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Uniform asymptotic stability of q-deformed conformable fractional systems with delay and application

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Abstract

In this article, we initiate the study of new concepts of conformable q-fractional calculus. The conformable fractional q-derivative and q-integral are defined and their fundamental theorems are also proved. The uniform asymptotic stability of the q-deformed conformable fractional system with constant delay is investigated by using the Lyapunov-Razumikhin method. For application, a new asymptotic stability necessary condition for the conformable q-fractional linear system with constant delay is obtained in term of linear matrix inequality (LMI). A numerical example is demonstrated for the results given to illustrate the effectiveness.

Keywords: q-deformed conformable fractional system, Lyapunov-Razumikhin theorem, uniform asymptotic stability, linear matrix inequality.

2020 MSC: 34K20, 34K25, 34K37.

1. Introduction

Many related properties and formulas in different fields ranging from natural science to social science of classical calculus are well-known to have been investigated extensively and intensively by Newton and Leibniz. Dating in a sense back to the early eighteenth century, the well-known mathematician Leonhard Euler (1707-1783) made pioneering discoveries in quantum calculus, which is focused on the study of infinite series and is the only beginning to see more usefulness in many branches of mathematical analysis as seen recently. With these fascinating ideas, Jackson [13, 14] introduced quantum derivative and quantum integral in 1910. Moreover, Carmichael [6] published the general theory of linear quantum difference equations which is devoted to an investigation of the existence and properties of solutions of linear quantum difference equations, see [16] for more details. In particular, quantum calculus has been applied in

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physics such as q-coulomb, q-hydrogen atom, general relativity, molecular, nuclear spectroscopy, chemical physics, and string theory.

In addition, Tariboon and Ntouyas [30] initiated the study of quantum calculus on finite intervals, which defined the q_k-derivative and q_k-integral of a function. Fractional calculus was the study derivative and integral of real or complex order. Recently, many researchers have been interested in this study and expanded the field of applications such as engineering, physics, chemistry, and mechanics. Interesting some recent articles on time-fractional equations can be found in [3–5, 22, 23]. The conformable fractional implicated a limit instead of an integral, which can be found in [1, 17, 24] for more information. Khalil [17] introduced the conformable fractional derivative to the product and quotient rules. We refer the readers to see also [33, 34], and the references cited therein. Howsoever, no results on the uniform asymptotic stability for q-deformed conformable fractional with delay have not been apprised to the best of the creators' knowledge. This is the driving force behind our current inquisition. The q-deformed conformable fractional was first studied by Chung [7]. We observed that in this case $\alpha = 1$ after that q-deformed conformable fractional reduces to Jackson q-derivative. It also has special functions and applies a deformed transform see in [11].

Time delay systems or delay differential equations [9] were often seen in many engineering systems, chemistry, biology, and economics. At present, many authors [20, 35] established the stability of problems of nonlinear fractional systems. There are also many authors [19, 21] who introduced the finite-time stability of fractional differential equations with time delay based on real problems considering the autonomous and non-autonomous fractional differential systems. Moreover, many researchers have been interested in studying about various stability criteria for a fractional neural network with time-delays [10, 26–29]. In [15], Jarad et al. initiated stability of q-fractional non-autonomous systems by using Lyapunov's direct method. In [2], a Caputo type q-fractional initial value problem was solved, and its solution was expressed by q-Mittag–Leffler function.

A few years later, Koca and Demirci [18] introduced local asymptotic stability of q-fractional nonlinear dynamical systems, which are interested in q-fractional order nonlinear dynamical model. In 2017, Abdourazek Souahi [25] introduced the stability and asymptotic stability of conformable fractional-order nonlinear systems by using the Lyapunov function. Lyapunov-Razumikhin functional was born in the framework of Teel [31] who studied input of state stability of time-delay systems in terms of the Lyapunov functions. For other papers on the subject, see [8, 12].

In this article, the new definitions of conformable q-fractional calculus are proposed. Some fundamental basic properties of conformable fractional q-derivative and q-integration are proved. Moreover, Lyapunov-Razumikhin stability theorem for q-deformed conformable fractional system with constant delay is presented. The nonlinear inequalities and some inequalities are used in condition with a Lyapunov-Razumikhin functional. Moreover, a delay-dependent asymptotic stability criterion for conformable q-fractional linear system with constant delay is obtained and formulated in the form of linear matrix inequality. Finally, an example is given to present that it is very convenient to check the stability of a practical system by using our proposed method.

2. Preliminaries

In this section, we recommend some basic definitions of the conformable fractional calculus of order $0 < \alpha \leqslant 1$.

Definition 2.1 ([24])**.** The conformable fractional derivative of a differentiable function $\vartheta : [s_0, \infty) \to \mathbb{R}$ is given by

$$ {}_{s_0}D^{\alpha}\vartheta(s) = \lim_{\epsilon \to 0} \frac{\vartheta(s + \epsilon(s - s_0)^{1-\alpha}) - \vartheta(s)}{\epsilon}, \quad s > s_0, $$

and ${}_{s_0}D^{\alpha}\vartheta(s) = (s - s_0)^{1-\alpha}\vartheta'(s)$.

