique number $-\infty \leq \sigma < \infty$ such that the integral (2.1) converges if $\Re(u) > \sigma$ and diverges if $\Re(u) < \sigma$ where $\Re(u)$ denotes the real part of u . The number σ is called the abscissa of convergence and denoted by σ_f . It is well known that $|F(u)| \rightarrow 0$ as $\Re(u) \rightarrow \infty$.

Definition 2.3 (Convolution of two functions, [1]). The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and it is defined as

$$f(t) * g(t) = f * g = \int_0^t f(s) g(t-s) ds = \int_0^t f(t-s) g(s) ds.$$

Theorem 2.4 (Convolution theorem for Mahgoub transforms, [1]). Suppose that $f(t)$ and $g(t)$ are given functions defined for $t \geq 0$. If $\mathcal{M}\{f(t)\} = F(u)$ and $\mathcal{M}\{g(t)\} = G(u)$, then

$$\mathcal{M}\{f(t) * g(t)\} = \frac{1}{u} \mathcal{M}\{f(t)\} \mathcal{M}\{g(t)\} = \frac{1}{u} F(u) G(u).$$

Definition 2.5 (Inverse Mahgoub transform, [1]). If $\mathcal{M}\{f(t)\} = F(u)$ then $f(t)$ is called the inverse Mahgoub Transform of $F(u)$ and mathematically it is defined as $f(t) = \mathcal{M}^{-1}\{F(u)\}$, where \mathcal{M}^{-1} is the inverse Mahgoub transform operator.

Inverse Mahgoub transform is also defined as follows:

$$\mathcal{M}^{-1}(T)(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(a+ix)} e^{(a+ix)v} T(a+ix) dx.$$

Definition 2.6. The differential equation (1.1) has the Hyers-Ulam stability, if there exists a constant $K > 0$, which has the following properties: For every $\epsilon > 0$ and an n times continuously differentiable function $x(t)$ satisfying the inequality

$$\left| x^{(n)}(v) + \sum_{\kappa=0}^{n-1} a_{\kappa} x^{(\kappa)}(v) - \psi(v) \right| \leq \epsilon,$$

there exists some $y : (0, \infty) \rightarrow \mathbb{F}$ satisfying the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon$, for any $t > 0$. We call such K as the Hyers-Ulam stability constant for (1.1).

3. Hyers-Ulam stability

Lemma 3.1. Let $Q_1(u) = \sum_{i=0}^n a_i u^i$ and $Q_2(u) = \sum_{j=0}^m b_j u^j$, where n and m are non-negative integers with $n > m$ and a_i, b_j are scalars. Then there exists $\phi : (0, \infty) \rightarrow \mathbb{T}$ which is an infinitely differentiable mapping, such that

$$\mathcal{M}(\phi) = \frac{Q_2(u)}{Q_1(u)}, \quad (\Re(u) > d_q),$$

and

$$\phi^{(i)}(0) = \begin{cases} \frac{b_m}{a_n}, & i = n - m, \\ 0, & i = 0, 1, \dots, n - m - 1, \end{cases}$$

where $d_q = \max\{\Re(u) : Q_1(u) = 0\}$.

Proof. Take $\sigma = n - m$. We write

$$Q_1(u) = a_n(u - u_1)^{n_1}(u - u_2)^{n_2} \dots (u - u_\kappa)^{n_\kappa},$$

where u_i are complex numbers $i = 1, 2, \dots, \kappa$ and n_j are integers, with $n = n_1 + \dots + n_\kappa$,

$$\frac{Q_2(u)}{Q_1(u)} = \sum_{i=1}^{\kappa} \sum_{j=1}^{n_i} \frac{\zeta_{ij}}{(u - u_i)^j}, \tag{3.1}$$

where ζ_{ij} are scalars. Let

$$\mu_{ij}(v) = \frac{1}{(j - 1)!} * v^{j-1} e^{u_i v},$$

where i, j are integers, $1 \leq i \leq \kappa$ and $1 \leq j \leq n_i$. Let

$$\phi(v) = \sum_{i=1}^{\kappa} \sum_{j=1}^{n_i} \zeta_{ij} \mu_{ij}(v).$$

Taking Mahgoub transform (3.1), we get

$$M(\phi) = \frac{Q_2(u)}{Q_1(u)}$$

for all u with $R(u) > d_q$, where $d_q = \max\{R(u_i) : i = 1, 2, \dots, \kappa\}$. Moreover, by Maclaurin’s series, we have

$$\phi(v) = \phi(0) + \phi'(0)v + \dots + \frac{\phi^{(n-1)}(0)}{(n - 1)!} v^{n-1} + \mu(v),$$

where $\mu(v) = \sum_{i=n}^{\infty} \frac{\phi^{(i)}(0)}{i!} v^i$. Note that $M(\mu) = \frac{P(u)}{u^n}$, where P is a complex function and

$$M(\phi) = \phi(0) + \frac{\phi'(0)}{u} + \frac{\phi''(0)}{u^2} + \dots + \frac{\phi^{(n-1)}(0)}{u^{n-1}} + \frac{P(u)}{u^n}.$$

