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Generalizations of some weighted Opial-type inequalities in conformable calculus

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Abstract

In this paper, we prove new α-fractional inequalities of Opial type using conformable calculus. From our results we obtain classical integral inequalities as special cases.

Keywords: Opial type inequalities, conformable fractional calculus, Hölder inequalities, chain rule.

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1. Introduction

In 1960, Opial in [23] proved that

$$\int_a^b \left|f(x) f'(x)\right| dx \leqslant \frac{b-a}{4} \int_a^b \left(f'(x)\right)^2 dx, \tag{1.1}$$

where $f \in C^1[a, b]$ and $f(x) > 0$, $f(a) = f(b) = 0$. The constant $1/4$ is the best constant. Also he proved that

$$\int_0^b \left|f(x) f'(x)\right| dx \leqslant \frac{b}{2} \int_0^b \left(f'(x)\right)^2 dx,$$

where $f(0) = 0$.

In 1962, Beesack in [5] generalized (1.1) and proved that

$$\int_a^b |f(x)| \left|f'(x)\right| dx \leqslant \frac{1}{2} \int_a^b \frac{1}{h(x)} dx \int_a^b h(x) \left(f'(x)\right)^2 dx, \tag{1.2}$$

where f is an absolutely continuous function on $[a, b]$, $f(a) = 0$ and h is a continuous and positive function such that $\int_a^b \left(\frac{1}{h(x)}\right) dx < \infty$.

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In 1966, Yang in [27] presented a simple proof of inequality (1.2) and obtained an extension of the form

$$\int_a^b s(x)\,|f(x)|\,|f'(x)|\,dx \leqslant \frac{1}{2}\int_a^b \frac{1}{h(x)}dx \int_a^b s(x)\,h(x)\,(f'(x))^2\,dx,$$

where f is an absolutely continuous function on $[a, b]$ and $f(a) = 0$, and s is a bounded, nonincreasing and positive function on $[a, b]$, and h is a positive and continuous function such that $\int_a^b \left(\frac{1}{h(x)}\right) dx < \infty$. Also, Yong in [27] proved that

$$\int_a^b |f(x)|^\lambda\,|f'(x)|^\mu\,dx \leqslant \frac{\mu}{\lambda+\mu}(b-a)^\lambda \int_a^b |f'(x)|^{\lambda+\mu}\,dx, \text{ for } \lambda,\ \mu \geqslant 1,$$

where f is an absolutely continuous function on $[a,\ b]$ with $f(a) = 0$.

In 1967, Boyd and Wong in [8] proved that

$$\int_0^a s(x)\,|f(x)|^\lambda\,|f'(x)|\,dx \leqslant \frac{1}{\beta_0(\lambda+1)}\int_0^a r(x)\,|f'(x)|^{\lambda+1}\,dx,$$

where β_0 is the smallest eigenvalue of the boundary value problem

$$\left(r(x)\,(g'(x))^\lambda\right)' = \lambda s'(x)\,g^\lambda(x),$$

where $r,\ s \in C^1[0,\ a]$ are nonnegative functions such that

$$r(a)\,(g'(a))^\lambda = \lambda s'(a)\,g^\lambda(a) \text{ and } g(0) = 0 \text{ for } 0 < g' \in [0, a].$$

In 1968, Beesack and Das in [6] proved the following inequality

$$\int_0^a \nu(x)\,|f(x)|^\lambda\,|f'(x)|^\mu\,dx \leqslant K(a,\ \lambda,\ \mu)\int_0^a h(x)\,|f'(x)|^{\lambda+\mu}\,dx, \tag{1.3}$$

where f is an absolutely continuous function on $[0,\ a]$, $f(0) = 0$, f' is of constant sign, λ, μ are real numbers with $\lambda\mu > 0$ and either $\lambda+\mu < 0$ or $\lambda+\mu > 1$, h, ν are nonnegative measurable functions with $\int_0^a h^{\frac{-1}{\lambda+\mu-1}}(t)\,dt < \infty$, and

$$K(a,\ \lambda,\ \mu) = \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left(\int_0^a s^{\frac{\lambda+\mu}{\lambda}}(x)\,r^{\frac{-\mu}{\lambda}}(x)\left(\int_0^x r^{\frac{-1}{\lambda+\mu-1}}(u)\,du\right)^{\lambda+\mu-1}dx\right)^{\frac{\lambda}{\lambda+\mu}}.$$