Definition 2.2 ([24])**.** The conformable fractional integral of a integrable function $\vartheta : [s_0, \infty) \to \mathbb{R}$ is given by

$$_{s_0}I^{\alpha}\vartheta(s) = \int_{s_0}^{s} (\theta - s_0)^{\alpha-1}\vartheta(\theta)d\theta, \quad s > s_0.$$

Next, we present the new definitions of conformable q-fractional calculus. The q-number of $[n]_q$ is given by

$$[n]_q = \frac{1-q^n}{1-q},$$

where $0 < q < 1$ and $n \in \mathbb{R}$. For example, $[3]_q = 1 + q + q^2$.

Definition 2.3. Let ϑ be a continuous function on $[s_0, \infty)$. Then the conformable fractional q-derivative of order $\alpha \in (0, 1]$ is defined by

$$_{s_0}D_q^{\alpha}\vartheta(s) = \frac{[\alpha]_q(\vartheta(s) - \vartheta(qs + (1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}} = (s-s_0)^{1-\alpha}{}_{s_0}D_q\vartheta(s), \quad s > s_0,$$

where

$$_{s_0}D_q\vartheta(s) = \frac{\vartheta(s) - \vartheta(qs + (1-q)s_0)}{(1-q)(s-s_0)}$$

and

$$_{s_0}D_q^{\alpha}\vartheta(s_0) = \lim_{s\to s_0} {}_{s_0}D_q^{\alpha}\vartheta(s).$$

Definition 2.4. Let ϑ be a continuous function on $[s_0, \infty)$. Then the conformable fractional q-integral of order $\alpha \in (0, 1]$ is defined by

$$\int_{s_0}^{s} \vartheta(\theta)d_q^{\alpha}\theta = \frac{1}{[\alpha]_q}(1-q^{\alpha})(s-s_0)^{\alpha}\sum_{i=0}^{\infty} q^{i\alpha}\vartheta(q^i s + (1-q^i)s_0) = \int_{s_0}^{s}(\theta - s_0)^{\alpha-1}\vartheta(\theta)d_q\theta,$$

where $\int_{s_0}^{s} \vartheta(\theta)d_q\theta = (1-q)(s-s_0)\sum_{i=0}^{\infty} q^i\vartheta(q^i s + (1-q^i)s)$.

Example 2.5. Let $\vartheta(s) = (s-s_0)^{\beta}$ with $s \geqslant s_0$, where β is a constant, then

$$_{s_0}D_q^{\alpha}\vartheta(s) = {}_{s_0}D_q^{\alpha}(s-s_0)^{\beta} = \frac{[\alpha]_q\left((s-s_0)^{\beta} - (qs+(1-q)s_0 - s_0)^{\beta}\right)}{(1-q^{\alpha})(s-s_0)^{\alpha}} = [\alpha]_q\frac{(1-q^{\beta})}{(1-q^{\alpha})}(s-s_0)^{\beta-\alpha}.$$

Example 2.6. Let $\vartheta(s) = (s-s_0)^{\gamma}$ with $s \geqslant s_0$, where γ is a constant, then

$$\begin{aligned}\int_{s_0}^{s} \vartheta(\theta)d_q^{\alpha}\theta &= \frac{1}{[\alpha]_q}(1-q^{\alpha})(s-s_0)^{\alpha}\sum_{i=0}^{\infty} q^{i\alpha}(q^i s + (1-q^i)s_0 - s_0)^{\gamma} \\ &= \frac{1}{[\alpha]_q}(1-q^{\alpha})(s-s_0)^{\alpha+\gamma}\sum_{i=0}^{\infty} q^{(\alpha+\gamma)i} = \frac{1}{[\alpha]_q}\frac{(1-q^{\alpha})}{(1-q^{(\alpha+\gamma)})}(s-s_0)^{\alpha+\gamma}.\end{aligned}$$

Theorem 2.7. *Suppose that* $\vartheta, \rho : [s_0, \infty) \to \mathbb{R}$ *is* q*-differentiable on* $[s_0, \infty)$*. Then*

(i) $\vartheta + \rho : [s_0, \infty) \to \mathbb{R}$ *is* q*-differentiable on* $[s_0, \infty)$*, and*

$$_{s_0}D_q^{\alpha}(\vartheta(s) + \rho(s)) = {}_{s_0}D_q^{\alpha}\vartheta(s) + {}_{s_0}D_q^{\alpha}\rho(s);$$

(ii) $\lambda\vartheta : [s_0, \infty) \to \mathbb{R}$ *is* q*-differentiable on* $[s_0, \infty)$ *for any constant* λ*, and*

$$_{s_0}D_q^{\alpha}(\lambda\vartheta)(s) = \lambda {}_{s_0}D_q^{\alpha}\vartheta(s);$$

(iii) $\vartheta\rho : [s_0,\infty) \to \mathbb{R}$ *is* q*-differentiable on* $[s_0,\infty)$*, and*

$$ {}_{s_0}D_q^{\alpha}(\vartheta\rho)(s) = \vartheta(qs+(1-q)s_0)\,{}_{s_0}D_q^{\alpha}\rho(s) + \rho(s)\,{}_{s_0}D_q^{\alpha}\vartheta(s); $$