Thus

$$\phi(0) + \frac{\phi'(0)}{u} + \frac{\phi''(0)}{u^2} + \dots + \frac{\phi^{(n-1)}(0)}{u^{n-1}} + \frac{P(u)}{u^n} = \frac{b_0 + b_1 u + \dots + b_m u^m}{a_0 + a_1 u + \dots + a_{m+\sigma} u^{m+\sigma}}.$$

If $\sigma \geq 1$, multiply both sides of the above equation by $1, u, u^2, \dots, u^\sigma$ and take $u \rightarrow \infty$ we get $\phi(0) = \phi'(0) = \dots = \phi^{(\sigma-1)}(0) = 0$ and $\phi^\sigma(0) = \frac{b_m}{a_n}$. Hence we complete the proof. □

Lemma 3.2. Let $n > 1$ be an integer, $\psi : (0, \infty) \rightarrow T$ be a continuous mapping and let $Q_1(u)$ be an n degree complex polynomial. Then there exists $\mu : (0, \infty) \rightarrow T$ which is an n times continuously differentiable function such that

$$M(\mu) = \frac{M(\psi)}{Q_1(u)}, \quad (R(u) > \max\{d_q, d_j\}),$$

where $d_q = \max\{R(u) : Q_1(u) = 0\}$ and d_j is the abscissa of convergence for ψ . In particular, $\mu^{(i)}(0) = 0$ for all $i = 0, 1, 2, \dots, n$.

Proof. By Lemma 3.1, if $Q_2(u) = u$ and $Q_1(u) = a_0 + a_1 u + \dots + a_n u^n$, then the Mahgoub transform of $\phi : (0, \infty) \rightarrow T$ is defined by

$$M(\phi) = \frac{u}{Q_1(u)}, \quad (R(u) > d_q),$$

and if $i = 0, 1, 2, \dots, n - 1$, then $\phi^{(i)}(0) = 0$ and $\phi^{(n)}(0) = \frac{1}{a^n}$. Now we define $\mu = \phi * \psi$. Then we obtain $M(\mu) = \frac{M(\psi)}{Q_1(u)}$ and

$$\mu^{(i)}(v) = \phi^{(i-1)}(0)\psi(v) + \int_0^v \phi^{(i)}(v-s)\psi(s)ds = \int_0^v \phi^{(i)}(v-s)\psi(s)ds$$

for all $i = 1, 2, \dots, n$. Then, we have $\mu(0) = \mu'(0) = \dots = \mu^{(n)}(0) = 0$. □

Theorem 3.3. Let α be a scalar. If a function $x : (0, \infty) \rightarrow \mathbb{T}$ satisfies the equality

$$|x'(v) + \alpha x(v) - \psi(v)| \leq \epsilon \tag{3.2}$$

for every $v > 0$ and $\epsilon > 0$, then there exists a solution $x_\alpha : (0, \infty) \rightarrow \mathbb{T}$, which satisfies the differential equation

$$x'(v) + \alpha x(v) = \psi(v) \tag{3.3}$$

such that

$$|x_\alpha(v) - x(v)| \leq \begin{cases} \frac{(1-e^{-R(\alpha)v})\epsilon}{R(\alpha)}, & R(\alpha) \neq 0, \\ \epsilon v, & R(\alpha) = 0, \end{cases}$$

for all $v > 0$.