In 1983, Yong in [28] proved that

$$\int_a^b r(x)\,|f(x)|^\lambda\,|f'(x)|^\mu\,dx \leqslant \frac{\mu}{\lambda+\mu}(b-a)^\lambda \int_a^b r(x)\,|f'(x)|^{\lambda+\mu}\,dx,$$

for $\lambda \geqslant 0$, $\mu \geqslant 1$, f is absolutely continuous on the interval $[a,\ b]$ such that $f(a) = 0$ and r is a bounded and positive function. For more details about Opial type inequalities, we refer readers to [3, 8, 13, 22, 23].

Various integral inequalities and their extensions are important in the study of qualitative behavior of differential equations (see, e.g., [7, 9, 10, 18, 19] for more details) and partial differential equations (see, e.g., [17, 20] for more details). Recently, Opial type inequalities and their extensions have become an important tool for all types of differential equations by using it to prove the uniqueness and existence of initial and boundary value problems.

By utilizing the conformable calculus, many authors proved some integral inequalities like Chebyshev's type [21, 26], Hardy's type [24], Hermite-Hadamard's type [2, 15, 16], and Iyengar's type [25].

The paper is organized as follows. In Section 2, we present some concepts on conformable calculus. In Section 3, we prove some Opial type inequalities for α-fractional differentiable functions and obtain the classical ones as special cases when $\alpha = 1$.

2. Preliminaries and basic lemmas

In this section, we present some basic definitions and lemmas on conformable calculus. The results are adapted from [14], for more details, we refer the reader to [1, 4, 14].

Definition 2.1. The conformable derivative of order α of a function $w : [0, \infty) \to \mathbb{R}$ is defined by

$$D_\alpha w(s) = \lim_{\epsilon \to 0} \frac{w\left(s + \epsilon s^{1-\alpha}\right) - w(s)}{\epsilon}, \quad \text{for all } s > 0,\ \alpha \in (0,1).$$

Definition 2.2. The conformable integral of order α of a function $w : [0, \infty) \to \mathbb{R}$ is defined by

$$(I_\alpha^a w)(x) = \int_a^x w(s)\, d_\alpha s = \int_a^x s^{\alpha-1} w(s)\, ds,\ 0 < \alpha \leqslant 1.$$

Theorem 2.3. *Let w and v be α-differentiable such that $x > 0$. Then for $\alpha \in (0,\ 1]$,*

1. $D_\alpha (aw + bv)(x) = aD_\alpha w(x) + bD_\alpha v(x)$;
2. $D_\alpha \left(x^\lambda\right) = \lambda x^{\lambda-\alpha}$ *for all* $\lambda \in \mathbb{R}$;
3. $D_\alpha(\theta) = 0$, *for all constant functions* $w(x) = \theta$;
4. $D_\alpha (wv)(x) = wD_\alpha v(x) + vD_\alpha w(x)$;
5. $D_\alpha \left(\frac{w}{v}\right)(x) = \frac{vD_\alpha w(x) - wD_\alpha v(x)}{v^2}$;
6. *if w is differentiable, then* $D_\alpha w(x) = x^{1-\alpha} \frac{dw(x)}{dx}$.

Lemma 2.4. *Let $v(x)$ be α-differentiable with respect to x and w be α-differentiable with respect to v. Then the chain rule by using conformable derivative is defined by*

$$D_\alpha w(v(x)) = v^{\alpha-1}(x)\left(D_\alpha w(v(x))\right) D_\alpha v(x). \tag{2.1}$$

Lemma 2.5. *Let w and v be α-differentiable with respect to x on $[a,\ b]$. Then the integration by parts using conformable calculus is defined as*

$$\int_a^b D_\alpha^a w(x) v(x)\, d_\alpha x = w(x) v(x)|_a^b - \int_a^b w(x)\left(D_\alpha^a v(x)\right) d_\alpha x.$$