(iv) *if* $\rho(s) \neq 0$*, then* $\frac{\vartheta}{\rho}$ *is* q*-differentiable on* $[s_0,\infty)$ *with*

$$ {}_{s_0}D_q^{\alpha}\left(\frac{\vartheta}{\rho}\right)(s) = \frac{\rho(s)\,{}_{s_0}D_q^{\alpha}\vartheta(s) - \vartheta(s)\,{}_{s_0}D_q^{\alpha}\rho(s)}{\rho(s)\rho(qs+(1-q)s_0)}. $$

Proof. (i) and (ii) are easy to prove. Therefor, we omit it. To prove (iii), from Definition 2.3, we have

$$ {}_{s_0}D_q^{\alpha}(\vartheta\rho)(s) = \frac{[\alpha]_q\left(\vartheta(s)\rho(s) - \vartheta(qs+(1-q)s_0)\rho(qs+(1-q)s_0)\right)}{(1-q^{\alpha})(s-s_0)^{\alpha}}. $$

By adding the term $\pm\vartheta(qs+(1-q)s_0)\rho(s)$, we obtain

$$ \begin{aligned} {}_{s_0}D_q^{\alpha}(\vartheta\rho)(s) &= \frac{[\alpha]_q\left(\vartheta(s)\rho(s) - \vartheta(qs+(1-q)s_0)\rho(s)\right)}{(1-q^{\alpha})(s-s_0)^{\alpha}} \\ &\quad + \frac{[\alpha]_q\left(\vartheta(qs+(1-q)s_0)\rho(s) - \vartheta(qs+(1-q)s_0)\rho(qs+(1-q)s_0)\right)}{(1-q^{\alpha})(s-s_0)^{\alpha}} \\ &= \rho(s)\left(\frac{[\alpha]_q(\vartheta(s)-\vartheta(qs+(1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}}\right) + \vartheta(qs+(1-q)s_0)\left(\frac{[\alpha]_q(\rho(s)-\rho(qs+(1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}}\right) \\ &= \vartheta(qs+(1-q)s_0)\,{}_{s_0}D_q^{\alpha}\rho(s) + \rho(s)\,{}_{s_0}D_q^{\alpha}\vartheta(s). \end{aligned} $$

To prove (iv), by using Definition 2.3, we obtain

$$ {}_{s_0}D_q^{\alpha}\left(\frac{\vartheta}{\rho}\right)(s) = \frac{[\alpha]_q\left(\frac{\vartheta(s)}{\rho(s)} - \frac{\vartheta(qs+(1-q)s_0)}{\rho(qs+(1-q)s_0)}\right)}{(1-q^{\alpha})(s-s_0)^{\alpha}}. $$

Adding the term $\pm\vartheta(s)\rho(s)$, we get

$$ \begin{aligned} {}_{s_0}D_q^{\alpha}\left(\frac{\vartheta}{\rho}\right)(s) &= \frac{[\alpha]_q(\vartheta(s)\rho(qs+(1-q)s_0) - \rho(s)\vartheta(qs+(1-q)s_0))}{\rho(s)\rho(qs+(1-q)s_0)(1-q^{\alpha})(s-s_0)^{\alpha}} \\ &= \frac{\rho(s)}{\rho(s)\rho(qs+(1-q)s_0)}\frac{[\alpha]_q(\vartheta(s)-\vartheta(qs+(1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}} \\ &\quad - \frac{\vartheta(s)}{\rho(s)\rho(qs+(1-q)s_0)}\frac{[\alpha]_q(\rho(s)-\rho(qs+(1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}} \\ &= \frac{\rho(s)\,{}_{s_0}D_q^{\alpha}\vartheta(s) - \vartheta(s)\,{}_{s_0}D_q^{\alpha}\rho(s)}{\rho(s)\rho(qs+(1-q)s_0)}. \end{aligned} $$

The proof is completed. □

Theorem 2.8. *Let* $\vartheta, \rho : [s_0,\infty) \to \mathbb{R}$ *be a continuous function. The following formulas hold:*

(i) ${}_{s_0}D_q^{\alpha}\int_{s_0}^{s}\vartheta(\theta)d_q^{\alpha}\theta = \vartheta(s)$;

(ii) $\int_{s_0}^{s}{}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta = \vartheta(s) - \vartheta(s_0)$;

(iii) $\int_{a}^{s}{}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta = \vartheta(s) - \vartheta(a)$*, where* $a \in (s_0, s)$.

Proof. To prove (i), by using Definitions 2.3 and 2.4, we get that

$$ {}_{s_0}D_q^{\alpha}\int_{s_0}^{s}\vartheta(\theta)d_q^{\alpha}\theta = {}_{s_0}D_q^{\alpha}\left[\frac{1}{[\alpha]_q}(1-q^{\alpha})(s-s_0)^{\alpha}\sum_{i=0}^{\infty}q^{i\alpha}\vartheta(q^i s + (1-q^i)s_0)\right] $$

$$
\begin{aligned}
&= \frac{(1-q^{\alpha})}{(1-q^{\alpha})(s-s_0)^{\alpha}}\Bigg[(s-s_0)^{\alpha}\sum_{i=0}^{\infty} q^{i\alpha}\vartheta(q^i s+(1-q^i)s_0) \\
&\quad -(qs+(1-q)s_0-s_0)^{\alpha}\sum_{i=0}^{\infty} q^{i\alpha}\vartheta(q^i(qs+(1-q)s_0)+(1-q^i)s_0)\Bigg] \\
&= \sum_{i=0}^{\infty} q^{i\alpha}\vartheta(q^i s+(1-q^i)s_0) - \sum_{i=0}^{\infty} q^{(1+i)\alpha}\vartheta(q^{1+i}s+(1-q^{1+i})s_0) = \vartheta(s).
\end{aligned}
$$