Proof. Let $\lambda(v) = x'(v) + \alpha x(v) - \psi(v)$, for all $v > 0$. Applying Mahgoub transform to $\lambda(v)$, we get

$$M(x) - \frac{x(0)u + M(\psi)}{u + \alpha} = \frac{M(\lambda)}{u + \alpha}. \tag{3.4}$$

Choosing $x_\alpha(v) = x(0)e^{-\alpha v} + (E_{-\alpha} * \psi)v$, we have $x_\alpha(0) = x(0)$, where $E_{-\alpha}(v) = e^{-\alpha v}$, therefore

$$M(x_\alpha) = \frac{x(0)u + M(\psi)}{u + \alpha} = \frac{x_\alpha(0)u + M(\psi)}{u + \alpha}.$$

Then, we have

$$M[x'_\alpha(v) + \alpha x_\alpha(v)] = M(\psi).$$

Since M is injective, $x'_\alpha(v) + \alpha x_\alpha(v) = \psi(v)$. Therefore, x_α is a solution of (3.2). Using (3.3) and (3.4), one can have

$$M[E_{-\alpha} * \lambda] = \frac{M(\lambda)}{u + \alpha}.$$

Hence, $M(x) - M(x_\alpha) = M[E_{-\alpha} * \lambda]$ and

$$x(v) - x_\alpha(v) = (E_{-\alpha} * \lambda)(v).$$

By (3.2), we get $|\lambda(v)| \leq \epsilon$ and by using convolution theorem on Mahgoub transforms, we get

$$|x(v) - x_\alpha(v)| = |(E_{-\alpha} * \lambda)(v)| \leq \epsilon e^{-R(\alpha)v} \int_0^v e^{-R(\alpha)s} ds.$$

This completes the proof. □

Theorem 3.4. Let α_i be scalars, where $i = 0, 1, \dots, n - 1$ and an integer $n > 1$. Then there exists a constant $N > 0$ such that for each mapping $x : (0, \infty) \rightarrow \mathbb{T}$ satisfying the equality

$$\left| x^{(n)}(v) + \sum_{\kappa=0}^{n-1} \alpha_\kappa x^{(\kappa)}(v) - \psi(v) \right| \leq \epsilon \tag{3.5}$$

for every $v > 0$ and $\epsilon > 0$, there exists $x_\alpha : (0, \infty) \rightarrow \mathbb{T}$ which is a solution of the differential equation (1.1) such that

$$|x_\alpha(v) - x(v)| \leq \epsilon N \frac{e^{\alpha v}}{\alpha}$$

for every $v > 0$ and $\alpha > \max[0, d_q, d_j]$, where d_q is defined in Lemma 3.1.

Proof. Using integration by parts repeatedly, we get

$$M[x^{(n)}] = u^n M(x) - \sum_{j=1}^n u^{n+1-j} x^{(j-1)}(0).$$

Let $a_n = 1$. So x_0 is a solution of (1.1) if and only if

$$M(\psi) = \rho_{n,0}(u)M(x_0) - \sum_{j=1}^n \rho_{n,j}(u)x_0^{(j-1)}(0)u, \tag{3.6}$$

where $\rho_{n,j}(u) = \sum_{\kappa=j}^n a_\kappa u^{\kappa-j}$ for $j = 0, 1, 2, \dots, n$. We consider

$$\mu(v) = x^n(v) + \sum_{\kappa=0}^{n-1} a_\kappa x^{(\kappa)}(v) - \psi(v)$$

for every $v > 0$. Then

$$M(\mu) = \rho_{n,0}(u)M(x) - \sum_{j=1}^n \rho_{n,j}(u)x^{(j-1)}(0)u - M(\psi).$$

Hence we get

$$M(x) - \frac{1}{\rho_{n,0}(u)} \left[\sum_{j=1}^n \rho_{n,j}(u)x^{(j-1)}(0)u + M(\psi) \right] = \frac{M(\mu)}{\rho_{n,0}(u)}. \tag{3.7}$$

Let d_j be the abscissa of convergence for ψ . Let u_1, u_2, \dots, u_n be the roots of $\rho_{n,0}$ and let

$$d_p = \max\{R(u_\kappa) : \kappa = 1, 2, \dots, n\}.$$

For any u with $R(u) > \max\{d_q, d_j\}$, we define

$$\Omega(u) = \frac{1}{\rho_{n,0}(u)} \left[\sum_{j=1}^n \rho_{n,j}(u)x^{(j-1)}(0)u + M(\psi) \right]. \tag{3.8}$$