Lemma 2.6. *Let $0 < \alpha \leqslant 1$ and $w, v : [a,\ b] \to \mathbb{R}$. Then the Hölder inequality by using conformable integral is defined by*

$$\int_a^b |w(x) v(x)|\, d_\alpha x \leqslant \left(\int_a^b |w(x)|^\beta\, d_\alpha x\right)^{\frac{1}{\beta}} \left(\int_a^b |v(x)|^\gamma\, d_\alpha x\right)^{\frac{1}{\gamma}}, \tag{2.2}$$

for $\frac{1}{\beta} + \frac{1}{\gamma} = 1$ *and* $\beta > 1$.

3. Main results

Theorem 3.1. *Let $\lambda, \mu \in \mathbb{R}^+$ such that $\lambda + \mu > 1$, $a, x \in \mathbb{R}$, g, h be nonnegative continuous functions on (a, x) with $\int_a^x g^{\frac{-1}{\lambda+\mu-1}}(s)\, d_\alpha s < \infty$, and $\Phi : [a, x] \to \mathbb{R}$ be α^{th} differentiable with $D_\alpha \Phi$ of constant sign in (a, x) and $\Phi(a) = 0$. Then*

$$\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu\, d_\alpha t \leqslant K_1(a,\ x,\ \lambda,\ \mu) \int_a^x g(t) |D_\alpha \Phi(t)|^{\lambda+\mu}\, d_\alpha t, \tag{3.1}$$

where

$$K_1(a,\ x,\ \lambda,\ \mu) = \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}} \left[\int_a^x h^{\frac{\lambda+\mu}{\lambda}}(t) \left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\lambda+\mu-1} d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}.$$

Proof. Suppose that

$$|\Phi(t)| = \int_a^t |D_\alpha \Phi(s)|\, d_\alpha s = \int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} g^{\frac{1}{\lambda+\mu}}(s) |D_\alpha \Phi(s)|\, d_\alpha s.$$

Since g is nonnegative on (a, x), then by using the Hölder inequality (2.2) such that $\beta = \lambda+\mu$ and $\gamma = \frac{\lambda+\mu}{\lambda+\mu-1}$ and

$$w(s) = g^{\frac{1}{\lambda+\mu}}(s) |D_\alpha \Phi(s)|, \text{ and } v(s) = \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)},$$

where that

$$\int_a^t |D_\alpha \Phi(s)|\, d_\alpha s \leqslant \left(\int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} d_\alpha s\right)^{\frac{1}{\lambda+\mu}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda+\mu-1}{\lambda+\mu}}.$$

This leads to

$$|\Phi(t)|^\lambda \leqslant \left(\int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} d_\alpha s\right)^{\frac{\lambda}{\lambda+\mu}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}.$$

Let

$$\Omega(t) := \int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} d_\alpha s.$$

Then $\Omega(a) = 0$, and

$$D_\alpha \Omega(t) = g(t) |D_\alpha \Phi(t)|^{\lambda+\mu} > 0.$$

Hence, we have that

$$|D_\alpha \Phi(t)|^\mu = \left(\frac{D_\alpha \Omega(t)}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \text{ and } |D_\alpha \Phi(t)|^{\lambda+\mu} = \frac{D_\alpha \Omega(t)}{g(t)}. \tag{3.2}$$

Since h is a nonnegative function on (a, x), then by using (3.2) we find that

$$\begin{aligned} h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu &\leqslant h(t) |D_\alpha \Phi(t)|^\mu \left(\int_a^t g(s) |D_\alpha \Phi(s)|^{\lambda+\mu} d_\alpha s\right)^{\frac{\lambda}{\lambda+\mu}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} \\ &= \Omega^{\frac{\lambda}{\lambda+\mu}}(t) (D_\alpha \Omega(t))^{\frac{\mu}{\lambda+\mu}} h(t) \left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}. \end{aligned}$$