To claim that (ii) from Definitions 2.3 and 2.4, we obtain

$$
\begin{aligned}
\int_{s_0}^{s} {}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta &= \int_{s_0}^{s}\left(\frac{[\alpha]_q(\vartheta(\theta)-\vartheta(q\theta+(1-q)s_0))}{(1-q^{\alpha})(s-s_0)^{\alpha}}\right)d_q^{\alpha}\theta \\
&= (1-q^{\alpha})(s-s_0)^{\alpha}\sum_{i=0}^{\infty} q^{i\alpha}\left(\frac{\vartheta(q^i s+(1-q)s_0)-\vartheta(q(q^i s+(1-q^i)s_0)+(1-q)s_0)}{(1-q^i)(q^i s+(1-q^i s_0)-s_0)^{\alpha}}\right) \\
&= \sum_{i=0}^{\infty}\vartheta(q^i s+(1-q)s_0) - \sum_{i=0}^{\infty}\vartheta(q^{i+1}s+(1-q^{i+1}s_0)) = \vartheta(s)-\vartheta(s_0).
\end{aligned}
$$

The part (ii) of this theorem implies that

$$
\int_{a}^{s} {}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta = \int_{s_0}^{s} {}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta - \int_{s_0}^{a} {}_{s_0}D_q^{\alpha}\vartheta(\theta)d_q^{\alpha}\theta = \vartheta(s)-\vartheta(a).
$$

Therefore, (iii) is proved. □

Definition 2.9 ([32])**.** A function $\vartheta(x,y)$ is called a homogeneous equation of degree k if $\vartheta(rx,ry) = r^k\vartheta(x,y)$ for all x,y and $r>0$.

Lemma 2.10. *Given a function $w:[s_0,\infty)\to\mathbb{R}$ be α-differentiable, where $\alpha\in(0,1]$ and M is a symmetric positive definite matrix, then ${}_{s_0}D_q^{\alpha}w^T(s)Mw(s)$ exists on $[s_0,\infty)$ and*

$$
{}_{s_0}D_q^{\alpha}w^T(s)Mw(s) = w^T(qs+(1-q)s_0)M\,{}_{s_0}D_q^{\alpha}w(s) + w^T(s)M\,{}_{s_0}D_q^{\alpha}w(s)
$$

for all $s>s_0$.

Proof. Applying the Theorem 2.7, we have

$$
{}_{s_0}D_q^{\alpha}w^T(s)Mw(s) = w^T(qs+(1-q)s_0)M\,{}_{s_0}D_q^{\alpha}w(s) + w^T(s)M\,{}_{s_0}D_q^{\alpha}w(s).
$$

□

3. Main result

We assume $C([a,b],\mathbb{R}^n)$ is the set of functions mapping the interval $[a,b]$ to $\mathbb{R}^n$. In many cases, we may have to analyze the maximum time delay ι of a system. In this situation, the set of continuous functions mappings $[-\iota,0]$ to $\mathbb{R}^n$ is attentive, for which we simplify the notation to $C=C([-\iota,0],\mathbb{R}^n)$. For all $J>0$ and continuous function of time $\Phi\in C([s_0-\iota,s_0+J],\mathbb{R}^n)$, $s_0\leqslant s\leqslant s_0+J$, suppose that $\Phi_s\in C$ is a segment of function Φ given by $\Phi_s(\eta)=\Phi_s(s+\eta)$, where $-\iota\leqslant\eta\leqslant 0$.

We investigate the Lyapunov-Razumikhin theorem of uniform stability and uniform asymptotical stability for q-deformed conformable fractional system with constant delay of the form

$$
{}_{s_0}D_q^{\alpha}w(s) = f(s,w(s-\iota)), \tag{3.1}
$$

where $s \geqslant s_0, 0 < \alpha \leqslant 1, w(s) \in \mathbb{R}^n$ is the state vector and $f : \mathbb{R} \times C \to \mathbb{R}^n$. For each the solution $w(s)$ of (3.1), we let the initial condition

$$w(s_0 + \theta) = \Phi(\theta),$$

where $\theta \in [-\iota, 0]$ and $\Phi \in C$.

Definition 3.1 ([32]). The trivial solution of a system (3.1) is said to be stable if for every $\epsilon > 0$, there is an $\vartheta = (s_0, \epsilon) > 0$ such that $\|w_{s_0}\| < \vartheta$ implies $\|w_s\| < \epsilon$ for all $s > s_0$.

It is said to be asymptotically stable if it is stable and $s_0 \in \mathbb{R}$ and for every $\epsilon > 0$, there is an $\vartheta_0 = \vartheta_0(s_0, \epsilon) > 0$ such that $\|w_{s_0}\| < \vartheta_0$ implies $\lim_{s\to\infty} w(s) = 0$.