By Lemma 3.2,

$$M(\psi_0) = \frac{M(\psi)}{\rho_{n,0}(u)}$$

for every u with $R(u) > \max\{d_q, d_j\}$ and $\psi_0(0) = \psi'_0(0) = \dots = \psi_0^{(n)}(0) = 0$ for $j = 1, 2, \dots, n$. So

$$\frac{\rho_{n,j}(u)}{\rho_{n,0}(u)} = \frac{1}{u^j} - \frac{\sum_{\kappa=0}^{j-1} a_\kappa u^\kappa}{u^j (\rho_{n,0}(u))} \tag{3.9}$$

for all u with $R(u) > \max\{0, d_q\}$. By Lemma 3.1, $Q_2(u) = \sum_{\kappa=0}^{j-1} a_\kappa u^\kappa$ and $Q_1(u) = u^j \rho_{n,0}(u)$. For a differentiable function ϕ_j ,

$$M(\phi_j) = \frac{\sum_{\kappa=0}^{j-1} a_\kappa u^\kappa}{u^j \rho_{n,0}(u)}$$

and $\phi_j(0) = \phi_j'(0) = \dots = \phi_j^{(n)} = 0$. Let

$$\psi_j(v) = \frac{v^{j-1}}{(j-1)!} - \phi_j(v)u \tag{3.10}$$

for $j = 1, 2, \dots, n$. Then we get

$$\psi_j^{(i)}(0) = \begin{cases} 0, & i = 0, 1, 2, \dots, j-2, j, j+1, \dots, n, \\ 1, & i = j-1. \end{cases}$$

If we define

$$x_a(v) = \sum_{j=1}^n x^{j-1}(0)\psi_j(v) + \psi_0(v),$$

then we get $x_a^{(i)}(0) = x^{(i)}(0)$ for all $i = 0, 1, 2, \dots, n$. Using (3.8) to (3.10) that $M(x_a) = \Omega(u)$ and we get

$$M(x_a) = \frac{1}{\rho_{n,0}(u)} \left[\sum_{j=1}^n \rho_{n,j}(u)x_a^{(j-1)}(0)u + M(\phi) \right] \tag{3.11}$$

for all u with $R(u) > \max\{0, d_q, d_j\}$. Now using (3.6) we get x_0 is a solution of (1.1). Again considering (3.7) and (3.11), we get

$$M(x) - M(x_a) = \frac{M(\mu)}{\rho_{n,0}(u)}$$

and so

$$|x(v) - x_a(v)| = \left| M^{-1} \left(\frac{M(\mu)}{\rho_{n,0}(u)} \right) \right|$$

for $v > 0$. Using (3.5) and the definition of μ , we get that $|\mu(v)| \leq \epsilon$ for all $v > 0$ and so

$$|M(\mu)| \leq \int_0^\infty |e^{-uv}| |\mu(v)| dv \leq \frac{\epsilon}{R(u)}$$

for all u with $R(u) > 0$. Finally, it follows from the formula for the inverse Mahgoub transform that

$$\begin{aligned} |x(v) - x_a(v)| &= \left| M^{-1} \left(\frac{M(\mu)}{\rho_{n,0}(u)} \right) \right| \\ &= \frac{1}{2\pi} \left| \int_{-\infty}^\infty \frac{e^{(a+ix)v} M(\mu)(a+ix)}{(a+ix)\rho_{n,0}(a+ix)} dx \right| \\ &\leq \frac{1}{2\pi} \int_{-\infty}^\infty e^{av} \frac{\epsilon}{a |a+ix| |\rho_{n,0}(a+ix)|} dx \\ &\leq \frac{\epsilon e^{av}}{2\pi a} \int_{-\infty}^\infty \frac{1}{|a+ix| |\rho_{n,0}(a+ix)|} dx \leq \epsilon N \frac{e^{av}}{a} \end{aligned}$$

for every $v > 0$ and any $a > \max\{0, d_q, d_j\}$, where

$$N = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{|a+ix| |\rho_{n,0}(a+ix)|} dx < \infty$$

because $n > 1$ is an integer. □

4. Discussion

In this section, now we discuss the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation (1.1).

Definition 4.1 ([14]). The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k,$$

where $z, \alpha \in \mathbb{C}$ and $\operatorname{Re}(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + 1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k} = e^z.$$

Remark 4.2. If we replace ϵ by $E_\alpha(z)\epsilon$ in Theorem 3.4, then the differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam stability.

Remark 4.3. If we replace ϵ by $E_\alpha(z)\phi(t)\epsilon$ in Theorem 3.4, then the differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability.

5. Conclusion

In this paper, we proved the Hyers-Ulam stability of the linear differential equations of higher order with constant coefficient using Mahgoub transform method. That is, we established the sufficient criteria for Ulam's stability of the linear differential equation of n th order. Additionally, this paper also provides another new method to study the Hyers-Ulam stability of differential equations. Also, this is the first attempt of using Mahgoub transform to obtain the Ulam stability for the linear differential equation of higher order and this paper shows that the Mahgoub transform method is more convenient to study the Ulam's stability problem for the linear differential equation with constant coefficients.

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