Integrating the above inequality from a to x, we have

$$\begin{aligned} &\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu d_\alpha t \\ &\leqslant \int_a^x \Omega^{\frac{\lambda}{\lambda+\mu}}(t) (D_\alpha \Omega(t))^{\frac{\mu}{\lambda+\mu}} h(t) \left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} d_\alpha t. \end{aligned} \tag{3.3}$$

By employing the Hölder inequality (2.2) on the right side of integral of (3.3) with $\beta = (\lambda+\mu)/\mu$ and $\gamma = (\lambda+\mu)/\lambda$, we have

$$\begin{aligned} &\int_a^x h(t) |\Phi(t)|^\lambda |D_\alpha \Phi(t)|^\mu d_\alpha t \\ &\leqslant \left[\int_a^x \Omega^{\frac{\lambda}{\mu}}(t) (D_\alpha \Omega(t))\, d_\alpha t\right]^{\frac{\mu}{\lambda+\mu}} \left[\int_a^x h^{\frac{\lambda+\mu}{\lambda}}(t) \left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}} \left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\lambda+\mu-1} d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}. \end{aligned} \tag{3.4}$$

By using the chain rule (2.1), we get that

$$\begin{aligned}D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right) &= D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}\right)(\Omega(t))\, D_\alpha(\Omega(t))\,\Omega^{\alpha-1}(t)\\ &= \frac{\lambda+\mu}{\mu}\Omega^{\frac{\lambda+\mu}{\mu}-\alpha}(t)\, D_\alpha(\Omega(t))\,\Omega^{\alpha-1}(t) = \frac{\lambda+\mu}{\mu}\Omega^{\frac{\lambda}{\mu}}(t)\, D_\alpha(\Omega(t)).\end{aligned}$$

Then we have

$$\Omega^{\frac{\lambda}{\mu}}(t)\, D_\alpha(\Omega(t)) = \frac{\mu}{\lambda+\mu} D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right). \tag{3.5}$$

Since $\Omega(a) = 0$ and from (3.4) and (3.5), we deduce that

$$\begin{aligned}&\int_a^x h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\, d_\alpha t\\ &\leqslant \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left[\int_a^x D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right) d_\alpha t\right]^{\frac{\mu}{\lambda+\mu}}\left[\int_a^x h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}}\left(\int_a^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\lambda+\mu-1} d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}\\ &= K_1(a,\ x,\ \lambda,\ \mu)\int_a^x g(t)\,|D_\alpha\Phi(t)|^{\lambda+\mu}\, d_\alpha t,\end{aligned}$$

which is the inequality (3.1). The proof is complete. □

Corollary 3.2. *In Theorem 3.1, if $\alpha = 1$ and $a = 0$, then we get*

$$\int_0^x h(t)\,|\Phi(t)|^\lambda\,\left|\Phi'(t)\right|^\mu dt \leqslant K_2(x,\ \lambda,\ \mu)\int_0^x g(t)\,\left|\Phi'(t)\right|^{\lambda+\mu} dt,$$

where

$$K_2(x,\ \lambda,\ \mu) = \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left[\int_0^x h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}}\left(\int_0^t \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} ds\right)^{\lambda+\mu-1} dt\right]^{\frac{\lambda}{\lambda+\mu}},$$

which is the inequality of (1.3).

Theorem 3.3. *Let $\lambda,\ \mu \in \mathbb{R}^+$ such that $\lambda+\mu > 1$, $x,\ b \in \mathbb{R}$, $g,\ h$ be nonnegative continuous functions on (x, b) with $\int_x^b g^{\frac{-1}{\lambda+\mu-1}}(s)\, d_\alpha s < \infty$, and $\Phi : [x, b] \to \mathbb{R}$ be α^{th} differentiable with $D_\alpha\Phi$ of constant sign in (x, b) and $\Phi(b) = 0$. Then*

$$\int_x^b h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\, d_\alpha t \leqslant K_3(x,\ b,\ \lambda,\ \mu)\int_x^b g(t)\,|D_\alpha\Phi(t)|^{\lambda+\mu}\, d_\alpha t, \tag{3.6}$$

where

$$K_3(x,\ b,\ \lambda,\ \mu) = \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left[\int_x^b h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}}\left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\lambda+\mu-1} d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}.$$