Theorem 3.2. *Let $\kappa_1, \kappa_2, \kappa_3 : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous non-decreasing functions, κ_1 and κ_2 be positive for all $\theta > 0$ and $\kappa_1(0) = \kappa_2(0) = 0$ with κ_2 being strictly increasing. If there is a differentiable functional $\nabla : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$ such that*

$$\kappa_1(\|w\|) \leqslant \nabla(s, w) \leqslant \kappa_2(\|w\|), \quad \text{for} \quad s \in \mathbb{R},$$

the q-derivative of ∇ with the solution $w(s)$ of conformable q-fractional systems (3.1) satisfies

$$_{s_0}D_q^\alpha \nabla(s, w(s)) \leqslant -\kappa_3(\|w(s)\|), \tag{3.2}$$

for $w \in \mathbb{R}^n$ and for all $s_0 \in \mathbb{R}$ and

$$\nabla(s + \zeta, w(s + \zeta)) \leqslant \nabla(s, w(s)),$$

for all $\zeta \in [-\iota, 0]$, then q-deformed conformable fractional system (3.1) is uniformly stable.

If $\kappa_3(\theta) > 0$ for all $\theta > 0$, and there is a continuous non-decreasing function $\varphi(\theta) > \theta$ for all $\theta > 0$ such that

$$_{s_0}D_q^\alpha \nabla(s, w(s)) \leqslant -\kappa_3(\|w(s)\|),$$

and

$$\nabla(s + \zeta, w(s + \zeta)) \leqslant \varphi(\nabla(s, w(s))),$$

for all $\zeta \in [-\iota, 0]$, then q-deformed conformable fractional system (3.1) is uniformly asymptotically stable.

Proof. We assume that $w(s) = w(s, s_0, \Phi), \nabla(s) = \nabla(s, w(s))$ and

$$\nabla^*(s) = \sup_{-\iota\leqslant\zeta\leqslant 0} \nabla(s + \zeta, w(s + \zeta)),$$

are functions. There exists $\hat{\zeta} \in [-\iota, 0]$ such that

$$\nabla^* = \nabla(s + \hat{\zeta}, w(s + \hat{\zeta})),$$

and either $\hat{\zeta} = 0$ or $\hat{\zeta} < 0$, and

$$\nabla(s + \zeta, w(s, \zeta)) \leqslant \nabla(s + \hat{\zeta}, w(s + \hat{\zeta})), \quad \text{for} \quad \hat{\zeta} \leqslant \zeta \leqslant 0.$$

Now, we will show that

$$_{s_0}D_q^\alpha \nabla^*(s, w(s)) \leqslant 0. \tag{3.3}$$

In the case $\hat{\zeta} < 0$, then for a function $\nabla^*(s + \Delta s, w(s + \Delta s)) = \nabla^*(s, w(s))$, there exists a $\Delta s > 0$, and thus $_{s_0}D_q^\alpha \nabla^*(s, w(s)) = 0$. From (3.2), if $\hat{\zeta} = 0$, then for a function $\nabla^*(s) = \nabla(s, w(s))$ and $_{s_0}D_q^\alpha \nabla^*(s, w(s)) =_{s_0} D_q^\alpha \nabla(s, w(s)) \leqslant 0$. Therefore, (3.3) holds and

$$\kappa_1(\|w(s)\|) \leqslant \nabla(s, w(s)) \leqslant \nabla^*(s, w(s)) \leqslant \nabla(s_0, w(s_0)) \leqslant \kappa_2(\|w(s_0)\|). \tag{3.4}$$

For all $\epsilon > 0$, there is $\vartheta > 0$ such that $\kappa_2(\vartheta) < \kappa_1(\epsilon)$. Suppose that $\|w_{s_0}\| < \vartheta$. From (3.4), it follows that

$$\kappa_1(\|w(t)\|) \leqslant \kappa_2(\|w(s_0)\|) \leqslant \kappa_2(\|w_{s_0}\|) \leqslant \kappa_2(\vartheta) < \kappa_1(\epsilon).$$

This implies $\|w(s)\| < \epsilon$. Therefore, q-deformed conformable fractional system (3.1) is uniformly stable. We assume that $\vartheta > 0$ and $G > 0$ such that $\nu(\vartheta) = u(G)$. Since $\|\Phi\| \leqslant \vartheta$, it follows that $\|w_{s_0}\| \leqslant G$, $\nabla(s, w(s)) < \nu(\vartheta)$ for $s \geqslant s_0 - \iota$. Let $0 < \psi \leqslant G$ be arbitrary. From the function $\varphi(\theta)$, there exists $\zeta > 0$ such that $\varphi(\theta) - \theta > \zeta$ for $u(\psi) \leqslant \theta \leqslant \nu(\vartheta)$. We let the smallest integer M such that $u(\psi) + M_\zeta \geqslant \nu(\vartheta)$ and let $T = \frac{M\nu(\vartheta)}{\varpi}$, where $\varpi = \inf_{\psi \leqslant \theta \leqslant G} w(\theta)$.

We will show that

$$\nabla(s, w(s)) \leqslant u(\psi) + (M-1)\zeta,$$

for all $s \geqslant s_0 + (\frac{\nu(\vartheta)}{\varpi})$. If $u(\psi) + (M-1)\zeta < \nabla(s, w(s))$ for $s_0 - \iota \geqslant s < s_0 + (\frac{\nu(\vartheta)}{\varpi})$, then $\nabla(s, w(s)) \leqslant \nu(\vartheta)$ for all $s \geqslant s_0 - \iota$, we get

$$\varphi(\nabla(s, w(s))) > \nabla(s, w(s)) + \zeta \geqslant u(\psi) + M\zeta \geqslant \nu(\vartheta) \geqslant \nabla(s + \xi, w(s + \xi)), \quad s_0 - \iota \geqslant s \geqslant s_0 + \frac{\nu(\vartheta)}{\varpi},$$

where $\xi \in [-\iota, 0]$. Consequently, we have

$${}_{s_0}D_q^\alpha \nabla(s, w(s)) \leqslant -w(|w(s)|) \leqslant -\varpi,$$

for $s_0 \geqslant s < s_0 + (\frac{\nu(\vartheta)}{\varpi})$, and

$$\nabla(s, w(s)) \leqslant \nabla(s_0, w(s_0)) - \varpi(s - s_0) \leqslant \nu(\vartheta) - \varpi(s - s_0).$$

Then for an inequality $\nabla(s, w(s)) \leqslant u(\psi) + (M-1)\zeta$ at $s_1 = s_0 + \frac{\nu(\vartheta)}{\varpi}$ implies $\nabla(s, w(s)) \leqslant u(\psi) + (M-1)\zeta$ for all $s \geqslant s_0 + \frac{\nu(\vartheta)}{\varpi}$ since ${}_{s_0}D_q^\alpha \nabla(s, w(s))$ is negative, when $\nabla(s, w(s)) = u(\psi) + (M-1)\zeta$.

Next, let $\bar{s_j} = j\frac{\nu(\vartheta)}{\varpi}, j = 1, 2, \ldots, M, \bar{s_0}$ and for some integer $k \geqslant 1$, where $\bar{s}_{k-1} - r \leqslant s - s_0 \leqslant \bar{s_k}$, we have

$$u(\psi) + (M-k)\zeta \leqslant \nabla(s, w(s)) \leqslant u(\psi) + (M-k+1)\zeta.$$

Thus

$${}_{s_0}D_q^\alpha \nabla(s, w(s)) \leqslant -\varpi, \quad \bar{s}_{k-1} \leqslant s - s_0 \leqslant \bar{s_k},$$

and

$$\nabla(s, w(s)) \leqslant \nabla(s_0 + \bar{s}_{k-1}, w(s_0 + \bar{s}_{k-1})) - \varpi(s - s_0 - \bar{s}_{k-1}) \leqslant \nu(\vartheta) - \varpi(s - s_0 - \bar{s}_{k-1}) \leqslant 0,$$

where $s - s_0 - \bar{s}_{k-1} \geqslant \frac{\nu(\vartheta)}{\varpi}$. Therefore, $\nabla(s_0 + \bar{s}_{k-1}, w(s_0 + \bar{s}_{k-1})) \leqslant u(\psi) + (M-k)\zeta$, then $\nabla(s, w(s)) \leqslant u(\psi) + (M-k)\zeta$ for all $s \geqslant s_0 + \bar{s}_{k-1}$. Finally, $\nabla(s, w(s)) \leqslant u(\psi)$ for all $s \geqslant s_0 + M\frac{\nu(\vartheta)}{\varpi}$. Therefore, q-deformed conformable fractional system (3.1) is uniformly asymptotically stable. □

Consider the q-deformed conformable fractional linear system with constant delay and w is a homogeneous function of the form

$${}_0D_q^\alpha w(s) = -Aw(s) + Bf(w(s - \iota)), \quad s \geqslant 0, \tag{3.5}$$

where $0 < \alpha \leqslant 1$, $w(s) \in \mathbb{R}^n$ is the state vector, let A and B be known real constant matrices and let ι be a positive real constant. For each solution $w(s)$ of (3.5), we suppose the initial condition

$$w(s) = \Phi(s), \quad s \in [-\iota, 0],$$

where $\Phi \in C([-\iota, 0]; \mathbb{R}^n)$.

The uncertainty $f(.)$ is the non-linear parameter perturbation with respect to the state $w(s)$ and satisfies

$$f^T(w(s-\iota))f(w(s-\iota)) \leqslant \vartheta^2 w^T(s-\iota)w(s-\iota), \tag{3.6}$$

where ϑ is a given real constant.