Proof. Suppose that

$$|\Phi(t)| = \int_t^b |D_\alpha\Phi(s)|\, d_\alpha s = \int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu}}(s)} g^{\frac{1}{\lambda+\mu}}(s)\,|D_\alpha\Phi(s)|\, d_\alpha s.$$

Since h is nonnegative on (x, b), then by using the Hölder inequality (2.2) such that $\beta = \lambda + \mu$ and $\gamma = \frac{\lambda+\mu}{\lambda+\mu-1}$ and

$$w(s) = g^{\frac{1}{\lambda+\mu}}(s)\left|D_\alpha \Phi(s)\right|, \text{ and } v(s) = \frac{1}{r^{\frac{1}{\lambda+\mu}}(s)},$$

where that

$$\int_t^b \left|D_\alpha \Phi(s)\right| d_\alpha s \leqslant \left(\int_t^b g(s)\left|D_\alpha \Phi(s)\right|^{\lambda+\mu} d_\alpha s\right)^{\frac{1}{\lambda+\mu}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda+\mu-1}{\lambda+\mu}}.$$

This gets us that

$$|\Phi(t)|^\lambda \leqslant \left(\int_t^b g(s)\left|D_\alpha \Phi(s)\right|^{\lambda+\mu} d_\alpha s\right)^{\frac{\lambda}{\lambda+\mu}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}.$$

Letting

$$\Omega(t) := \int_t^b g(s)\left|D_\alpha \Phi(s)\right|^{\lambda+\mu} d_\alpha s,$$

then we see that $\Omega(b) = 0$, and

$$D_\alpha \Omega(t) = -g(t)\left|D_\alpha \Phi(t)\right|^{\lambda+\mu} < 0.$$

Hence we have that

$$\left|D_\alpha \Phi(t)\right|^\mu = \left(\frac{-D_\alpha \Omega(t)}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \text{ and } \left|D_\alpha \Phi(t)\right|^{\lambda+\mu} = \frac{-D_\alpha \Omega(t)}{g(t)}. \tag{3.7}$$

Since h is a nonnegative function on (x, b), then by using (3.7) we find that

$$\begin{aligned} h(t)|\Phi(t)|^\lambda \left|D_\alpha \Phi(t)\right|^\mu &\leqslant h(t)\left|D_\alpha \Phi(t)\right|^\mu \left(\int_t^b g(s)\left|D_\alpha \Phi(s)\right|^{\lambda+\mu} d_\alpha s\right)^{\frac{\lambda}{\lambda+\mu}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} \\ &= \Omega^{\frac{\lambda}{\lambda+\mu}}(t)\left(-D_\alpha \Omega(t)\right)^{\frac{\mu}{\lambda+\mu}} h(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}}. \end{aligned}$$

Integrating the above inequality from x to b, we have

$$\begin{aligned} &\int_x^b h(t)|\Phi(t)|^\lambda \left|D_\alpha \Phi(t)\right|^\mu d_\alpha t \\ &\leqslant \int_x^b \Omega^{\frac{\lambda}{\lambda+\mu}}(t)\left(-D_\alpha \Omega(t)\right)^{\frac{\mu}{\lambda+\mu}} h(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda+\mu}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\frac{\lambda(\lambda+\mu-1)}{\lambda+\mu}} d_\alpha t. \end{aligned} \tag{3.8}$$

By using the Hölder inequality (2.2) on the right side of integral of (3.8) such that $\beta = (\lambda+\mu)/\mu$ and $\gamma = (\lambda+\mu)/\lambda$, we have

$$\begin{aligned} &\int_x^b h(t)|\Phi(t)|^\lambda \left|D_\alpha \Phi(t)\right|^\mu d_\alpha t \\ &\leqslant \left[\int_x^b \Omega^{\frac{\lambda}{\mu}}(t)\left(-D_\alpha \Omega(t)\right) d_\alpha t\right]^{\frac{\mu}{\lambda+\mu}} \left[\int_x^b h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}} \left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)} d_\alpha s\right)^{\lambda+\mu-1} d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}. \end{aligned} \tag{3.9}$$