Theorem 3.3. *For a given positive real constant* ϑ, (3.5) *is asymptotically stable if there exists symmetric positive definite matrix* K *and a positive real constant* ϵ *such that the following symmetric linear matrix inequality holds,*

$$\begin{bmatrix} -(1+q)KA + \iota\alpha K & 0 & KB \\ * & \epsilon\vartheta^2 I - \iota\alpha K & 0 \\ * & * & -\epsilon I \end{bmatrix} < 0. \tag{3.7}$$

Proof. We let the symmetric positive definite matrix K. Consider the Lyapunov functional of the form

$$\nabla(s) = w^T(s)Kw(s).$$

Taking the q-derivative of $\nabla(s)$ along the trajectory solution of the system (3.5) yields

$$\begin{aligned} {}_0D_q^\alpha \nabla(s) &= w^T(qs)K{}_0D_q^\alpha w(s) + w(s)^T K{}_0D_q^\alpha w(s) \\ &= (1+q)w^T(s)K{}_0D_q^\alpha w(s) = (1+q)w^T(s)K[-Aw(s) + Bf(w(s-\iota))]. \end{aligned}$$

From (3.6), we get

$$0 \leqslant \epsilon\vartheta^2 w^T(s-\iota)w(s-\iota) - \epsilon f(w(s-\iota))f^T(w(s-\iota)), \quad \text{for} \quad \epsilon > 0.$$

When $\nabla(s+\zeta, w(s+\zeta)) \leqslant \nabla(s, w(s))$ for all $\zeta \in [-\iota, 0]$, we get

$$0 \leqslant \iota\alpha w^T(s)Kw(s) - \iota\alpha w^T(s-\iota)Kw(s-\iota).$$

According to (3.5) and (3.7), it is straightforward to see that

$${}_{s_0}D_q^\alpha \nabla(s) \leqslant \xi^T(s) \begin{bmatrix} -(1+q)KA + \iota\alpha K & 0 & KB \\ * & \epsilon\vartheta^2 I - \iota\alpha K & 0 \\ * & * & -\epsilon I \end{bmatrix} \Omega(s),$$

where $\Omega(s) = \text{col}\{w(s), w(s-\iota), f(w(s-\iota))\}$. If (3.7) holds, then the system (3.5) is asymptotically stable. The proof of the theorem is completed. □

4. Numerical example

Example 4.1. Consider the following conformable q-fractional linear system of the form

$${}_{s_0}D_q^\alpha w(s) = -Aw(s) + Bf(w(s-\iota)). \tag{4.1}$$

We can solve the LMI (3.7) when

$$A = \begin{bmatrix} 1.9 & 0 \\ 0 & 0.8 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \iota = 0.3, \alpha = 0.7, q = 0.3,$$

we obtain the parameters $\epsilon = 1957$, $K = \begin{bmatrix} 7176 & -1924 \\ -1924 & 1328 \end{bmatrix}$ guaranteeing asymptotic stability of system (4.1) with $\vartheta = 0.2$.

Then, the maximum upper bound of the nonlinear parameter ϑ which guarantees the asymptotic stability of system (4.1) is 0.2840.

Table 1 represents the least upper bounds of the nonlinear parameter ϑ of this example for various values of ι, α, and q.

Table 1: Least upper bounds of the nonlinear parameter ϑ in Example 4.1.

	$q=\iota=0.2$	$q=\iota=0.3$	$q=\iota=0.4$
$\alpha=0.6$	0.2152	0.2673	0.3128
$\alpha=0.7$	0.2299	0.2840	0.3300
$\alpha=0.8$	0.2429	0.2984	0.3455

5. Conclusion

In this paper, we introduced the new definitions of conformable q-fractional calculus. The basic properties such as the conformable fractional q-derivative of a sum and of a product or quotient of two functions, q-integration by parts, and fundamental theorems were derived. We presented uniform asymptotic stability theorem for the q-deformed conformable fractional system with delay using the Lyapunov-Razumikhin method. We used the nonlinear inequality and some inequalities in condition with a Lyapunov-Razumikhin functional. Thus, new asymptotic stability necessary condition is constructed in terms of linear matrix inequality to guarantee asymptotic stability for q-deformed conformable fractional linear system with constant delay. Moreover, a numerical example illustrated our obtained result.

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References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66. 1
[2] T. Abdeljawad, D. Baleanu, *Caputo q-fractional initial value problems and a q-analogue mittag-leffler function*, Commun. Nonlinear Sci. Numer. Simul., **16** (2011), 4682–4688. 1
[3] T. Akram, M. Abbas, A. Ali, A. Iqbal, D. Baleanu, *A Numerical Approach of a Time Fractional Reaction-Diffusion Model with a Non-Singular Kernel*, Symmetry, **12** (2020), 16 pages. 1
[4] T. Akram, M. Abbas, A. Iqbal, D. Baleanu, J. H. Asad, *Novel Numerical Approach Based on Modified Extended Cubic B-Spline Functions for Solving Non-Linear Time-Fractional Telegraph Equation*, Symmetry, **12** (2020), 11 pages.
[5] M. Amin, M. Abbas, M. K. Iqbal, D. Baleanu, *Numerical Treatment of Time-Fractional Klein-Gordon Equation Using Redefined Extended Cubic B-Spline Fuctions*, Front. Phys., **8** (2020), 12 pages. 1
[6] R. D. Carmichael, *The general theory of linear q-difference equations*, Amer. J. Math., **32** (1912), 147–168. 1
[7] W. S. Chung, *On the q-deformed conformable fractional calculus and the q-deformed generalized conformable fractional calculus*, preprint, **2016** (2016), 11 pages. 1
[8] S. Dashkovskiy, L. Naujok, *Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems for interconnected ISS time-delay systems*, In Proceedings of the 19th international symposium on mathematical theory of networks and systems (MTNS), **2010** (2010), 5–9. 1
[9] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*, Birkhäuser/Springer, Cham, (2014). 1
[10] X. Han, M. Hymavathi, S. Sanober, B. Dhupia, M. S. Ali, *Robust Stability of Fractional Order Memristive BAM Neural Networks with Mixed and Additive Time Varying Delays*, Fract. Fract., **6** (2022), 20 pages. 1
[11] O. Herscovici, T. Mansour, *q-Deformed conformable fractional natural tranform*, arXiv, **2018** (2018), 16 pages. 1
[12] C. H. Hou, J. X. Qian, *On an estimate of the decay rate for applications of Razumikhin-type theorems*, IEEE Trans. Automat. Control, **43** (1998), 958–960. 1
[13] F. H. Jackson, *q-difference equations*, Amer. J. Math., **32** (1910), 305–314. 1
[14] D. O. Jackson, T. Fukuda, O. Dunn, E. Majors, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203. 1
[15] F. Jarad, T. Abdeljawad, D. Baleanu, *Stability of q-fractional non-autonomous systems*, Nonlinear Anal. Real World Appl., **14** (2013), 780–784. 1
[16] V. Kac, P. Cheung, *Quantum Calculus*, Springer-Verlag, New York, (2001). 1
[17] R. Khalil, A. L. M. Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70. 1
[18] I. Koca, E. Demirci, *On local asymptotic stability of q-fractional nonlinear dynamical systems*, Appl. Appl. Math., **11** (2016), 174–183. 1

[19] C. P. Li, F. R. Zhang, *A survey on the stability of fractional differential equations*, Eur. Phys. J. Special Topics, **193** (2011), 27–47. 1

[20] T. D. Liu, F. Wang, W. C. Lu, X. H. Wang, *Global stabilization for a class of nonlinear fractional-order systems*, Int. J. Model. Simul. Sci. Comput., **10** (2019), 11 pages. 1

[21] L. Liu, S. Zhong, *Finite-time stability analysis of fractional-order with multi-state time delay*, Int. J. Math. Comput. Sci., **5** (2011), 9 pages. 1

[22] A. Majeed, M. Kamran, M. Abbas, M. Y. Bin Misro, *An Efficient Numerical Scheme for The Simulation of Time-Fractional Nonhomogeneous Benjamin-Bona-Mahony-Burger Model*, Phys. Scr., **8** (2021), 10 pages. 1

[23] A. Majeed, M. Kamran, M. Abbas, J. Singh, *An Efficient Numerical Technique for Solving Time-Fractional Generalized Fisher's Equation*, Front. Phys., **8** (2020), 8 pages. 1

[24] M. Musraini, E. Rustam, L. Endang, H. Ponco, *Classical properties on conformable fractional calculus*, Pure Appl. Math. J., **8** (2019), 83–87. 1, 2.1, 2.2

[25] A. Souahi, A. Ben Makhlouf, M. A. Hammami, *Stability analysis of conformable fractional-order nonlinear systems*, Indag. Math. (N.S.), **28** (2017), 1265–1274. 1

[26] M. Syed Ali, M. Hymavathi, S. A. Kauser, G. Rajchakit, P. Hammachukiattikul, N. Boonsatit, *Synchronization of Fractional Order Uncertain BAM Competitive Neural Networks*, Fract. Fract., **6** (2022), 17 pages. 1

[27] M. Syed Ali, G. Narayanan, V. Shekher, A. Alsaedi, B. Ahmad, *Global Mittag-Leffler stability analysis of impulsive fractional-order complex-valued BAM neural networks with time varying delays*, Commun. Nonlinear Sci. Numer. Simul., **83** (2020), 22 pages.

[28] M. Syed Ali, G. Narayanan, V. Shekher, H. Alsulami, T. Saeed, *Dynamic stability analysis of stochastic fractional-order memristor fuzzy BAM neural networks with delay and leakage terms*, Appl. Math. Comput., **369** (2020), 23 pages.

[29] M. Syed Ali, G. Narayanan, V. Shekher, S. Arik, *Global stability analysis of fractional-order fuzzy BAM neural networks with time delay and impulsive effects*, Commun. Nonlinear Sci. Numer. Simul., **78** (2019), 12 pages. 1

[30] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ., **2013** (2013), 19 pages. 1

[31] A. R. Teel, *Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem*, IEEE Trans. Automat. Control, **43** (1998), 960–964. 1

[32] M. Tenenbaum, H. Pollard, *Ordinary differential equations*, Dover Publications, New York, (1986). 2.9, 3.1

[33] F. Usta, M. Z. Sarikaya, *Explicit bounds on certain integral inequalities via conformable fractional calculus*, Cogent Math., **4** (2017), 8 pages. 1

[34] F. Usta, M. Z. Sarikaya, *Some improvements of conformable fractional integral inequalities*, Int. J. Anal. Appl., **14** (2017), 162–166. 1

[35] X. H. Wang, *Mittag-Leffler stabilization of fractional-order nonlinear systems with unknown control coefficients*, Adv. Differ. Equ., **16** (2018), 14 pages. 1