Using the chain rule (2.1), we get that

$$D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right) = D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}\right)(\Omega(t))\,D_\alpha(\Omega(t))\,\Omega^{\alpha-1}(t)$$
$$= \frac{\lambda+\mu}{\mu}\Omega^{\frac{\lambda+\mu}{\mu}-\alpha}(t)\,D_\alpha(\Omega(t))\,\Omega^{\alpha-1}(t) = \frac{\lambda+\mu}{\mu}\Omega^{\frac{\lambda}{\mu}}(t)\,D_\alpha(\Omega(t)).$$

Then we have

$$\Omega^{\frac{\lambda}{\mu}}(t)\,D_\alpha(\Omega(t)) = \frac{\mu}{\lambda+\mu}D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right). \tag{3.10}$$

Since $\Omega(b) = 0$ and from (3.9) and (3.10), we deduce that

$$\int_x^b h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\,d_\alpha t$$
$$\leqslant \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left[-\int_x^b D_\alpha\left(\Omega^{\frac{\lambda+\mu}{\mu}}(t)\right)d_\alpha t\right]^{\frac{\mu}{\lambda+\mu}}\left[\int_x^b h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}}\left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)}d_\alpha s\right)^{\lambda+\mu-1}d_\alpha t\right]^{\frac{\lambda}{\lambda+\mu}}$$
$$= K_3(x,\ b,\ \lambda,\ \mu)\int_x^b g(t)\,|D_\alpha\Phi(t)|^{\lambda+\mu}\,d_\alpha t,$$

which is the inequality (3.6). The proof is complete. □

Corollary 3.4. *In Theorem 3.3, if $\alpha = 1$, then we have the following inequality*

$$\int_x^b h(t)\,|\Phi(t)|^\lambda\,\left|\Phi'(t)\right|^\mu dt \leqslant K_4(x,\ b,\ \lambda,\ \mu)\int_x^b g(t)\,\left|\Phi'(t)\right|^{\lambda+\mu}dt,$$

where

$$K_4(x,\ b,\ \lambda,\ \mu) = \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{\mu}{\lambda+\mu}}\left[\int_x^b h^{\frac{\lambda+\mu}{\lambda}}(t)\left(\frac{1}{g(t)}\right)^{\frac{\mu}{\lambda}}\left(\int_t^b \frac{1}{g^{\frac{1}{\lambda+\mu-1}}(s)}ds\right)^{\lambda+\mu-1}dt\right]^{\frac{\lambda}{\lambda+\mu}}.$$

Assume that there exists $x \in (a,\ b)$ which is the unique solution of the equation

$$K(\lambda,\ \mu) = K_1(a,\ x,\ \lambda,\ \mu) = K_3(x,\ b,\ \lambda,\ \mu) < \infty,$$

where $K_1(a, x, \lambda, \mu)$ and $K_3(x, b, \lambda, \mu)$ are given in Theorems 3.1 and 3.3, now since

$$\int_a^b h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\,d_\alpha t = \int_a^x h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\,d_\alpha t + \int_x^b h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\,d_\alpha t,$$

then we have the following theorem.

Theorem 3.5. *Let $\lambda, \mu \in \mathbb{R}^+$ such that $\lambda\mu > 0$ and $\lambda+\mu > 1$, $a, b \in \mathbb{R}$, g, h be nonnegative continuous functions on (a, b) with $\int_a^b g^{\frac{-1}{\lambda+\mu-1}}(s)\,d_\alpha s < \infty$, and $\Phi : [a, b] \to \mathbb{R}$ be α^{th} differentiable thus $D_\alpha\Phi$ of constant sign in (a, b), and $\Phi(a) = 0 = \Phi(b)$. Then*

$$\int_a^b h(t)\,|\Phi(t)|^\lambda\,|D_\alpha\Phi(t)|^\mu\,d_\alpha t \leqslant K(\lambda,\ \mu)\int_a^b g(t)\,|D_\alpha\Phi(t)|^{\lambda+\mu}\,d_\alpha t.$$

Proof. The proof can be obtained by making a combination of the proof of Theorems 3.1 and 3.3. □

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